Verifying Global Convergence of a Digital Phase-Locked Loop with Z3

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I. Problem: Global Convergence of a PLL

Phase-locked loops (PLL) are ubiquitous in analog and mixed-signal designs.

- Showing convergence to this locked behaviour is hard:
  - A large, continuous state space of possible starting conditions.
  - Simulations of convergence are impractical: time-to-lock is very large compared with the time-scales required for accurate circuit simulation.
  - Coverage achieved by the simulations is quite small.

II. A State-of-the-Art Digital PLL (from CICC 2010)

- PLL
  - Always Lock
  - Continuous
  - Global Convergence
  - Dynamical System
  - Analog
  - Digital

III. Lyapunov Function

- Continuous counterpart of a ranking function for discrete progress arguments.

IV. Lyapunov Function for Simple Nonlinear ODE

- Linear ODE: \( \dot{x} = Ax \)
- Let \( P \) be the solution of \( A^T P + PA = -I \). A possible Lyapunov function becomes \( \Psi(x) = X^T P X \).
- If \( P \) is positive definite, then the system \( \dot{x} = Ax \) globally converges to \( x = 0 \).

V. Verifying a Simplified Nonlinear Model using Z3

- Strong candidate for automation using symbolic/programatic differentiation
- Z3 easily proves global convergence.
- Fixed procedure: promising for automation.

VI. Improve the Model 1

Parameters with ranges:
- Inequality ranges for parameters \( \alpha \in 1 \pm 0.2, \beta \in 1 \pm 0.2, \gamma_0 \in 1 \pm 0.2 \) and \( c_0 \in 1 \pm 0.2 \).
- Strategy: Check where the parameters are used, try simplify non-linear part.

Example: Simplifying an element of the Jacobian matrix:

- The approximation is based on the observation that \( \alpha \in 1 \pm 0.2 \).

VII. Improve the Model 2

Adding quantization error:
- Need to show: \( \forall x \in Q_0 - Q_T, \forall \eta \in Err. f(x + \eta)^T P x < 0 \), where \( f \) is nonlinear. This creates nonlinear terms for the components of \( \eta \).
- Strategy: This is equivalent to: \( \forall x \in (Q_0 - Q_T) \oplus Err. \forall \eta \in Err. (x + \eta \in Q_0 - Q_T) \Rightarrow (h(x)^T P (x + \eta) < 0) \)

The Minkowski sum of two sets, \( A \oplus B \), is the set of elements that can be obtained as the sum of an element from \( A \) and an element from \( B \):

\( A \oplus B = \{ z \mid \exists a \in A, \exists b \in B, z = a + b \} \)

VIII. Conclusion & Future Work

- Conclusion:
  - Using a simplified model, we showed convergence where specifications for components are interval bounds using Z3.
  - SMT-based methods can address these problems more effectively than traditional simulation techniques.

- Future work:
  - Provide bounds on lock time.
  - Examine other digital PLL architectures to assess the reusability and automatability of this verification.
  - Component validation: formalize the connection between the models used and those used in other phases of the analog and mixed-signal design process.