Simultaneous Visibility Representations of Plane st-graphs Using L-shapes ⋆

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Abstract. Let $\langle G_r, G_b \rangle$ be a pair of plane st-graphs with the same vertex set V. A simultaneous visibility representation with L-shapes of $\langle G_r, G_b \rangle$ is a pair of bar visibility representations $\langle \Gamma_r, \Gamma_b \rangle$ such that, for every vertex $v \in V$, $\Gamma_r(v)$ and $\Gamma_b(v)$ are a horizontal and a vertical segment, which share an end-point. In other words, every vertex is drawn as an L-shape, every edge of G_r is a vertical visibility segment, and every edge of G_b is a horizontal visibility segment. Also, no two L-shapes intersect each other. An L-shape has four possible rotations, and we assume that each vertex is given a rotation for its L-shape as part of the input. Our main results are: (i) a characterization of those pairs of plane st-graphs admitting such a representation, (ii) a cubic time algorithm to recognize them, and (iii) a linear time drawing algorithm if the test is positive.

1 Introduction

Let G_r and G_b be two plane graphs with the same vertex set. A *simultaneous embedding (SE)* of $\langle G_r, G_b \rangle$ consists of two planar drawings, Γ_r of G_r and Γ_b of G_b, such that every edge is a simple Jordan arc, and every vertex is the same point both in Γ_r and in Γ_b. The problem of computing SEs has received a lot of attention in the Graph Drawing literature, partly for its theoretical interest and partly for its application to the visual analysis of dynamically changing networks on a common (sub)set of vertices. For example, it is known that any two plane graphs with the same vertex set admit a SE where the edges are polylines with at most two bends, which are sometimes necessary [8]. If the edges are straight-line segments, the representation is called a *simultaneous geometric embedding (SGE)*, and many graph pairs do not have an SGE: a tree and a path [1], a planar graph and a matching [6], and three paths [5]. On the positive side, the discovery of graph pairs that have an SGE is still a fertile research topic. The reader can refer to the survey by Bläsius, Kobourov and Rutter [21] for references and open problems.

Only a few papers study simultaneous representations that adopt a drawing paradigm different from SE and SGE. A seminal paper by Jampani and Lubiw initiates the study of *simultaneous intersection representations (SIR)* [16]. In an intersection representation of a graph, each vertex is a geometric object and there is an edge between two

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vertices if and only if the corresponding objects intersect. Let $\langle G_r, G_b \rangle$ be two graphs that have a subgraph in common. A SIR of $\langle G_r, G_b \rangle$ is a pair of intersection representations where each vertex in $G_r \cap G_b$ is mapped to the same object in both realizations. Polynomial-time algorithms for testing the existence of SIRs for chordal, comparability, interval, and permutation graphs have been presented [4, 15, 16].

We introduce and study a different type of simultaneous representation, where each graph is realized as a *bar visibility representation* and two segments representing the same vertex share an end-point. A bar visibility representation of a plane graph G is an embedding preserving drawing Γ where the vertices of G are non-overlapping horizontal segments, and two segments are joined by a vertical visibility segment if and only if there exists an edge in G between the two corresponding vertices (see, e.g., [18, 22]). A visibility segment has thickness $\epsilon > 0$ and does not intersect any other segment.

A *simultaneous visibility representation with L-shapes* of $\langle G_r, G_b \rangle$ is a pair of bar visibility representations $\langle \Gamma_r, \Gamma_b \rangle$ such that for every vertex $v \in V$, $\Gamma_r(v)$ and $\Gamma_b(v)$ are a horizontal and a vertical segment that share an end-point. In other words, every vertex is an L-shape, and every edge of G_r (resp., G_b) is a vertical (resp., horizontal) visibility segment. Also, no two L-shapes intersect. A simultaneous visibility representation with L-shapes of $\langle G_r, G_b \rangle$ where the rotation of the L-shape of each vertex in V is defined by a function $\Phi : V \to \mathcal{H} = \{\llcorner, \lrcorner, \urcorner, \ulcorner\}$, is called a *Φ-LSVR* in the following. While this definition does not assume any particular direction on the edges of G_r (resp., G_b), the resulting representation does induce a bottom-to-top (resp., left-to-right) st-orientation. In this paper, we assume that G_r and G_b are directed and this direction must be preserved in the visibility representation. Also, the two graphs have been augmented with distinct (dummy) sources and sinks. More formally, G_r=$(V \cup \{s_r, t_r\}, E_r)$ and G_b=$(V \cup \{s_b, t_b\}, E_b)$ are two plane st-graphs with sources s_r, s_b, and sinks t_r, t_b.

In terms of readability, this kind of simultaneous representation has the following advantages: (i) The edges are depicted as straight-line segments (as in SGE) and the edge-crossings are rectilinear; (ii) The edges of the two graphs are easy to distinguish, since they consistently flow from bottom to top for one graph and from left to right for the other graph. Having rectilinear crossing edges is an important benefit in terms of readability, as shown in [14], which motivated a relevant amount of research on right-angle crossing (RAC) drawings, see [9] for a survey.

Our main contribution is summarized by the following theorem.

Theorem 1. *Let G_r and G_b be two plane st-graphs defined on the same set of n vertices V and with distinct sources and sinks. Let $\Phi : V \to \mathcal{H} = \{\llcorner, \lrcorner, \urcorner, \ulcorner\}$. There exists an $O(n^3)$-time algorithm to test whether $\langle G_r, G_b \rangle$ admits a Φ-LSVR. Also, in the positive case, a Φ-LSVR can be computed in $O(n)$ time.*

This result relates to previous studies on topological rectangle visibility graphs [20] and transversal structures (see, e.g., [12, 13, 17, 19]). Also, starting from a Φ-LSVR of $\langle G_r, G_b \rangle$, we can compute a *simultaneous RAC embedding* of the two graphs with at most two bends per edge, improving the general upper bound by Bekos *et al.* [3] for those pairs of graphs that can be directed and augmented to admit a Φ-LSVR.

The proof of Theorem 1 is based on a characterization described in Section 3, which allows for an efficient testing algorithm presented in Section 4. Due to space restrictions, some proofs are omitted or only sketched in the text; full proofs can be found in [11].

2 Preliminaries

A graph $G = (V, E)$ is *simple*, if it contains neither loops nor multiple edges. We consider simple graphs, if not otherwise specified. A *drawing* Γ of G maps each vertex of V to a point of the plane and each edge of E to a Jordan arc between its two end-points. We only consider *simple drawings*, i.e., drawings such that the arcs representing two edges have at most one point in common, which is either a common end-vertex or a common interior point where the two arcs properly cross. A drawing is *planar* if no two arcs representing two edges cross. A planar drawing subdivides the plane into topologically connected regions, called *faces*. The unbounded region is called the *outer face*. A *planar embedding* of a graph is an equivalence class of planar drawings that define the same set of faces. A graph with a given planar embedding is a *plane* graph. For a non-planar drawing, we can still derive an embedding considering that the boundary of a face may consist also of edge segments between vertices and/or crossing points of edges. The unbounded region is still called the outer face.

A graph is *biconnected* if it remains connected after removing any one vertex. A directed graph (a digraph for short) is biconnected if its underlying undirected graph is biconnected. The *dual graph* D of a plane graph G is a plane multigraph whose vertices are the faces of G with an edge between two faces if and only if they share an edge. If G is a digraph, D is also a digraph whose dual edge e^* for a primal edge e is conventionally directed from the face, $\mathit{left}_G(e)$, on the left of e to the face, $\mathit{right}_G(e)$, on the right of e. Since we also use the opposite convention, we let $D^{\rightarrow}$ (resp., $D^{\leftarrow}$) be the dual whose edges cross the primal edges from left to right (resp., right to left).

A *topological numbering* of a digraph is an assignment, X, of numbers to its vertices such that $X(u) < X(v)$ for every edge (u, v). A graph admits a topological numbering if and only if it is acyclic. An acyclic digraph with a single source s and a single sink t is called an *st-graph*. In such a graph, for every vertex v, there exists a directed path from s to t that contains v [22]. A *plane st-graph* is an st-graph that is planar and embedded such that s and t are on the boundary of the outer face. In any st-graph, the presence of the edge (s, t) guarantees that the graph is biconnected. In the following we consider st-graphs that contain the edge (s, t), as otherwise it can be added without violating planarity. Let G be a plane st-graph, then for each vertex v of G the incoming edges appear consecutively around v, and so do the outgoing edges. Vertex s only has outgoing edges, while vertex t only has incoming edges. This is a particular transversal structure (see Section 3) known as a *bipolar orientation* [18, 22]. Each face f of G is bounded by two directed paths with a common origin and destination, called the *left path* and *right path* of f. For all vertices v and edges e on the left (resp., right) path of f, we let $\mathit{right}_G(v) = \mathit{right}_G(e) = f$ (resp., $\mathit{left}_G(v) = \mathit{left}_G(e) = f$).

Tamassia and Tollis [22] proved the following lemma.

Lemma 1 ([22]). *Let G be a plane st-graph and let $D^{\rightarrow}$ be its dual graph. Let u and v be two vertices of G. Then exactly one of the following four conditions holds:* $(\mathbf{i})$ *G has a path from u to v, or* $(\mathbf{ii})$ *from v to u;* $(\mathbf{iii})$ *$D^{\rightarrow}$ has a path from $\mathit{right}_G(u)$ to $\mathit{left}_G(v)$, or* $(\mathbf{iv})$ *from $\mathit{right}_G(v)$ to $\mathit{left}_G(u)$.*

Let v be a vertex of G, then denote by $B(v)$ (resp., $T(v)$) the set of vertices that can reach (resp., can be reached from) v. Also, denote by $L(v)$ (resp., $R(v)$) the set of vertices that are to the left (resp., to the right) of every path from s to t through v. By Lemma 1, these four sets partition the vertices of $G \setminus \{v\}$. In every planar drawing of G, they are contained in four distinct regions of the plane that share point v. The vertices of $B(v)$ are in the region delimited by the leftmost and the rightmost paths from s to v, while the vertices of $T(v)$ are in the region delimited by the leftmost and the rightmost paths from v to t. Edge (s,t) separates the two regions containing the vertices of $L(v)$ and $R(v)$, as in Fig. 1. Refer to [7] for further details.

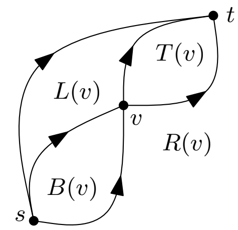


Fig. 1. Vertex sets $B(v)$, $T(v)$, $L(v)$, and $R(v)$ and their corresponding regions of the plane.

3 Characterization

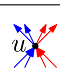
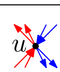
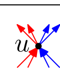
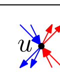
A *transversal structure* of a plane graph G, is a coloring and an orientation of the inner edges (i.e., those edges that do not belong to the outer face) of the graph that obey some local and global conditions. Transversal structures have been widely studied and important applications have been found. Bipolar orientations (also known as *st-orientations*) of plane graphs have been used to compute bar visibility representations [18, 22]. Further applications can be found in [12, 13, 17, 19], see also [10] for a survey.

To characterize those pairs of graphs that admit a Φ-LSVR, we introduce a new transversal structure for the union of the two graphs (which may be non-planar) and show that it is in bijection with the desired representation. In what follows $G_r = (V_r = V \cup \{s_r, t_r\}, E_r)$ and $G_b = (V_b = V \cup \{s_b, t_b\}, E_b)$ are two plane st-graphs with duals $D_r^{\rightarrow}$ and $D_b^{\leftarrow}$, respectively.

Definition 1. *Given $\Phi : V \rightarrow \mathcal{H} = \{$└, ┘, ┐, ┌$\}$, a* (4-polar) Φ-transversal *is a drawing of a directed (multi)graph on the vertex set $V \cup \{s_r, t_r, s_b, t_b\}$ whose edges are partitioned into red edges, blue edges, and the four special edges (s_r, s_b), (s_b, t_r), (t_r, t_b), and (t_b, s_r) forming the outer face, in clockwise order. In addition, the Φ-transversal obeys the following conditions:*

c1. *The red (resp., blue) edges induce an st-graph with source s_r (resp., s_b) and sink t_r (resp., t_b).*

c2. *For every vertex $u \in V$, the clockwise order of the edges incident to u forms four non-empty blocks of monochromatic edges, such that all edges in the same block are either all incoming or all outgoing with respect to u. The four blocks are encountered around u depending on $\Phi(u)$ as in the following table.*

└ ⇨ 	┘ ⇨ 	┐ ⇨ 	┌ ⇨ 
$f_b(u) = right_b(u)$ $f_r(u) = left_r(u)$	$f_b(u) = right_b(u)$ $f_r(u) = right_r(u)$	$f_b(u) = left_b(u)$ $f_r(u) = right_r(u)$	$f_b(u) = left_b(u)$ $f_r(u) = left_r(u)$

c3. *Only blue and red edges may cross and only if blue crosses red from left to right.*

A pair of plane st-graphs $\langle G_r, G_b \rangle$ *admits* a Φ-transversal if there exists a Φ-transversal G_{rb} such that restricting $G_{rb} \setminus \{s_b, t_b\}$ to the red edges realizes the planar embedding G_r and restricting $G_{rb} \setminus \{s_r, t_r\}$ to the blue edges realizes the planar embedding G_b.

Let u be a vertex of V, then the edges of a single color enter and leave u by the same face in the embedding of the other colored graph. In other words, as condition **c2** indicates, $\Phi(u)$ defines the face of G_b (resp., G_r), denoted by $f_b(u)$ (resp., $f_r(u)$), by which the edges of G_r (resp., G_b) incident to u enter and leave u, in the Φ-transversal. Also, condition **c3** implies that edges $\{(s_r, s_b), (s_b, t_r), (t_r, t_b), (t_b, s_r)\}$ are not crossed, because they are not colored.

In the remainder of this section we will prove the next theorem.

Theorem 2. *Let G_r and G_b be two plane st-graphs defined on the same set of vertices V and with distinct sources and sinks. Let $\Phi : V \to \mathcal{H} = \{$└ , ┘ , ┐ , ┌ $\}$. Then $\langle G_r, G_b \rangle$ admits a Φ-LSVR if and only if it admits a Φ-transversal.*

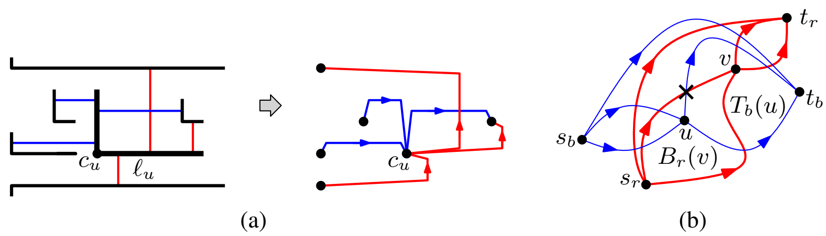


Fig. 2. (a) The replacement of the L-shape, ℓ_u, for vertex u with its corner point c_u and the drawing of u's adjacent edges with 2 bends per edge when constructing a Φ-transversal from a Φ-LSVR. Only ℓ_u's visibilities are shown. (b) Illustration for the proof of Lemma 3: the case when u is in $B(v)$ and v is in $T(u)$.

The necessity of the Φ-transversal is easily shown. Let $\langle \Gamma_r, \Gamma_b \rangle$ be a Φ-LSVR of $\langle G_r, G_b \rangle$ with two additional horizontal bars at the bottommost and topmost sides of the drawing that represent s_r and t_r, and two additional vertical bars at the leftmost and rightmost sides of the drawing that represent s_b and t_b. From such a representation we can compute a Φ-transversal G_{rb} as follows. Since the four vertices s_r, t_r, s_b, and t_b are represented by the extreme bars in the drawing, these four vertices belong to the outer face, and the four edges on the outer face can be added without crossings. Also, we color red all inner edges represented by vertical visibilities (directed from bottom to top), and blue all inner edges represented by horizontal visibilities (directed from left to right). To see that conditions **c1**, **c2** and **c3** are satisfied, let G_{rb} be a polyline drawing computed as follows. Let c_u be the corner of the L-shape, ℓ_u, representing vertex u. For every edge (u, v), replace its visibility segment by a polyline from c_u to c_v that has two bends, both contained in the visibility segment and each at distance δ from a different one of its endpoints, for an arbitrarily small, fixed $\delta > 0$. See Fig. 2(a). Finally, replace every L-shape ℓ_u with its corner c_u. Since each bar visibility representation preserves the embedding of the input graph, **c1** is respected. Also, **c2** and **c3** are clearly satisfied by the embedding derived from G_{rb}. We remark that, by construction, each edge is represented by a polyline with two bends and two edges cross only at right angles; this observation will be used in Section 5.

To prove sufficiency, assume $\langle G_r, G_b \rangle$ admits a Φ-transversal G_{rb}. We present an algorithm, Φ`LSVRDrawer`, that takes as input G_{rb} and returns a Φ-LSVR $\langle \Gamma_r, \Gamma_b \rangle$ of $\langle G_r, G_b \rangle$. We first recall the algorithm by Tamassia and Tollis (TT in the following) to compute an embedding preserving bar visibility representation of a plane st-graph G, see [7, 22]:

1. Compute the dual $D^{\rightarrow}$ of G.
2. Compute a pair of topological numberings Y of G and X of $D^{\rightarrow}$.
3. Draw each vertex v as a horizontal bar with y-coordinate $Y(v)$ and between x-coordinates $X(\mathit{left}_G(v))$ and $X(\mathit{right}_G(v)) - \epsilon$.
4. Draw each edge $e = (u, v)$ as a vertical segment at x-coordinate $X(\mathit{left}_G(e))$, between y-coordinates $Y(u)$ and $Y(v)$, and with thickness ϵ.

We are now ready to describe algorithm Φ`LSVRDrawer`.

Step 1: Compute the dual graphs $D_r^{\rightarrow}$ of G_r and $D_b^{\leftarrow}$ of G_b.
Step 2: Compute a pair of topological numberings n_r of G_r and n_b of G_b.
Step 3: Compute a pair of topological numberings n_r^* of $D_r^{\rightarrow}$ and n_b^* of $D_b^{\leftarrow}$.
Step 4: Compute a bar visibility representation Γ_r of G_r by using the TT algorithm with $X(u)$=$X_r(u) = n_r^*(u)$ and $Y(u)$=$Y_r(u)$=$n_b^*(f_b(u)) + n_r(u)\delta$, for each vertex u. Also, shift the horizontal segment for each vertex u to the left by $n_b(u)\delta$.
Step 5: Compute a bar visibility representation Γ_b' of G_b by using the TT algorithm with $X(u)$=$X_b(u)$=$n_b^*(u)$ and $Y(u)$=$Y_b(u)$=$n_r^*(f_r(u)) + n_b(u)\delta$, for each vertex u. Then turn Γ_b' into a vertical bar visibility representation, Γ_b, by drawing every horizontal segment $((x_0, y), (x_1, y))$ in Γ_b' as the vertical segment $((y, x_0), (y, x_1))$ in Γ_b. Finally, shift the vertical segment for each vertex u up by $n_r(u)\delta$.

Lemma 2 guarantees that Y_r and Y_b are valid topological numberings, and thus, that Γ_r and Γ_b are two bar visibility representations. Also, Lemma 3 ensures the union of Γ_r and Γ_b is a Φ-LSVR. The shifts performed at the end of **Steps 4-5** are to prevent the bars of two L-shapes from coinciding. The value $\delta > 0$ is chosen to be less than ϵ and less than the smallest difference between distinct numbers divided by the largest number from any topological numbering n_r, n_b, n_r^*, or n_b^*. This choice of δ guarantees that all visibilities are preserved after the shift, and that no new visibilities are introduced.

Lemma 2. *Y_r is a valid topological numbering of G_r and Y_b is a valid topological numbering of G_b.*

Proof. Let (u, v) be a red edge from u to v. We know that $n_r(u) < n_r(v)$. Let e_0, e_1, $\ldots$, e_k be the blue edges crossed by (u, v) in G_{rb}. Due to conditions **c2** and **c3**, there exists a path $\{f_b(u) = \mathit{right}_b(e_0), \mathit{left}_b(e_0) = \mathit{right}_b(e_1), \ldots, \mathit{left}_b(e_{k-1}) = \mathit{right}_b(e_k),$

$\mathit{left}_b(e_k) = f_b(v)\}$ in $D_b^{\leftarrow}$. Thus, we also know that $n_b^*(f_b(u)) \leq n_b^*(f_b(v))$. Since $Y_r(u) = n_b^*(f_b(u)) + n_r(u)\delta$ and $\delta > 0$, it follows that $Y_r(u) < Y_r(v)$. A symmetric argument shows $Y_b(u) < Y_b(v)$ if (u, v) is a blue edge. □

Lemma 3. *Each vertex u of V is represented by an L-shape ℓ_u in $\langle \Gamma_r, \Gamma_b \rangle$ as defined by the function Φ. Also no two L-shapes intersect each other.*

Proof. Suppose $\Phi(u)=$ └ , as the other cases are similar. Then, $f_b(u)=\mathit{right}_b(u)$ and $f_r(u)=\mathit{left}_r(u)$. The horizontal bar representing u in Γ_r is the segment $[p_0(u), p_1(u)]$, where the two points $p_0(u)$ and $p_1(u)$ are $p_0(u) = (n_r^*(\mathit{left}_r(u)) + n_b(u)\delta, Y_r(u))$, and $p_1(u) = (n_r^*(\mathit{right}_r(u)) + n_b(u)\delta, Y_r(u))$. Note that, $n_r^*(\mathit{left}_r(u)) < n_r^*(\mathit{right}_r(u))$. The vertical bar representing u in Γ_b is the segment $[q_0(u), q_1(u)]$, where the two points $q_0(u)$ and $q_1(u)$ are $q_0(u) = (Y_b(u), n_b^*(\mathit{right}_b(u)) + n_r(u)\delta)$, and $q_1(u) = (Y_b(u), n_b^*(\mathit{left}_b(u)) + n_r(u)\delta)$. Note that, $n_b^*(\mathit{right}_b(u)) < n_b^*(\mathit{left}_b(u))$. Since $Y_r(u) = n_b^*(f_b(u)) + n_r(u)\delta = n_b^*(\mathit{right}_b(u)) + n_r(u)\delta$, the bottom coordinate of the vertical bar representing u matches the y-coordinate of the horizontal bar representing u. Since $Y_b(u) = n_r^*(f_r(u)) + n_b(u)\delta = n_r^*(\mathit{left}_r(u)) + n_b(u)\delta$, the left coordinate of the horizontal bar representing u matches the x-coordinate of the vertical bar representing u. Thus the two bars form the L-shape └ .

We now show that no two L-shapes properly intersect each other. Suppose by contradiction that the vertical bar of a vertex u, properly intersects the horizontal bar of a vertex v. Based on Φ, the vertical bar of u involved in the intersection is either a left vertical bar or a right vertical bar, and it is drawn at x-coordinate $n_r^*(\mathit{left}_r(u)) + n_b(u)\delta$ or $n_r^*(\mathit{right}_r(u)) + n_b(u)\delta$, respectively. Suppose it is a left vertical bar, as the other case is symmetric. Since u's vertical bar properly intersects v's horizontal bar, we know by construction that $n_r^*(\mathit{left}_r(v)) + n_b(v)\delta < n_r^*(\mathit{left}_r(u)) + n_b(u)\delta < n_r^*(\mathit{right}_r(v)) + n_b(v)\delta$. Proper intersection implies that these inequalities are strict, that there is a path in the red dual $D_r^{\rightarrow}$ from $\mathit{left}_r(v)$ to $\mathit{left}_r(u)$ to $\mathit{right}_r(v)$, and that the three faces are distinct. This implies that u belongs either to $B_r(v)$ or to $T_r(v)$, and it lies in the corresponding regions of the plane, with $f_r(u)$ (and hence the start/end of curves representing blue edges incident to u) inside the region. Similarly, by considering the blue dual $D_b^{\leftarrow}$, $n_b^*(\mathit{right}_b(u)) + n_r(u)\delta < n_b^*(f_b(v)) + n_r(v)\delta < n_b^*(\mathit{left}_b(u)) + n_r(u)\delta$, we know that v belongs either to $B_b(u)$, or to $T_b(u)$, and it lies in the corresponding regions of the plane, with $f_b(v)$ (and hence the start/end of curves representing red edges incident to v) inside the region. No matter which region, $B_r(v)$ or $T_r(v)$, vertex u lies in, or which region, $B_b(u)$ or $T_b(u)$, vertex v lies in, the directed boundary of the blue region ($B_b(u)$ or $T_b(u)$) containing v crosses the directed boundary of the red region ($B_r(v)$ or $T_r(v)$) containing u from right to left. This either violates condition **c3** (if edges of the boundaries cross) or it violates condition **c2** (if the boundaries share a vertex). See Fig. 2(b) for an illustration. □

Theorem 3. *Let G_r and G_b be two plane st-graphs defined on the same set of n vertices V and with distinct sources and sinks. Let $\Phi : V \rightarrow \mathcal{H} = \{$└ , ┘ , ┐ , ┌ $\}$. If $\langle G_r, G_b \rangle$ admits a Φ-transversal, then algorithm Φ`LSVRDrawer` computes a Φ-LSVR of $\langle G_r, G_b \rangle$ in $O(n)$ time.*

Proof. Lemmas 2 and 3 imply that Φ`LSVRDrawer` computes a Φ-LSVR of $\langle G_r, G_b \rangle$. Computing the dual graphs and the four topological numberings (**Steps 1-3**), as well as computing the two bar visibility representations and shifting each segment (**Steps 4-5**), can be done in $O(n)$ time, as shown in [7, 22]. $\square$

4 Testing Algorithm

In this section we show how to test whether two plane st-graphs with the same set of vertices admit a Φ-LSVR for a given function ϕ. In [11] it is shown a pair of graphs $\langle G_r, G_b \rangle$ that does not admit a Φ-LSVR for any function Φ. This emphasizes the need for an efficient testing algorithm. Our algorithm exploits the interplay between the primal of the blue (red) graph and the dual of the red (blue) graph. Given the circular order of the edges around each vertex imposed by the function ϕ, we aim to compute a suitable path in the red dual for each blue edge. Such paths will then be used to route the blue edges. Finally, we check that no two blue edges cross.

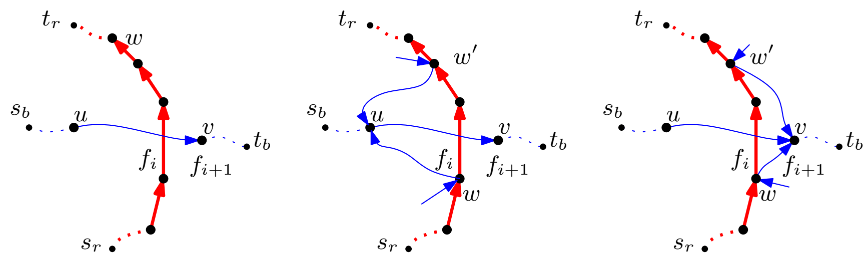
We first introduce a few definitions. Let G and $D^{\rightarrow}$ be a plane st-graph and its dual. Let f and g be two faces of G that share an edge $e = (x, z)$ of G, such that e belongs to the right (resp., left) path of f (resp., g). Let e^* be the dual edge in $D^{\rightarrow}$ corresponding to e. Let w be a vertex on the right path of f (or, equivalently, on the left path of g). Then w is *cut from above* (resp., *below*) by e^*, if w precedes z (resp., succeeds x) along the right path of f, i.e., all vertices that precede z (including x) are cut from above, while all vertices that succeed x (including z) are cut from below by e^*.

Let G_r and G_b be a pair of plane st-graphs with the same vertex set V and with distinct sources and sinks. Let $\Phi : V \rightarrow \mathcal{H} = \{\llcorner, \lrcorner, \urcorner, \ulcorner\}$. Recall that, for a given vertex u of G_b, with the notation $L_b(u)$, $R_b(u)$, $T_b(u)$ and $B_b(u)$ we represent the set of vertices to the left, to the right, that are reachable from, and that can reach u in G_b, respectively (see Section 2). Then consider an edge $e = (u, v)$ of G_b and a path[3] $\pi_e = \{f_r(u) = f_0, e^*_0, f_1, \ldots, f_{k-1}, e^*_{k-1}, f_r(v) = f_k\}$ in $D^{\rightarrow}_r$, where f_i $(0 \leq i \leq k)$ are the faces traversed by the path, and e^*_i $(0 \leq i < k)$ are the dual edges used by the path to go from f_i to f_{i+1}. Path π_e is a *traversing path* for e, if $\pi_e = \{f_r(u) = f_r(v)\}$, or for all $0 \leq i < k$ and all vertices w in the right path of f_i:

p1. If $w \in L_b(u)$ then w is cut from below by e^*_i. See Fig. 3(a).
p2. If $w \in R_b(u)$ then w is cut from above by e^*_i.
p3. If $w \in B_b(u)$ and $\Phi(w) = \llcorner$ (resp., $\Phi(w) = \ulcorner$) then w is cut from above (resp., below) by e^*_i. See Fig. 3(b).
p4. If $w \in T_b(u)$ and $\Phi(w) = \ulcorner$ (resp., $\Phi(w) = \llcorner$) then w is cut from above (resp., below) by e^*_i.
p5. If $w \in B_b(v)$ and $\Phi(w) = \urcorner$ (resp., $\Phi(w) = \lrcorner$) then w is cut from above (resp., below) by e^*_i. See Fig. 3(c).
p6. If $w \in T_b(v)$ and $\Phi(w) = \lrcorner$ (resp., $\Phi(w) = \urcorner$) then w is cut from above (resp., below) by e^*_i.

We now show that if $\langle G_r, G_b \rangle$ admits a Φ-transversal, then for each blue edge (the same argument would apply for red edges) there exists a unique traversing path.

[3] Since $D^{\rightarrow}_r$ is a multigraph, to uniquely identify π_e we specify the edges that are traversed.



(a) **p1** (b) **p3** (c) **p5**

Fig. 3. Illustration for some of the properties of a traversing path π_e.

Lemma 4. *Let G_r and G_b be two plane st-graphs with the same vertex set V and with distinct sources and sinks. Let $\Phi : V \to \mathcal{H} = \{\llcorner, \lrcorner, \urcorner, \ulcorner\}$. If $\langle G_r, G_b \rangle$ admits a Φ-transversal, then for every edge e of G_b there is a unique traversing path π_e in $D_r^{\rightarrow}$.*

Proof. If $\langle G_r, G_b \rangle$ admits a Φ-transversal G_{rb}, then for every edge $e = (u, v)$ of G_b there exists a path $\pi_e = \{f_r(u) = f_0, e_0^*, f_1, \ldots, f_{k-1}, e_{k-1}^*, f_r(v) = f_k\}$ in $D_r^{\rightarrow}$, which is the path used by e to go from $f_r(u)$ to $f_r(v)$ in G_{rb}.

If f_0 and f_k coincide, then π_e is a traversing path. Otherwise, we would have a cycle $\pi_e = \{f_0 = f_k, \ldots, f_0 = f_k\}$, which is not possible since $D_r^{\rightarrow}$ is acyclic, being the dual of a plane st-graph.

If f_0 and f_k do not coincide, let w be a vertex in the right path of f_i. First, if w belongs to $L_b(u)$, then it is cut from below. Otherwise, if w was cut from above, since edge $e = (u, v)$ cannot cross the right path of f_i twice (by condition **c3**), it would belong to $R_b(u)$, a contradiction with the fact that the embedding of G_b is preserved. Thus **p1** is respected by π_e. With a symmetric argument we can also prove **p2**. Suppose now that w belongs to $B_b(u)$, then $f_r(w) = f_i = \mathit{left}_r(w)$, otherwise if $f_r(w) = f_{i+1} = \mathit{right}_r(w)$, the blue path from w to u would violate **c3**. In other words, either $\Phi(w) = \llcorner$ or $\Phi(w) = \ulcorner$. Furthermore, if $\Phi(w) = \llcorner$, then w must be cut from above, while if $\Phi(w) = \ulcorner$, then w must be cut from below, as otherwise the incoming blue edges to w must enter a region delimited by the blue path from w to u, the blue edge (u, v), and part of the (red) right path of f_i, which violates the planarity of the embedding of G_b or condition **c2** (see Fig. 3(b)). Thus **p3** is respected by π_e. With similar arguments one can prove **p4** – **p6**. Hence, π_e is a traversing set. To prove that π_e is unique, note that any possible traversing set for e must start from f_0 and leave this face. Hence, any vertex w on the right path of f_0 must be cut from either above or below, according to properties **p1** – **p6** (which cover all possible cases for w). The only edge that can satisfy the cut condition for all vertices on the right path of f_0, is an edge e_0^* whose corresponding red primal edge, denoted by (x, z), is such that all vertices on the right path of f_0 above x must be cut from below and all those below z must be cut from above. Clearly, this edge is unique. By repeatedly applying this argument for each face f_i $(0 \leq i < k)$, the traversing path π_e is uniquely identified. □

The next theorem concludes the proof of Theorem 1.

Theorem 4. *Let G_r and G_b be two plane st-graphs with the same set of n vertices V and with distinct sources and sinks. Let $\Phi : V \rightarrow \mathcal{H} = \{\llcorner, \lrcorner, \urcorner, \ulcorner\}$. There exists an $O(n^3)$-time algorithm to test whether $\langle G_r, G_b \rangle$ admits a Φ-transversal.*

Proof sketch: Our testing algorithm aims to compute (if it exists) a Φ-transversal G_{rb} for $\langle G_r, G_b \rangle$. We fix the circular order of the edges restricted to the blue edges (resp., red edges) around each vertex u of G_{rb} to satisfy **c1** and to maintain the planar embedding of G_b (resp., G_r). We then fix the circular order of the blue edges with respect to the red edges around each vertex u of G_{rb} to satisfy **c2** (i.e., to obey $\Phi(u)$). Then, we first check if for every blue edge e there exists a traversing path π_e; if so, we verify that by routing every blue edge e through π_e no two blue edges cross each other. If this procedure succeeds then $\langle G_r, G_b \rangle$ admits Φ-transversal, because, by construction, the resulting embedding of G_{rb} satisfies conditions **c1**, **c2** and **c3**. Otherwise, either there exists a blue edge with no traversing path, or two traversing paths are such that the two corresponding edges of G_b cross if routed through them. In the first case $\langle G_r, G_b \rangle$ does not admit a Φ-transversal by Lemma 4. In the second case, since the traversing paths are unique, condition **c2** cannot be satisfied, and again a Φ-transversal cannot be found. The testing algorithm works in two phases as follows.

Phase 1. For every edge $e = (u, v) \in E_b$. If $f_r(u) = f_r(v)$, we have found a traversing path. Otherwise, we label each vertex on the right path of $f_r(u)$, by A if it must be cut from above or by B if it must be cut from below, according to properties **p1** – **p6**. Then we check if the sequence of labels along the path is a nonzero number of A's followed by a nonzero number of B's. If so, then the dual edge of the traversing path is the one whose corresponding primal edge has the two end-vertices with different labels (which is unique). If this is not the case, then a traversing path for e does not exist. In the positive case, we add the dual edge we found and the next face we reach through this edge to π_e and we iterate the algorithm until we reach either $f_r(v)$ or the outer face of $D_r^{\rightarrow}$. In the former case π_e is a traversing path for e, while in the latter case, since the edges of the outer face of G_{rb} cannot be crossed by definition of Φ-transversal, we have that again no traversing path can be found.

Phase 2. We now check that by routing every edge $e \in E_b$ through its corresponding traversing path π_e, no two of these edges cross each other. Consider the dual graph $D_r^{\rightarrow}$, which is a plane st-graph. Construct a planar drawing Γ of $D_r^{\rightarrow}$. Consider any two traversing paths π_e and π'_e, which corresponds to two paths in Γ, and let $e = (u, v)$ and $e' = (w, z)$ be the two corresponding edges of G_b. Denote by $\hat{\pi}_e = \{u\} \cup \pi_e \cup \{v\}$ and $\hat{\pi}'_e = \{w\} \cup \pi'_e \cup \{z\}$ the two enriched paths. Enrich Γ by adding the four edges $(x, f_r(x))$, where $x \in \{u, v, w, z\}$, in a planar way respecting the original embedding of G_b. Consider now the subdrawing Γ' of Γ induced by $\hat{\pi}_e \cup \hat{\pi}'_e$. If e and e' cross each other, then $\pi_e \cap \pi'_e$ cannot be empty. Moreover, the intersection $\pi_e \cap \pi'_e$ must be a single subpath, as otherwise the two traversing paths would not be unique. Let f be the first face and let g be the last face in this subpath. Let e_u be the incoming edge of f that belongs to the subpath of $\hat{\pi}_e$ from u to f; and let e_w be the incoming edge of f that belongs to the subpath of $\hat{\pi}'_e$ from w to f. Also, let e_v be the outgoing edge of g that belongs to the subpath of $\hat{\pi}_e$ from g to v; and let e_z be the outgoing edge of g that belongs to the subpath of $\hat{\pi}'_e$ from g to z. Then e and e' cross if and only if walking clockwise along $\pi_e \cup \pi'_e$ from f to g and back to f these four edges are encountered in the circular order e_u, e_z, e_v, e_w. Note that, e_u and e_w may coincide if $u = w$, and similarly for e_v and e_z. $\square$

5 Final Remarks and Open Problems

In this paper we have introduced and studied the concept of simultaneous visibility representation with L-shapes of two plane st-graphs. We remark that it is possible to include in our theory the case when the vertices can also be drawn as rectangles. Nevertheless, this would not enlarge the class of representable pairs of graphs. In fact, for every vertex v drawn as a rectangle $\mathcal{R}_v$, we can replace $\mathcal{R}_v$ with any L-shape by keeping only two adjacent sides of $\mathcal{R}_v$ in the drawing and prolonging the visibilites incident to the removed sides of $\mathcal{R}_v$. The converse is not true. Indeed, roughly speaking, L-shapes can be nested, whereas rectangles cannot. To give an example, if a vertex v must see a vertex u both vertically and horizontally, this immediately implies that the two corresponding rectangles need to overlap, while two L-shapes could instead be nested. Several extensions of the model introduced in this paper can also be studied, e.g., the case where every edge is represented by a T-shape, or more generally by a $+$-shape.

Our results can also be used to shed more light on the problem of computing a simultaneous RAC embedding (SRE) [2, 3]. Given two planar graphs with the same vertex set, an SRE is a simultaneous embedding where crossings between edges of the two graphs occur at right angles. Argyriou *et al.* proved that it is always possible to construct an SRE with straight-line edges of a cycle and a matching, while there exist a wheel graph and a cycle that do not admit such a representation [2]. This motivated recent results about SRE with bends along the edges. Namely, Bekos *et al.* show that two planar graphs with the same vertex set admit an SRE with at most six bends per edge in both graphs [3]. We observe that any pair of graphs that admit a simultaneous visibility representation with L-shapes also admits an SRE with at most two bends per edge. This is obtained with the technique used in Section 3 to compute a Φ-transversal from a Φ-LSVR, see Fig. 2(a). Thus, a new approach to characterize graph pairs that have SREs with at most two bends per edge is as follows: Given two planar graphs with the same vertex set, add to each of them a unique source and a unique sink, and look for two st-orientations (one for each of the two graphs) and a function Φ such that the two graphs admit a Φ-LSVR. In [11], we show an alternative proof of another result by Bekos *et al.* that a wheel graph and a matching admit an SRE with at most two bends for each edge of the wheel, and no bends for the matching edges [3].

Three questions that stem from this paper are whether the time complexity of the testing algorithm in Section 4 can be improved; what is the complexity of deciding if two given plane st-graphs admit a Φ-LSVR for some function Φ, which is not part of the input; and what is the complexity of deciding if two undirected graphs admit a Φ-LSVR for some function Φ.

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