SEFE without Mapping via Large Induced Outerplane Graphs in Plane Graphs*

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Abstract. We show that every n-vertex planar graph admits a simultaneous embedding without mapping and with fixed edges with any $(n/2)$-vertex planar graph. In order to achieve this result, we prove that every n-vertex plane graph has an induced outerplane subgraph containing at least $n/2$ vertices. Also, we show that every n-vertex planar graph and every n-vertex planar partial 3-tree admit a simultaneous embedding without mapping and with fixed edges.

1 Introduction

Simultaneous embedding is a flourishing area of research studying topological and geometric properties of planar drawings of multiple graphs on the same point set. The seminal paper in the area is the one of Braß *et al.* [7], in which two types of simultaneous embedding are defined, *with mapping* and *without mapping*. In the former variant, a bijective mapping between the vertex sets of any two graphs G_1 and G_2 to be drawn is part of the problem's input, and the goal is to construct a planar drawing of G_1 and a planar drawing of G_2 so that corresponding vertices are mapped to the same point. In the latter variant, the drawing algorithm is free to map any vertex of G_1 to any vertex of G_2 (still the n vertices of G_1 and the n vertices of G_2 have to be placed on the same n points). Simultaneous embeddings have been studied with respect to two different drawing standards: In a *geometric simultaneous embedding*, edges are required to be straight-line segments. In a *simultaneous embedding with fixed edges* (also known as SEFE), edges can be arbitrary curves, but each edge that belongs to both G_1 and G_2 must be represented by the same curve in the drawing of G_1 and in the drawing of G_2.

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Many papers deal with the problem of constructing geometric simultaneous embeddings and simultaneous embeddings with fixed edges of pairs of planar graphs in the variant *with mapping*. Typical considered problems include: (i) determining notable classes of planar graphs that always or may not always admit a simultaneous embedding; (ii) designing algorithms for constructing simultaneous embeddings within small area and with few bends on the edges; (iii) determining the time complexity of testing the existence of a simultaneous embedding for a given set of graphs. We refer the reader to the recent survey by Bläsius *et al.* [4].

In contrast to the large number of papers dealing with simultaneous embedding *with mapping*, little progress has been made on the *without mapping* version of the problem. Braß *et al.* [7] showed that, for any $k \geq 1$, planar graphs $G_1, \dots, G_k$ admit a geometric simultaneous embedding without mapping if $G_2, \dots, G_k$ are outerplanar. They left open the following attractive question: Do every two n-vertex planar graphs admit a geometric simultaneous embedding without mapping? Cardinal *et al.* [8] have shown that a constant number of graphs is the most we could hope for by demonstrating a collection of 7,393 n-vertex planar graphs ($n = 35$) that do not admit a simultaneous geometric embedding without mapping.

In this paper we initiate the study of simultaneous embeddings with fixed edges and without mapping, called SefeNoMap for brevity. In this setting, the natural counterpart of the Braß *et al.* [7] question reads as follows: Do every two n-vertex planar graphs admit a SefeNoMap?

Since answering this question seems to be an elusive goal, we tackle the following generalization of the problem: What is the largest $k \leq n$ such that every n-vertex planar graph and every k-vertex planar graph admit a SefeNoMap? That is: What is the largest $k \leq n$ such that every n-vertex planar graph G_1 and every k-vertex planar graph G_2 admit two planar drawings Γ_1 and Γ_2 with their vertex sets mapped to point sets P_1 and P_2, respectively, so that $P_2 \subseteq P_1$ and so that if edges e_1 of G_1 and e_2 of G_2 have their end-vertices mapped to the same two points p_a and p_b, then e_1 and e_2 are represented by the same curve between p_a and p_b in Γ_1 and in Γ_2? In this paper we prove that $k \geq n/2$, that is:

Theorem 1. *Every n-vertex planar graph and every $\lceil n/2 \rceil$-vertex planar graph have a* SefeNoMap.

Observe that the previous theorem would be easily proved if $\lceil n/2 \rceil$ were replaced with $\lceil n/4 \rceil$: First, consider an $\lceil n/4 \rceil$-vertex independent set I of any n-vertex planar graph G_1 (which always exists, as a consequence of the four color theorem [14,15]). Then construct any planar drawing Γ_1 of G_1, and let $P(I)$ be the point set on which the vertices of I are mapped in Γ_1. Finally, construct a planar drawing Γ_2 of any $\lceil n/4 \rceil$-vertex planar graph G_2 on point set $P(I)$ (e.g. using Kaufmann and Wiese's technique [13]). Since I is an independent set, any bijective mapping between the vertex set of G_2 and I ensures that G_1 and G_2 share no edges. Thus, Γ_1 and Γ_2 are a SefeNoMap of G_1 and G_2.

In order to get the $\lceil n/2 \rceil$ bound, we study the problem of finding a large induced outerplane graph in a plane graph. A *plane graph G* is a planar graph

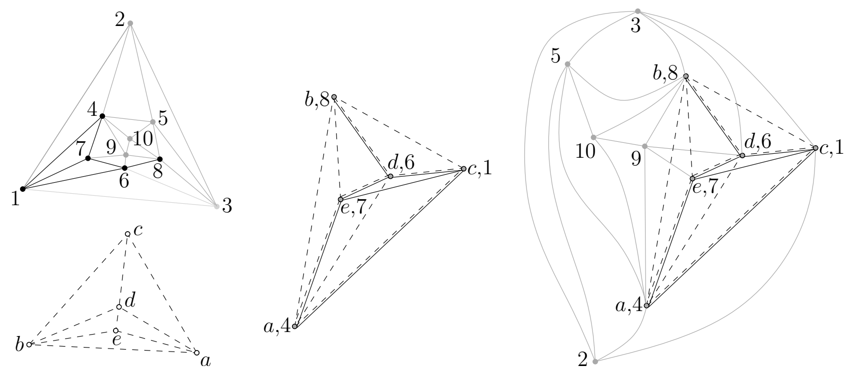

Fig. 1. (a) A 10-vertex planar graph G_1 (solid lines) and a 5-vertex planar graph G_2 (dashed lines). A 5-vertex induced outerplane graph $G_1[V']$ in G_1 is colored black. Vertices and edges of G_1 not in $G_1[V']$ are colored gray. (b) A straight-line planar drawing $\Gamma(G_2)$ of G_2 with no three collinear vertices, together with a straight-line planar drawing of $G_1[V']$ on the point set P_2 defined by the vertices of G_2 in $\Gamma(G_2)$. (c) A SefeNoMap of G_1 and G_2.

together with a *plane embedding*, that is, an equivalence class of planar drawings of G, where two planar drawings Γ_1 and Γ_2 are equivalent if: (1) the *rotation systems* of G in Γ_1 and in Γ_2 coincide, i.e., the clockwise order of the edges incident to each vertex of G is the same in Γ_1 and in Γ_2; (2) each face has the same *facial boundaries* in Γ_1 and in Γ_2, i.e., for each face f the lists of vertices determined by clockwise traversing the walks delimiting f are the same in Γ_1 and in Γ_2; and (3) Γ_1 and Γ_2 have the same *outer face*. Observe that, for planar drawings of connected graphs, condition (2) is implied by condition (1). An *outerplane graph* is a graph together with an *outerplane embedding*, that is a plane embedding where all the vertices are incident to the outer face. An *outerplanar graph* is a graph that admits an outerplane embedding; a plane embedding of an outerplanar graph is not necessarily outerplane. Consider a plane graph G and a subset V' of its vertex set. The *induced plane graph* $G[V']$ is the subgraph of G induced by V' together with the plane embedding *inherited* from G, i.e., the embedding obtained from the plane embedding of G by removing all the vertices and edges not in $G[V']$. We show the following result.

Theorem 2. *Every n-vertex plane graph $G(V,E)$ has a vertex set $V' \subseteq V$ with $|V'| \geq n/2$ such that $G[V']$ is an outerplane graph.*

Theorem 2 and the results of Gritzmann *et al.* [10] yield a proof of Theorem 1, as follows.

Proof of Theorem 1: Consider any n-vertex plane graph G_1 and any $\lceil n/2 \rceil$-vertex plane graph G_2 (see Fig. 1(a)). Let $\Gamma(G_2)$ be any straight-line planar drawing of G_2 in which no three vertices are collinear. Denote by P_2 the set of

$\lceil n/2 \rceil$ points to which the vertices of G_2 are mapped in $\Gamma(G_2)$. Consider any $\lceil n/2 \rceil$-vertex subset $V' \subseteq V(G_1)$ such that $G_1[V']$ is an outerplane graph. Such a set exists by Theorem 2. Construct a straight-line planar drawing $\Gamma(G_1[V'])$ of $G_1[V']$ in which its vertices are mapped to P_2 so that the resulting drawing has the same (outerplane) embedding as $G_1[V']$. Such a drawing exists by results of Gritzmann *et al.* [10]; also it can be found efficiently by results of Bose [6] (see Fig. 1(b)). Construct any planar drawing $\Gamma(G_1)$ of G_1 in which the drawing of $G_1[V']$ is $\Gamma(G_1[V'])$. Such a drawing exists, given that $\Gamma(G_1[V'])$ is a planar drawing of a plane subgraph $G_1[V']$ of G_1 preserving the embedding of $G_1[V']$ in G_1 (see Fig. 1(c)). Both $\Gamma(G_1)$ and $\Gamma(G_2)$ are planar, by construction. Also, the only edges that are possibly shared by G_1 and G_2 are those between two vertices that are mapped to P_2. However, such edges are drawn as straight-line segments both in $\Gamma(G_1)$ and in $\Gamma(G_2)$. Thus, $\Gamma(G_1)$ and $\Gamma(G_2)$ are a SefeNoMap of G_1 and G_2. □

By the straightforward observation that the vertices in the odd (or even) levels of a breadth-first search tree of a planar graph induce an *outerplanar* graph, we know that G has an induced outerplanar graph with at least $n/2$ vertices. However, since its embedding in G may not be outerplane, this seems insufficient to prove the existence of a SefeNoMap of every n-vertex and every $\lceil n/2 \rceil$-vertex planar graph.

Theorem 2 might be of independent interest, as it is related to (in fact it is a weaker version of) a famous and long-standing graph theory conjecture:

Conjecture 1. (Albertson and Berman 1979 [2]) Every n-vertex planar graph $G(V, E)$ has a vertex set $V' \subseteq V$ with $|V'| \geq n/2$ such that $G[V']$ is a forest.

Conjecture 1 would prove the existence of an $\lceil n/4 \rceil$-vertex independent set in a planar graph without using the four color theorem [14,15]. The best known partial result related to Conjecture 1 is that every planar graph has a vertex subset with 2/5 of its vertices inducing a forest, which is a consequence of the *acyclic* 5-*colorability* of planar graphs [5]. Variants of the conjecture have also been studied where G is further restricted to be bipartite [1] or outerplanar [12], or where each connected component of the induced forest is required to be a path [17,18].

The topological structure of an outerplane graph is arguably much closer to that of a forest than the one of a non-outerplane graph. Thus the importance of Conjecture 1 may justify the study of induced outerplane graphs in plane graphs in its own right.

To complement the results of the paper, we also show the following. A *plane* 3-*tree* is inductively defined as follows: (1) The complete graph K_3 together with its unique plane embedding is the only plane 3-tree with three vertices; and (2) every plane 3-tree G_n with $n \geq 4$ vertices can be obtained from a plane 3-tree G_{n-1} with $n-1$ vertices by inserting a vertex w inside an internal face (u, v, z) of G_{n-1} and connecting w with u, v, and z. A *planar* 3-*tree* is a graph that admits a plane embedding as a plane 3-tree. A *partial planar* 3-*tree* is a subgraph of a planar 3-tree. A *planar partial* 3-*tree* is a planar graph with tree-width at most

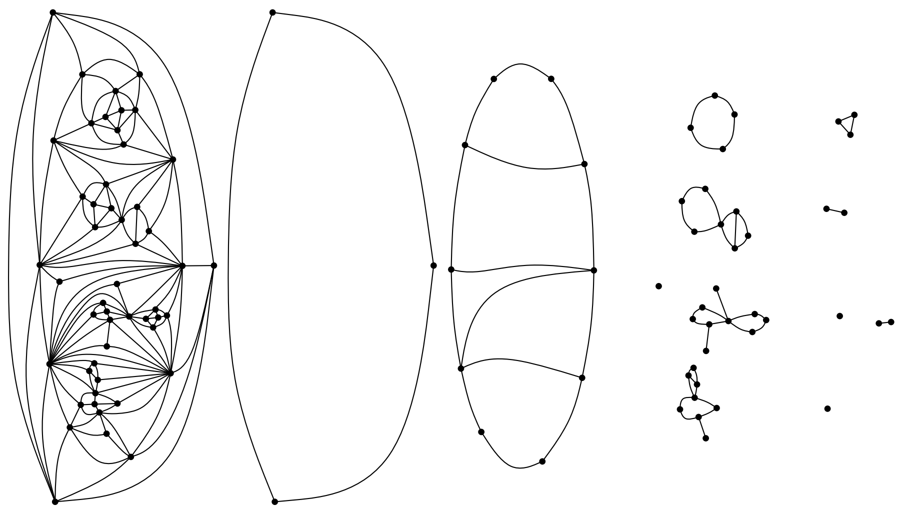

Fig. 2. A maximal plane graph G with outerplanarity index 4 and its levels.

three. The class of partial planar 3-trees and the class of planar partial 3-trees are equal [16]. We, typically, use the latter term for the class.

Theorem 3. *Every n-vertex planar graph and every n-vertex planar partial* 3*-tree have a* SefeNoMap*.*

The rest of the paper is organized as follows. In Section 2 we prove Theorem 2; in Section 3 we prove Theorem 3; finally, in Section 4 we conclude and suggest some open problems.

2 Proof of Theorem 2

In this section we prove Theorem 2, that is, we show an algorithm that receives as input an n-vertex plane graph $G(V, E)$ and constructs a set $V' \subseteq V$ with $|V'| \geq n/2$ such that $G[V']$ is an outerplane graph.

We assume that G is a *maximal* plane graph, that is, a plane graph whose faces are all delimited by 3-cycles. If that is not the case, dummy edges can be added to G in order to make it a maximal plane graph G'. Then the vertex set V' of an induced outerplane graph $G'[V']$ in G' induces an outerplane graph in G, as well.

Outerplane levels. Our algorithm will use a natural plane graph decomposition, which consists of "peeling" a plane graph by repeatedly removing the vertices incident to its outer face and their incident edges. Formally, let G be a maximal plane graph, let $G_1^* = G$ and, for any $i \geq 1$, let G_{i+1}^* be the plane graph obtained by removing from G_i^* the set V_i of vertices incident to the outer

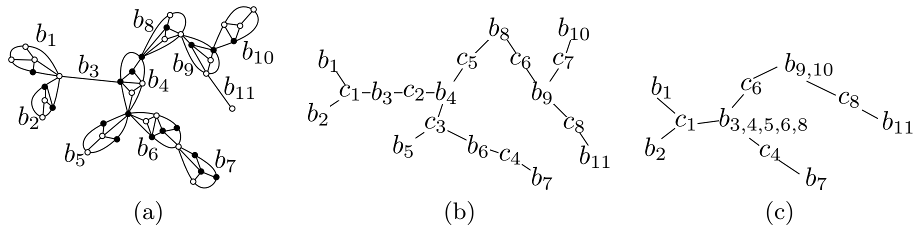

Fig. 3. (a) A connected internally-triangulated plane graph G with a 2-coloring ψ, (b) the block-cutvertex tree $\mathcal{BC}(G)$, and (c) the contracted block-cutvertex tree $\mathcal{CBC}(G, \psi)$.

face of G_i^* and their incident edges. Vertex set V_i is the *i-th outerplane level* of G. Denote by k the maximum index such that V_k is non-empty; then k is the *outerplanarity index* of G. For any $1 \leq i \leq k$, graph $G[V_i]$ is a (not necessarily connected) outerplane graph and graph G_i^* is a (not necessarily connected) *internally-triangulated* plane graph, that is, a plane graph whose internal faces are all delimited by 3-cycles. See Fig. 2. Since G is maximal, for any $1 \leq i \leq k$ and for any internal face f of $G[V_i]$, at most one connected component of G_{i+1}^* lies inside f.

Colorings. In order to define a set $V' \subseteq V$ such that $G[V']$ is an outerplane graph, our algorithm will color the vertices in V, in such a way that a vertex is in V' if it is colored white, and it is not in V' if it is colored black. Formally, a 2-*coloring* $\psi = (W, B)$ of G is a partition of V into two sets W and B. We say that the vertices in W are *white* and the ones in B are *black*. Further, an edge is *white* if both its end-vertices are white. We also say that $G[W]$ is *strongly outerplane* if it is outerplane and it contains no black vertex inside any of its internal faces. Finally, we define the *surplus* of ψ as $s(G, \psi) = |W| - |B|$. Observe that the existence of a set $V' \subseteq V$ with $|V'| \geq n/2$ such that $G[V']$ is an outerplane graph is equivalent to the existence of a 2-coloring $\psi = (W, B)$ of G such that $s(G, \psi) \geq 0$ and such that $G[W]$ is an outerplane graph.

Block decompositions. The outerplane graph $G[V_i]$ induced by an outerplane level V_i of G is not necessarily connected; we will handle the decomposition of each connected component of $G[V_i]$ into 2-connected components with the aid of a well-known data structure, called the *block-cutvertex tree*, and of a suitably-defined variation of it, which we call the *contracted block-cutvertex tree*.

A *cutvertex* in a connected graph G is a vertex whose removal disconnects G. A *maximal* 2-*connected component* of G, also called a *block* of G, is an induced subgraph $G[V']$ of G such that $G[V']$ is 2-connected and there exists no $V'' \subseteq V(G)$ where $V' \subset V''$ and $G[V'']$ is 2-connected.

The *block-cutvertex tree* $\mathcal{BC}(G)$ of G is a tree that represents the arrangement of the blocks of G (see Figs. 3(a) and 3(b) and refer to [11,19]). Namely, $\mathcal{BC}(G)$ contains a $\mathcal{B}$-*node* for each block of G and a $\mathcal{C}$-*node* for each cutvertex of G; further, there is an edge between a $\mathcal{B}$-node b and a $\mathcal{C}$-node c if c is a vertex of b.

Given a 2-coloring $\psi = (W, B)$ of G, the *contracted block-cutvertex tree* $\mathcal{CBC}(G, \psi)$ of G is the tree obtained from $\mathcal{BC}(G)$ by identifying all the $\mathcal{B}$-nodes that are adjacent to the same black cut-vertex c, and by removing c and its incident edges (see Fig. 3(c)). Each node of $\mathcal{CBC}(G, \psi)$ is either a $\mathcal{C}$-node c or a $\mathcal{BU}$-node b. In the former case, c corresponds to a white $\mathcal{C}$-node in $\mathcal{BC}(G)$. In the latter case, b corresponds to a maximal connected subtree $\mathcal{BC}(G(b))$ of $\mathcal{BC}(G)$ only containing $\mathcal{B}$-nodes and black $\mathcal{C}$-nodes. The *subgraph $G(b)$ of G associated with a $\mathcal{BU}$-node b* is the union of the blocks of G corresponding to $\mathcal{B}$-nodes in $\mathcal{BC}(G(b))$. We denote by $H(b)$ the outerplane graph induced by the vertices incident to the outer face of $G(b)$.

Proof outline and main lemma. We now prove Theorem 2, that is, we show how to construct a 2-coloring $\psi = (W, B)$ of any plane graph G so that $s(G, \psi) \geq 0$ and $G[W]$ is an outerplane graph. The proof works by induction on the outerplanarity index of G. If the outerplanarity index of G is one, then we simply insert all the vertices of G in W. Otherwise, we remove from G all the vertices in its first outerplane level V_1, together with their incident edges, thus obtaining a plane graph K with one less outerplane level. Then we color K inductively and finally we assign colors to the vertices in V_1, thus obtaining ψ.

The core of the proof consists of designing a suitable inductive hypothesis that ensures simultaneously that $s(K, \psi) \geq 0$ and that sufficiently many vertices in V_1 can be colored white so that $s(G, \psi) \geq 0$. For example, the simple inductive hypothesis stating that $s(K, \psi) \geq 0$ would not ensure that sufficiently many vertices in V_1 can be colored white; in fact, K could contain "a lot of" white vertices incident to its outer face, hence it might be the case that no vertex in V_1 can be colored white without destroying the outerplanarity of $G[W]$.

The formalization of our inductive hypothesis is expressed in the following.

Lemma 1. *For any connected internally-triangulated plane graph G, there exists a 2-coloring $\psi = (W, B)$ of G such that:*

(1) the subgraph $G[W]$ of G induced by W is strongly outerplane; and
(2) for any $\mathcal{BU}$-node b in $\mathcal{CBC}(G, \psi)$, one of the following holds:
- *(a) $s(G(b), \psi) \geq |W \cap V(H(b))| + 1$ (that is, the number of white vertices in $G(b)$ minus the number of black vertices in $G(b)$ is strictly greater than the number of white vertices in $H(b)$);*
- *(b) $s(G(b), \psi) = |W \cap V(H(b))|$ (that is, the number of white vertices in $G(b)$ minus the number of black vertices in $G(b)$ is equal to the number of white vertices in $H(b)$) and there exists a white edge incident to the outer face of $G(b)$; or*
- *(c) $s(G(b), \psi) = 1$ and $G(b)$ is a single vertex.*

Lemma 1 implies Theorem 2 as follows: If G is an n-vertex maximal plane graph, it is 2-connected and internally-triangulated. By Lemma 1, there exists a 2-coloring $\psi = (W, B)$ of G such that $G[W]$ is an outerplane graph and $|W| - |B| \geq |W \cap V_1| \geq 0$, hence $|W| \geq n/2$.

We emphasize that Lemma 1 shows the existence of a large induced subgraph $G[W]$ of G satisfying an even stronger property than just being outerplane;

namely, the 2-coloring $\psi = (W, B)$ is such that $G[W]$ is outerplane and contains no black vertex in any of its internal faces. Although this property is not needed in order to prove Theorem 2 and it is not even needed in order for the upcoming inductive proof of Lemma 1 to work, we find it of some graph-theoretical interest, and hence we explicitly mention it in Lemma 1.

The reminder of the section is devoted to a proof of Lemma 1.

Notation. We introduce some definitions and notation. Let G be a connected internally-triangulated plane graph. We denote $H = G[V_1]$. Hence, H is a connected outerplane graph. Further, we denote by K the subgraph of G induced by the internal vertices of G. Consider any internal face f of H. We say that f is *empty* if it contains no vertex of K in its interior. If f is not empty, we denote by K_f the connected component of K in the interior of f (observe that K_f is connected and internally-triangulated, given that G is internally-triangulated), and by D_f the closed walk delimiting the outer face of K_f. Given a 2-coloring $\psi_f = (W_f, B_f)$ of K_f, we say that f is *trivial* if K_f is a single white vertex or all the vertices in D_f are black. Also, for any $\mathcal{BU}$-node b in the contracted block-cutvertex tree $\mathcal{CBC}(K_f, \psi_f)$, we denote by $D_f(b)$ the closed walk delimiting the outer face of $K_f(b)$.

Characterization. We present a characterization of the 2-colorings inducing a strongly outerplane graph in G, that will be used later in the algorithm.

Lemma 2. *Given a 2-coloring $\psi = (W, B)$ of a connected internally-triangulated plane graph G, we have that $G[W]$ is strongly outerplane if and only if, for every vertex v of G, there exists a path $(u_0 = v, u_1, \dots, u_k)$, for some $k \geq 0$, such that $u_i \in B$, for every $1 \leq i \leq k$, and such that u_k is incident to the outer face of G.*

Proof: In order to prove the sufficiency, we need to prove that every vertex v of G is incident to or lies in the outer face of $G[W]$, assuming that a path $(u_0 = v, u_1, \dots, u_k)$ exists as in the statement of the lemma. If $k = 0$, then v is incident to the outer face of G, and hence it is incident to or lies in the outer face of $G[W]$. Assume that $k \geq 1$. Since u_k is incident to the outer face of G and since all the vertices of path $(u_1, \dots, u_k)$ are black, it follows that all of $u_1, \dots, u_k$ lie in the outer face of $G[W]$. By planarity and since edge (v, u_1) exists in G, it follows that v is incident to or lies in the outer face of $G[W]$.

In order to prove the necessity, we need to prove that, for every vertex v of G, a path $(u_0 = v, u_1, \dots, u_k)$ as in the statement of the lemma exists, assuming that $G[W]$ is strongly outerplane. If v is incident to the outer face of G, then the desired path consists only of vertex v. Otherwise, v is an internal vertex of G. Denote by $s_1, \dots, s_l$ the clockwise order of the neighbors of v. We claim that v has at least one black neighbor s_i. For a contradiction, suppose that all of $s_1, \dots, s_l$ are white. Since G is internally-triangulated, cycle $C = (s_1, \dots, s_l)$ exists in G, hence C is a cycle in $G[W]$ containing v in its interior. This contradicts the assumption that $G[W]$ is strongly outerplane. It remains to prove the existence of a path in $G[B]$ connecting s_i to a vertex incident to the outer face of G. Indeed, if the connected component $G_i[B]$ of $G[B]$ containing s_i does not contain at least one vertex incident to the outer face of G, then there exists a cycle in

$G[W]$ containing $G_i[B]$ in its interior, contradicting the assumption that $G[W]$ is strongly outerplane. This concludes the proof of the lemma. $\square$

Coloring algorithm. We now prove Lemma 1 by induction on the outerplanarity index of G.

In the base case, the outerplanarity index of G is 1; color all the vertices of G white. Since the outerplanarity index of G is 1, $G[W] = G$ is an outerplane graph, thus satisfying Condition (1) of Lemma 1. Also, consider any $\mathcal{BU}$-node b in the contracted block-cutvertex tree $\mathcal{CBC}(G, \psi)$ (which coincides with the block-cutvertex tree $\mathcal{BC}(G)$, given that all the vertices of G are white). All the vertices of $G(b)$ are white, hence either Condition (2b) or Condition (2c) of Lemma 1 is satisfied, depending on whether $G(b)$ has or does not have an edge, respectively.

In the inductive case, the outerplanarity index of G is greater than 1.

First, we inductively construct a 2-coloring $\psi_f = (W_f, B_f)$ of K_f satisfying the conditions of Lemma 1, for each non-empty internal face f of H. The 2-coloring ψ of G extends these colorings, i.e., a vertex of K_f is white in ψ if and only if it is white in ψ_f. Then, in order to determine ψ, it suffices to describe how to color the vertices of H.

Second, we look at the internal faces of H one at a time. For a face f of H, denote by C_f the simple cycle delimiting f. When we look at a face f, we determine a subset X_f of the vertices in C_f that we will color black in ψ. We choose X_f in such a way that, for every vertex v in K_f, there exists a path $(u_0 = v, u_1, \dots, u_k)$ in G such that $u_i \in B_f$, for every $1 \leq i \leq k-1$, and such that $u_k \in X_f$. By Lemma 2 the existence of such a path implies that $G[W]$ is strongly outerplane. We remark that, when the vertices in a set $X_f \subseteq V(C_f)$ are colored black, the vertices in $V(C_f) \setminus X_f$ are not necessarily colored white, as a vertex in $V(C_f) \setminus X_f$ might belong to the set $X_{f'}$ of vertices that are colored black for a face $f' \neq f$ of H. In fact, only after the set X_f of vertices of C_f are colored black for *every* internal face f of H, are the remaining uncolored vertices in H colored white.

We now describe in more detail how to color the vertices of H. We show an algorithm, that we call *algorithm cycle-breaker*, that associates a set X_f to each internal face f of H as follows.

Empty faces: For any empty face f of H, let $X_f = \emptyset$.

Trivial faces: While there exists a vertex $v_{1,2}$ incident to two trivial faces f_1 and f_2 of H to which no sets X_{f_1} and X_{f_2} have been associated yet, let $X_{f_1} = X_{f_2} = \{v_{1,2}\}$. When no such vertex exists, for any trivial face f of H to which no set X_f has been associated yet, let v be any vertex of C_f and let $X_f = \{v\}$.

Non-trivial non-empty faces: Consider any non-trivial non-empty internal face f of H. By induction, for any $\mathcal{BU}$-node b in the contracted block-cutvertex tree $\mathcal{CBC}(K_f, \psi_f)$, it holds $s(K_f(b), \psi_f) \geq |W_f \cap V(D_f(b))| + 1$, or $s(K_f(b), \psi_f) = |W_f \cap V(D_f(b))|$ and $D_f(b)$ contains a white edge.

We consider the $\mathcal{BU}$-nodes of $\mathcal{CBC}(K_f, \psi_f)$ one at a time, in any order. When considering a $\mathcal{BU}$-node b, we insert some vertices of C_f in X_f, based on the

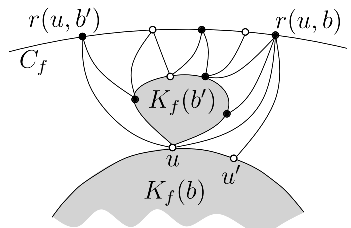

Fig. 4. The rightmost neighbors of u in C_f from b and from b'. Observe that, if $s(K_f(b), \psi_f) = |W_f \cap V(D_f(b))|$, it might be the case that $r(u, b)$ is white.

structure and the coloring of $K_f(b)$. We now describe how to perform such an insertion in more detail.

For every white vertex u in $D_f(b)$, we define the *rightmost neighbor* $r(u, b)$ *of* u *in* C_f *from* b as follows (see Fig. 4). Denote by u' the vertex following u in the clockwise order of the vertices along $D_f(b)$. Vertex $r(u, b)$ is the vertex preceding u' in the clockwise order of the neighbors of u. Observe that, since G is internally-triangulated, $r(u, b)$ belongs to C_f. Also, $r(u, b)$ is well-defined because u is not a cutvertex (in fact, it might be a cutvertex of K_f, but it is not a cutvertex of $K_f(b)$, since such a graph contains no white cut-vertex).

Suppose that $s(K_f(b), \psi_f) \geq |W_f \cap V(D_f(b))| + 1$. Then, for every white vertex u in $D_f(b)$, we add $r(u, b)$ to X_f.

Suppose that $s(K_f(b), \psi_f) = |W_f \cap V(D_f(b))|$ and $D_f(b)$ contains a white edge (v, v'). Assume, w.l.o.g., that v' follows v in the clockwise order of the vertices along $D_f(b)$. Then, for every white vertex $u \neq v$ in $D_f(b)$, we add $r(u, b)$ to X_f.

After the execution of algorithm cycle-breaker, a set X_f has been defined for every internal face f of H. Color black all the vertices in $\bigcup_f X_f$, where the union is over all the internal faces f of H. Also, color white all the vertices of H that are not colored black. Denote by $\psi = (W, B)$ the resulting coloring of G.

We have the following lemma, that completes the induction, and hence the proof of Lemma 1.

Lemma 3. *Coloring ψ satisfies Conditions (1) and (2) of Lemma 1.*

Proof: *We prove that ψ satisfies Condition (1) of Lemma 1.* By Lemma 2, it suffices to prove the following claim: For every vertex v in G, there exists a path $(u_0 = v, u_1, \dots, u_k)$ in G such that $u_i \in B$, for every $1 \leq i \leq k$, and such that u_k is incident to the outer face of G.

Denote by f any internal face of H such that v lies in the interior of f or is in C_f. If v is in C_f, then the desired path consists only of vertex v. Otherwise, v belongs to K_f. This implies that f is not an empty face.

If f is a trivial face and K_f consists of a single white vertex (which is v by the assumption that v is in K_f), then by construction there exists a black neighbor u_1 of v in C_f, hence the desired path is (v, u_1).

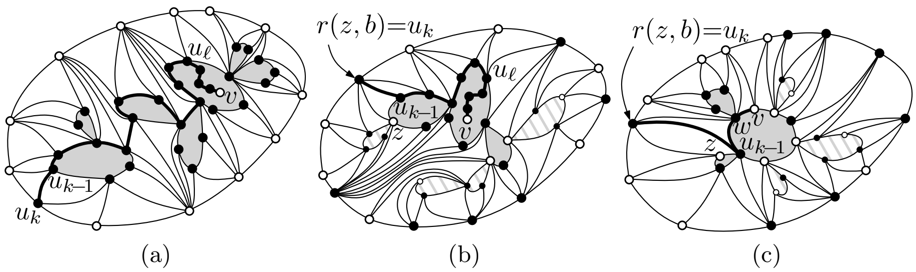

Fig. 5. Illustration for the proof that ψ satisfies Condition (1) of Lemma 1. The thick path is $(v, u_1, \ldots, u_k)$. (a) The case in which f is a trivial face and all the vertices in D_f are black. The graph whose interior is gray is K_f. (b)-(c) The case in which f is a non-trivial non-empty face. The graph whose interior is filled gray is $K_f(b)$. In (b) v is not in D_f, while in (c) v is a white vertex in D_f and w is black.

If f is a trivial face and all the vertices in D_f are black, as in Fig. 5(a), then by Lemma 2 applied to K_f there exists a path $(v = u_0, u_1, \ldots, u_\ell)$ in K_f such that $u_i \in B_f \subseteq B$, for every $1 \leq i \leq \ell$, and such that u_ℓ is in D_f. Then, D_f can be traversed until a vertex u_{k-1} is found that has a black neighbor u_k in C_f; vertices u_{k-1} and u_k exist by construction and since G is internally-triangulated. This defines an open walk $(v = u_0, u_1, \ldots, u_\ell, \ldots, u_{k-1}, u_k)$, where $u_1, \ldots, u_\ell, \ldots, u_{k-1}, u_k$ are black vertices. Removing cycles in such a walk determines the desired path.

If f is a non-trivial non-empty face and v is not in D_f, as in Fig. 5(b), or if v is a black vertex in D_f, then by Lemma 2 applied to K_f there exists a path $(u_0 = v, u_1, \ldots, u_\ell)$ in K_f such that $u_i \in B_f \subseteq B$, for every $1 \leq i \leq \ell$, and such that u_ℓ is in D_f. Let b be any node of $\mathcal{CBC}(K_f, \psi_f)$ such that $K_f(b)$ contains u_ℓ. Counterclockwise traverse $D_f(b)$ until two consecutive vertices u_{k-1} and z are encountered such that u_{k-1} is black and z is white. Observe that a black vertex in $D_f(b)$ exists by assumption (u_ℓ is one such vertex) and a white vertex in $D_f(b)$ exists since f is non-trivial. Since G is internally-triangulated, u_{k-1} and z are both neighbors of vertex $r(z, b)$. By construction $u_k = r(z, b)$ is a black vertex. This defines an open walk $(v = u_0, u_1, \ldots, u_\ell, \ldots, u_{k-1}, u_k)$, where $u_1, \ldots, u_\ell, \ldots, u_{k-1}, u_k$ are black vertices. Removing cycles in such a walk determines the desired path.

If f is a non-trivial non-empty face and v is a white vertex in D_f, then let b be any node of $\mathcal{CBC}(K_f, \psi_f)$ such that $K_f(b)$ contains v. If $r(v, b)$ is black, then $(v, r(v, b))$ is the desired path. If $r(v, b)$ is white, then denote by w the vertex following v in counter-clockwise direction in $D_f(b)$. By construction there is at most one white vertex in $D_f(b)$ whose rightmost neighbor in C_f from b is not black. Hence, if w is white, then vertex $r(w, b)$ is black; moreover, since G is internally-triangulated, $r(w, b)$ is adjacent to v, hence $(v, r(w, b))$ is the desired path. If w is black, as in Fig. 5(c), then let $u_1 = w$ and determine a path $(u_1, \ldots, u_k)$ as in the case in which v is not in D_f. Namely, counterclockwise traverse $D_f(b)$ from u_1 until two consecutive vertices u_{k-1} and z are encountered such that u_{k-1} is black and z is white. Since G is internally-triangulated, u_{k-1} and z are both neighbors of black vertex $u_k = r(z, b)$. This defines an open walk $(v = u_0, u_1, \ldots, u_\ell, \ldots, u_{k-1}, u_k)$, where $u_1, \ldots, u_\ell, \ldots, u_{k-1}, u_k$ are black vertices. Removing cycles in such a walk determines the desired path.

We prove that ψ satisfies Condition (2) of Lemma 1. Consider any $\mathcal{BU}$-node b in the contracted block-cutvertex tree $\mathcal{CBC}(G, \psi)$. Recall that $H(b)$ denotes the outerplane graph induced by the vertices incident to the outer face of $G(b)$.

We distinguish three cases. In *Case A*, graph $H(b)$ contains at least one non-trivial non-empty internal face; in *Case B*, all the faces of $H(b)$ are either trivial or empty, and there exists a vertex $v_{1,2}$ incident to two trivial faces f_1 and f_2 of $H(b)$; finally, in *Case C*, all the faces of $H(b)$ are either trivial or empty, and there exists no vertex incident to two trivial faces of $H(b)$. We prove that, in the first two cases Condition (2a) of Lemma 1 is satisfied, while in the third case Condition (2b) of Lemma 1 is satisfied.

In all cases, the surplus $s(G(b), \psi)$ is the sum of the surpluses $s(K_f, \psi)$ of the connected components K_f of K inside the internal faces of $H(b)$, plus the number $|W \cap V(H(b))|$ of white vertices in $H(b)$, minus the number $|B \cap V(H(b))|$ of black vertices in $H(b)$, which is equal to $|\bigcup_f X_f|$. Denote by n_a the number of trivial faces of $H(b)$ and by n_b the number of non-trivial non-empty faces of $H(b)$.

We first discuss *Case A*. Note that, the number of vertices inserted in $\bigcup_f X_f$ by algorithm cycle-breaker when looking at trivial faces of $H(b)$ is at most n_a, since at most one vertex is inserted into X_f for every trivial face f of $H(b)$. Also, the sum of the surpluses $s(K_f, \psi)$ of the connected components K_f of K inside trivial faces of $H(b)$ is at least n_a, given that each connected component K_f inside a trivial face is either a single white vertex, or it is such that all the vertices incident to the outer face of K_f are black (hence by induction $s(K_f, \psi) \geq |W \cap V(D_f)| + 1 = 1$).

Next, we will prove the following

Claim 1. For every non-trivial non-empty face f of $H(b)$ containing a connected component K_f of K in its interior, algorithm cycle-breaker inserts into X_f at most $s(K_f, \psi) - 1$ vertices.

We first show that Claim 1 implies that Condition (2a) of Lemma 1 is satisfied by $G(b)$. In fact:

1. the sum of the surpluses of the connected components of K inside the internal faces of $H(b)$ is $n_a + \sum_f s(K_f, \psi)$, where the sum is over every non-trivial non-empty internal face f of $H(b)$;
2. the number of white vertices in $H(b)$ is $|W \cap V(H(b))|$; and
3. the number of black vertices in $H(b)$ is at most $n_a + \sum_f (s(K_f, \psi) - 1)$, where the sum is over every non-trivial non-empty internal face f of $H(b)$.

Hence, $s(G(b), \psi) \geq n_a + \sum_f s(K_f, \psi) + |W \cap V(H(b))| - n_a - \sum_f (s(K_f, \psi) - 1) = |W \cap V(H(b))| + n_b$. By the assumption of *Case A*, we have $n_b \geq 1$, and Condition (2a) of Lemma 1 follows.

We now prove Claim 1. Consider any non-trivial non-empty face f of $H(b)$ containing a connected component K_f of K in its interior. Let n_c and n_d respectively denote the number of $\mathcal{C}$-nodes and the number of $\mathcal{BU}$-nodes in the contracted block-cutvertex tree $\mathcal{CBC}(K_f, \psi)$ of K_f. Let $b_1, b_2, \dots, b_{n_d}$ be the $\mathcal{BU}$-nodes of $\mathcal{CBC}(K_f, \psi)$ in any order.

We prove that, when algorithm cycle-breaker deals with $\mathcal{BU}$-node b_i, for any $1 \leq i \leq n_d$, it inserts into X_f a number of vertices which is at most $s(K_f(b_i), \psi) - 1$. Namely, if $s(K_f(b_i), \psi) \geq |W \cap V(D_f(b_i))| + 1$, then it suffices to observe that, for each white vertex in $D_f(b_i)$, at most one vertex is inserted into X_f; further, if $s(K_f(b_i), \psi) = |W \cap V(D_f(b_i))|$ and there exists a white edge e in $D_f(b_i)$, then, for each white vertex in $D_f(b_i)$, at most one vertex is inserted into X_f with the exception of one of the end-vertices of e, for which no vertex is inserted into X_f. Hence, the number of vertices inserted into X_f by algorithm cycle-breaker is at most $\sum_{i=1}^{n_d}(s(K_f(b_i), \psi) - 1) = \sum_{i=1}^{n_d} s(K_f(b_i), \psi) - n_d$.

In order to complete the proof of Claim 1, it remains to prove that $\sum_{i=1}^{n_d} s(K_f(b_i), \psi) - n_d = s(K_f, \psi) - 1$. Roughly speaking, this comes from the fact that white cutvertices in K_f belong to more than one graph $K_f(b_i)$, hence they give a contribution greater than 1 to $\sum_{i=1}^{n_d} s(K_f(b_i), \psi)$, while they give a contribution equal to 1 to $s(K_f, \psi)$. More precisely, every vertex in K_f which is not a white cutvertex contributes $+1$ or -1 to $s(K_f, \psi)$ if and only if it contributes $+1$ or -1, respectively, to $\sum_{i=1}^{n_d} s(K_f(b_i), \psi)$. Further, every white cutvertex in K_f gives a $+1$ contribution to $s(K_f, \psi)$; hence, the contribution of the white cutvertices in K_f to $s(K_f, \psi)$ is equal to n_c. Finally, every white cutvertex in K_f gives a contribution to $\sum_{i=1}^{n_d} s(K_f(b_i), \psi)$ equal to its degree in $\mathcal{CBC}(K_f, \psi)$; hence, the contribution of the white cutvertices in K_f to $\sum_{i=1}^{n_d} s(K_f(b_i), \psi)$ is equal to the number of edges of $\mathcal{CBC}(K_f, \psi)$, which is $n_c + n_d - 1$. Thus, $\sum_{i=1}^{n_d} s(K_f(b_i), \psi) - s(K_f, \psi) = (n_c + n_d - 1) - n_c = n_d - 1$.

We now discuss *Case B*. First, the sum of the surpluses $s(K_f, \psi)$ of the connected components K_f of K inside the internal faces of $H(b)$ is at least n_a, given that each connected component K_f is either a single white vertex, or it is such that all the vertices incident to the outer face of K_f are black (hence by induction $s(K_f, \psi) \geq |W \cap V(D_f)| + 1 = 1$).

Second, by the assumptions of *Case B* and by construction, algorithm cycle-breaker defines $X_{f_1} = X_{f_2} = \{v_{1,2}\}$ for two trivial faces f_1 and f_2 of $H(b)$ sharing a vertex $v_{1,2}$. Thus, $|B \cap V(H(b))| = |\bigcup_f X_f| < n_a$. In fact, each trivial face contributes at most one vertex to $\bigcup_f X_f$ and at least two trivial faces of $H(b)$ contribute a total of one vertex to $\bigcup_f X_f$.

Hence, $s(G(b), \psi) \geq n_a + |W \cap V(H(b))| - (n_a - 1) = |W \cap V(H(b))| + 1$, thus Condition (2a) of Lemma 1 is satisfied.

We finally discuss *Case C*. As in the previous case, the sum of the surpluses of the connected components K_f of K inside the internal faces of $H(b)$ is at least n_a.

Further, $|B \cap V(H(b))| = |\bigcup_f X_f| = n_a$, as each trivial face contributes one vertex to $\bigcup_f X_f$. (Notice that, since no two trivial faces share a vertex, no two trivial faces contribute the same vertex to $\bigcup_f X_f$.)

Hence, $s(G(b), \psi) = n_a + |W \cap V(H(b))| - n_a = |W \cap V(H(b))|$. Thus, in order to prove that Condition (2b) of Lemma 1 is satisfied, it remains to prove that there exists a white edge incident to the outer face of $G(b)$ or, equivalently, to the outer face of $H(b)$. In the reminder of the proof, to simplify the notation, we denote the graph $H(b)$ by L.

We first show how to restrict the attention to the case in which L is 2-connected. Let c be a cutvertex of L and let $L(b')$ be a block of L corresponding to a $\mathcal{B}$-node b' of the block-cutvertex tree $\mathcal{BC}(L)$ of L. We say that c *belongs to* $L(b')$ if c is a vertex of $L(b')$ and either (i) c is only incident to empty faces of L or (ii) the only trivial face incident to c belongs to $L(b')$. Observe that each cutvertex belongs to at least one block of L. Consider an orientation of $\mathcal{BC}(L)$ such that an edge (c, b') is oriented from the $\mathcal{C}$-node c to the $\mathcal{B}$-node b' if c belongs to b', otherwise it is oriented from b' to c. This orientation is acyclic (as it is an orientation of a tree), and it hence has a sink. However, each cutvertex has out-degree at least one. Thus, there exists a $\mathcal{B}$-node b' that is a sink; hence, every cutvertex of L in $L(b')$ belongs to b'. It follows that algorithm cycle-breaker does not insert any vertex of $L(b')$ into a set X_f for a face f not in $L(b')$. Hence, a white edge incident to the outer face of the 2-connected graph $L(b')$ implies the existence of a white edge incident to the outer face of L.

We can hence assume that L is 2-connected. Then there are at most $\lfloor \frac{|V(L)|}{3} \rfloor$ trivial faces in L, given that each of them has at least three vertices, and that no two of them share any vertex. Thus, algorithm cycle-breaker colors black at most $\lfloor \frac{|V(L)|}{3} \rfloor$ vertices in L. Further, the outer face of L is delimited by a cycle with $|V(L)|$ edges; hence, there have to be at least $\lceil \frac{|V(L)|}{2} \rceil$ black vertices in L in order for all these edges to be black. However, $\lceil \frac{|V(L)|}{2} \rceil > \lfloor \frac{|V(L)|}{3} \rfloor$ for $|V(L)| \geq 2$. Hence, a white edge incident to the outer face of L must exist.

This concludes the proof of the lemma. □

3 Proof of Theorem 3

In this section we prove Theorem 3, that is, we prove that every n-vertex plane graph G_1 and every n-vertex planar partial 3-tree G_2 have a SefeNoMap. We assume that G_1 is a *maximal* plane graph and that G_2 is a (*maximal*) plane 3-tree G_2. If that is not the case, then G_1 can be augmented to an n-vertex maximal plane graph G'_1 and G_2 can be augmented to an n-vertex plane 3-tree G'_2; the latter augmentation can always be performed [16]. Then a SefeNoMap can be constructed for G'_1 and G'_2, and finally the edges not in G_1 and G_2 can be removed, thus obtaining a SefeNoMap of G_1 and G_2.

Denote by $C_i = (u_i, v_i, z_i)$ the cycle delimiting the outer face of G_i, for $i = 1, 2$, where vertices u_i, v_i, and z_i appear in this clockwise order along C_i.

The outline of the proof is as follows. We start by constructing any planar drawing Γ_1 of G_1. In order to construct a planar drawing Γ_2 of G_2, we map u_2 to u_1, v_2 to v_1, and z_2 to z_1, and we let the closed curve representing C_2 in Γ_2 coincide with the closed curve representing C_1 in Γ_1. We construct the rest of Γ_2

by repeatedly performing the following operation: Consider a triangular face f of the subgraph of G_2 drawn so far that is not a face of G_2; insert a vertex inside f in Γ_2 and draw curves connecting the inserted vertex with the vertices on the boundary of f. The main tool we use to perform this operation argues about the drawability of three curves on top of a planar drawing of a graph. In the following we formally describe this tool; we will later return to its application for the construction of a SefeNoMap of G_1 and G_2.

Two open curves γ_1 and γ_2 *intersect* if they share a point; they *cross* if they share a point that is an interior point of at least one of them. If γ_1 and γ_2 represent edges in a drawing, then this definition agrees with the definition that two edges cross if they share a common point that is not a vertex incident to both. Analogously, an open curve γ_1 and a closed curve γ_2 *intersect* if they share a point; they *cross* if they share a point that is an interior point of γ_1.

Let G be a 2-connected internally-triangulated plane graph with $n-3$ internal vertices. Let C be the simple cycle delimiting the outer face of G and let u, v, and z be three vertices appearing in this clockwise order along C. Let $\mathcal{P}_{uv}$, $\mathcal{P}_{vz}$, and $\mathcal{P}_{zu}$ respectively denote the path composing C that connects u and v, v and z, and z and u. Further, let $n_{uv}, n_{vz}, n_{zu} \geq 0$ be integers with $n_{uv} + n_{vz} + n_{zu} = n - 4$. Finally, let Γ_C be a planar drawing of C. We have the following:

Lemma 4. *There exist a planar drawing Γ of G that coincides with Γ_C when restricted to C, an internal vertex w of G, and three curves s_{uw}, s_{vw}, and s_{zw} respectively connecting u, v, and z with w such that (see Fig. 6):*

- *Property (P1): s_{uw}, s_{vw}, and s_{zw} do not cross Γ_C and do not cross each other;*
- *Property (P2): if G contains edge (u,w) ((v,w), (z,w)), then s_{uw} (s_{vw}, s_{zw}, respectively) coincides with the drawing of such an edge in Γ;*
- *Property (P3): each of s_{uw}, s_{vw}, and s_{zw} intersects each edge of G at most once and does not contain any vertex of G in its interior;*
- *Property (P4): the closed curve $\mathcal{C}_{uvw}$ composed of $\mathcal{P}_{uv}$, s_{uw}, and s_{vw} contains in its interior n_{uv} vertices of G; the closed curve $\mathcal{C}_{vzw}$ composed of $\mathcal{P}_{vz}$, s_{vw}, and s_{zw} contains in its interior n_{vz} vertices of G; the closed curve $\mathcal{C}_{zuw}$ composed of $\mathcal{P}_{zu}$, s_{zw}, and s_{uw} contains in its interior n_{zu} vertices of G;*
- *Property (P5): if an edge e of G has both its end-vertices inside or on $\mathcal{C}_{uvw}$ ($\mathcal{C}_{vzw}$, $\mathcal{C}_{zuw}$), then the interior of e lies inside $\mathcal{C}_{uvw}$ ($\mathcal{C}_{vzw}$, $\mathcal{C}_{zuw}$, respectively).*

Proof: We prove the lemma by induction on $n_{uv} + n_{vz} + n_{zu}$.

In the base case, $n_{uv} + n_{vz} + n_{zu} = 0$. Let Γ be any planar drawing of G that coincides with Γ_C when restricted to C. Let w be the only internal vertex of G. If G contains edge (u,w) ((v,w), (z,w)), then s_{uw} (s_{vw}, s_{zw}, respectively) coincides with the drawing of such an edge in Γ. Draw the remaining curves among s_{uw}, s_{vw}, and s_{zw} in the interior of $\mathcal{C}$ with a minimum number of crossings. It is readily seen that, due to the minimality, these curves do not cross each other and they intersect each edge of G at most once. An algorithm to efficiently draw s_{uw}, s_{vw}, and s_{zw} can be found in [9].

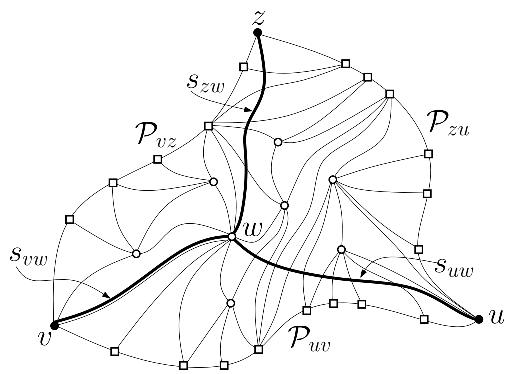

Fig. 6. Illustration for the statement of Lemma 4. White circles and squares represent, respectively, internal and external vertices of G. Curves s_{uw}, s_{vw}, and s_{zw} are thick. In this example $n_{uv} = 1$, $n_{vz} = 2$, and $n_{zu} = 4$.

In the inductive case, $n_{uv} + n_{vz} + n_{zu} > 0$. Suppose, w.l.o.g., that $n_{uv} > 0$, the other cases being analogous. Initialize Γ as any planar drawing of G that coincides with Γ_C. We perform a modification of G into a graph G' so that G' has an internal vertex with at least two neighbors in $\mathcal{P}_{uv}$.

If there exists an internal vertex t of G with two neighbors in $\mathcal{P}_{uv}$, then let $G' = G$ and $\Gamma' = \Gamma$.

If there is no internal vertex of G with two neighbors in $\mathcal{P}_{uv}$, as in Fig. 7(a), then G contains some *empty chords*, where an empty chord e is an edge satisfying the following conditions: (i) e connects two vertices of $\mathcal{C}$; (ii) one of the two cycles determined by $\mathcal{C}$ and by e contains no vertex in its interior and contains part of $\mathcal{P}_{uv}$ on its boundary. The existence of empty chords might prevent the existence of an internal vertex of G with at least two neighbors in $\mathcal{P}_{uv}$. We remove from G and Γ every empty chord. The graph obtained after all the removals has one non-triangular face g, where g has one incident internal vertex t of G. Triangulate g by inserting edges from t to every vertex of g. Draw these edges planarly in Γ, as in Fig. 7(b). Denote by G' the resulting plane graph and by Γ' its planar drawing. Observe that t has at least two neighbors in $\mathcal{P}_{uv}$.

Consider an internal vertex t of G' that has at least two neighbors in $\mathcal{P}_{uv}$. Traverse $\mathcal{P}_{uv}$ from u to v; let a and b respectively denote the first and the last encountered neighbor of t. We say that t is *close to* $\mathcal{P}_{uv}$ if the cycle $\mathcal{C}_t$ composed of edges (t, a) and (t, b) and of the subpath of $\mathcal{P}_{uv}$ between a and b contains no vertex in its interior. A vertex t close to $\mathcal{P}_{uv}$ always exists. Namely, among all the vertices with at least two neighbors in $\mathcal{P}_{uv}$, consider a vertex t such that $\mathcal{C}_t$ contains a minimum number n_t of vertices of G' in its interior. We claim that $n_t = 0$. Indeed, if $n_t > 0$, then $\mathcal{C}_t$ contains in its interior a vertex t' with at least two neighbors in $\mathcal{P}_{uv}$, given that G' is internally-triangulated. However, $\mathcal{C}_{t'}$ contains in its interior a number of vertices smaller than n_t, a contradiction to the assumed minimality of n_t.

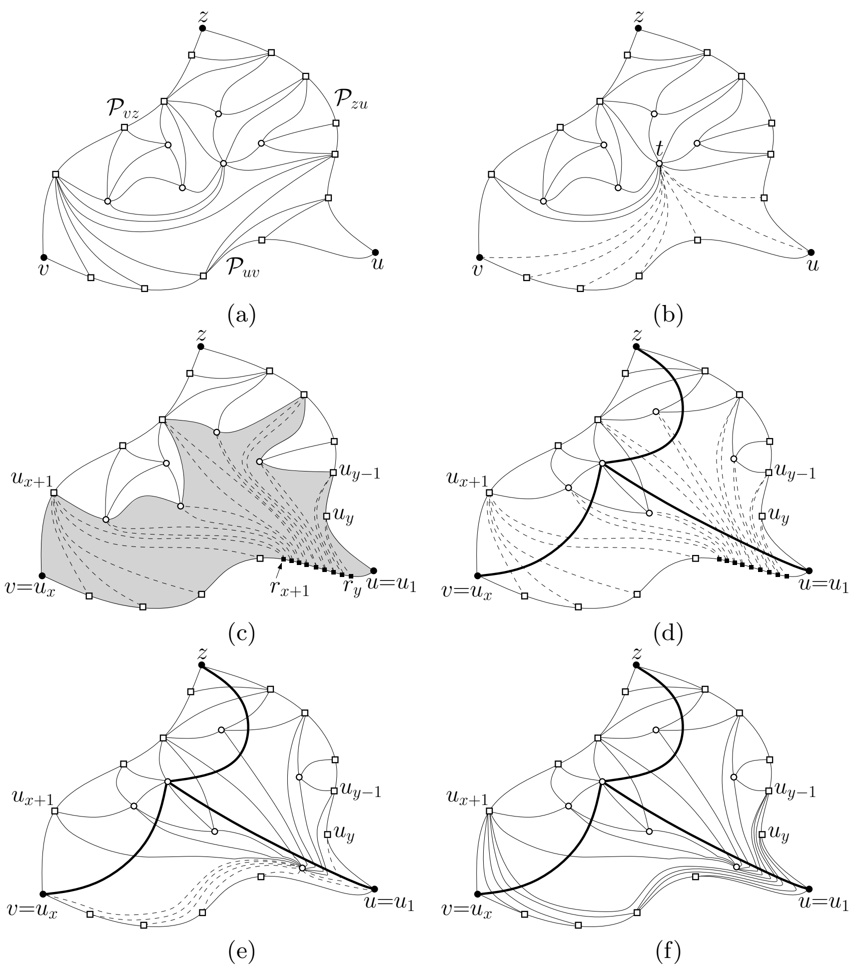

Fig. 7. (a) Drawing Γ of a graph G, which contains some empty chords. (b) Replacement of the empty chords with edges incident to an internal vertex t. (c) Removal of t and of its incident edges from G' and insertion of dummy vertices and edges to triangulate f. Face f is gray. (d) Inductive construction of s_{uw}, s_{vw}, s_{zw}, and Γ''. (e) Reintroduction of t and its incident edges. (f) Reintroduction of the empty chords of G.

Let t be any vertex close to $\mathcal{P}_{uv}$. Remove t and its incident edges from G'. Let f be the face of G' in which t used to lie and let C_f be the cycle delimiting f. Since t is close to $\mathcal{P}_{uv}$, the vertices in $\mathcal{P}_{uv}$ appear consecutively along C_f. Denote by $u_1, u_2, \ldots, u_y$ the clockwise order of the vertices along C_f, where $u_1, u_2, \ldots, u_x$, for some $x \geq 2$, are the vertices in $\mathcal{P}_{uv}$. Insert $y - x$ dummy vertices $r_y, r_{y-1}, \ldots, r_{x+1}$ in this order along edge (u_1, u_2). Insert dummy edges $e_{x+1}, e_{x+2}, \ldots, e_y$ inside f, where e_i connects r_i and u_i, for each $x + 1 \leq i \leq y$.

Moreover, insert edges inside f between u_{x+1} and $u_2, \ldots, u_{x-1}$, and between r_i and u_{i+1}, for each $x+1 \leq i \leq y-1$. See Fig. 7(c). These edges triangulate the interior of f. Denote by G'' the resulting 2-connected internally-triangulated graph.

Inductively construct a drawing Γ'' of G'' that coincides with Γ_C when restricted to C, and draw curves s_{uw}, s_{vw}, and s_{zw} so that Properties (P1)–(P5) are satisfied, where Property (P4) ensures that $\mathcal{C}_{uvw}$, $\mathcal{C}_{vzw}$, and $\mathcal{C}_{zuw}$ respectively contain $n_{uv}-1$, n_{vz}, and n_{zu} internal vertices of G'' in their interior. See Fig. 7(d).

Reinsert t at a point arbitrarily close to edge (u_1, u_2). Reintroduce the edges incident to t as follows. Draw curves connecting t and $u_1, u_2, \ldots, u_x$ inside f arbitrarily close to $\mathcal{P}_{uv}$. Also, for each $x+1 \leq i \leq y$, draw a curve connecting t and u_i as composed of two curves, the first one arbitrarily close to $\mathcal{P}_{uv}$, the second one coinciding with part of edge e_i. Remove all the inserted dummy vertices and edges from the drawing, thus obtaining a drawing of G'. See Fig. 7(e). Reintroduce the empty chords of G as edges arbitrarily close to cycle $\mathcal{C}$. See Fig. 7(f). This determines a drawing Γ of G. We prove that Γ together with the constructed drawings of s_{uw}, s_{vw}, and s_{zw} satisfy Properties (P1)–(P5).

- Property (P1) directly follows from the fact that Γ'' satisfies Property (P1), by induction.
- Property (P2) directly follows from the fact that Γ'' satisfies Property (P2), by induction, and from the fact that $w \neq t$ (hence edges (u,w), (v,w), and (z,w) belong to G if and only if they belong to G'').
- We prove Property (P3). That s_{uw}, s_{vw}, and s_{zw} do not contain any vertex of G in their interiors follows by induction and by the fact that t is in the interior of $\mathcal{C}_{uvw}$. We prove that s_{uw} intersects any edge e of G at most once; analogous proofs hold for s_{vw} and s_{zw}.
  If e is an empty chord, then it intersects s_{uw} at most once, given that e is arbitrarily close to $\mathcal{C}$.
  If e is not an empty chord and is not incident to t, then the statement follows by induction.
  Otherwise, e connects t and a vertex u_i, for some $1 \leq i \leq y$. If $1 \leq i \leq x$, then e is arbitrarily close to $\mathcal{P}_{uv}$, hence it intersects s_{uw} at most once (and only if $u_i = u$). If $x+1 \leq i \leq y$, then e is composed of two curves, the first one arbitrarily close to (u_1, u_2) and not incident to u (hence, such a curve does not intersect s_{uw} at all), the second one coinciding with part of edge e_i (hence such a curve intersects s_{uw} at most once, since Γ'' satisfies Property (P3), by induction).
- We prove Property (P4). Since Γ'' satisfies Property (P4), by induction, $\mathcal{C}_{uvw}$, $\mathcal{C}_{vzw}$, and $\mathcal{C}_{zuw}$ respectively contain in their interiors $n_{uv}-1$, n_{vz}, and n_{zu} internal vertices of G''. Since t is inserted inside $\mathcal{C}_{uvw}$, it follows that $\mathcal{C}_{uvw}$, $\mathcal{C}_{vzw}$, and $\mathcal{C}_{zuw}$ respectively contain in their interiors n_{uv}, n_{vz}, and n_{zu} internal vertices of G.
- We prove Property (P5). Consider any edge e of G.
  If e is an empty chord, then it has both its end-vertices inside or on $\mathcal{C}_{uvw}$ if and only if it connects two vertices on $\mathcal{P}_{uv}$. In this case, e is arbitrarily close

to $\mathcal{P}_{uv}$, hence the interior of e entirely lies inside $\mathcal{C}_{uvw}$. Also, if e is an empty chord, it does not have both its end-vertices in $\mathcal{C}_{vzw}$ or in $\mathcal{C}_{zuw}$.

If e is not an empty chord and is not incident to t, then it satisfies Property (P5) since Γ'' satisfies Property (P5) by induction.

If e is incident to t, then it has at least one of its end-vertices inside $\mathcal{C}_{uvw}$. We show that, if the second end-vertex of e is inside or on $\mathcal{C}_{uvw}$, then the interior of e is inside $\mathcal{C}_{uvw}$. If the second end-vertex of e is one of $u_1, \dots, u_x$, then e lies arbitrarily close to $\mathcal{P}_{uv}$, hence the interior of e is inside $\mathcal{C}_{uvw}$. If the second end-vertex of e is w, then the interior of e is not inside $\mathcal{C}_{uvw}$ only if e intersects at least twice s_{uw} or s_{vw}. However, this would violate Property (P3) on Γ'', given that e is composed of two parts, one of which does not intersect s_{uw} or s_{vw} at all, and one of which coincides with the drawing of an edge of G'' in Γ''. If the second end-vertex of e is not u_1, u_2, $\dots$, u_x, or w, then the interior of e is not inside $\mathcal{C}_{uvw}$ only if e intersects one of s_{uw} and s_{vw} twice (or any positive even number of times) or it intersects each of s_{uw} and s_{vw} once (or any positive odd number of times). However, this would violate Property (P3) or Property (P5) of Γ'', given that e is composed of two parts, one of which does not intersect s_{uw} or s_{vw} at all, and one of which coincides with the drawing of an edge of G'' in Γ''.

This concludes the proof of the lemma. □

We now go back to the proof of Theorem 3. We exploit the construction for plane 3-trees, which states that G_2 can be constructed, starting from the cycle (u_2, v_2, z_2) delimiting its outer face, by repeatedly performing the following operation. Select an internal triangular face f of the so far constructed subgraph of G_2. Insert a vertex inside f and connect this vertex to the three vertices incident to f.

We show how to construct a SefeNoMap of G_1 and G_2. Construct any planar drawing of G_1. Map cycle (u_2, v_2, z_2) to the closed curve representing cycle (u_1, v_1, z_1) in the constructed drawing of G_1, with u_2, v_2, and z_2 mapped to u_1, v_1, and z_1, respectively.

If G_2 has not been entirely drawn yet, denote by G'_2 the subgraph of G_2 drawn so far. Then there exists a not-yet-drawn vertex d of G_2 that has to be inserted inside an internal face (a, b, c) of G'_2. We assume that the already constructed drawings Δ_1 and Δ'_2 of G_1 and G'_2, respectively, form a SefeNoMap of G_1 and G'_2 satisfying properties analogous to the ones in the statement of Lemma 4. Namely, we assume that any two edges of G_1 and G'_2 intersect at most once and that no edge contains a vertex in its interior. Moreover, we assume that any face f of G'_2 contains in its interior a number of vertices of G_1 equal to the number of vertices internal to the cycle delimiting f in G_2. Finally, we assume that if an edge e of G_1 has both its end-vertices inside or on the border of a face f of G'_2, then the interior of e is inside f. All these properties are trivially satisfied once G'_2 coincides with cycle (u_2, v_2, z_2).

We now proceed to draw d and edges (a, d), (b, d), and (c, d). Replace each crossing between an edge of G_1 and the edges of cycle (a, b, c) with a dummy vertex. Denote by $\mathcal{C}$ cycle (a, b, c) subdivided by the insertion of the dummy vertices. Denote by G_1' the subgraph of G_1 whose vertices and edges are those inside or on $\mathcal{C}$. Also, let $\mathcal{P}_{ab}$, $\mathcal{P}_{bc}$, and $\mathcal{P}_{ca}$ respectively denote the path (of the three paths that compose $\mathcal{C}$) that connects a and b, b and c, and c and a. Further, let n_{ab}, n_{bc}, and n_{ca} be the number of vertices in the subgraphs of G_2 whose vertices and edges are those inside cycle $\mathcal{P}_{ab} \cup (a,d) \cup (b,d)$, inside cycle $\mathcal{P}_{bc} \cup (b,d) \cup (c,d)$, and inside cycle $\mathcal{P}_{ca} \cup (c,d) \cup (a,d)$, respectively. Insert dummy edges to triangulate the internal faces of G_1'. Now G_1' is 2-connected and internally-triangulated.

By Lemma 4, there exist a planar drawing Γ_1' of G_1' in which cycle $\mathcal{C}$ has the same drawing as in Δ_1, an internal vertex w of G_1', and three curves s_{aw}, s_{bw}, and s_{cw} respectively connecting a, b, and c with w satisfying Properties (P1)–(P5). Namely, s_{aw}, s_{bw}, and s_{cw} lie in the interior of $\mathcal{C}$ and do not cross each other; further, if G_1' contains edge (a,w) ((b,w), (c,w)), then s_{aw} (s_{bw}, s_{cw}, respectively) coincides with the drawing of such an edge in Γ_1'; also, each of s_{aw}, s_{bw}, and s_{cw} intersects each edge of G_1' at most once and does not contain any vertex of G_1' in its interior; moreover, cycles $\mathcal{C}_{abw} = \mathcal{P}_{ab} \cup s_{aw} \cup s_{bw}$, $\mathcal{C}_{bcw} = \mathcal{P}_{bc} \cup s_{bw} \cup s_{cw}$, and $\mathcal{C}_{caw} = \mathcal{P}_{ca} \cup s_{cw} \cup s_{aw}$ respectively contain in their interiors n_{ab}, n_{bc}, and n_{ca} internal vertices of G_1'; finally, if an edge e of G_1' has both its end-vertices inside or on $\mathcal{C}_{abw}$ ($\mathcal{C}_{bcw}$, $\mathcal{C}_{caw}$), then the interior of e lies inside $\mathcal{C}_{abw}$ ($\mathcal{C}_{bcw}$, $\mathcal{C}_{caw}$, respectively). Thus, the drawing of G_1' in Δ_1 can be replaced with Γ_1', vertex d can be mapped to w, and edges (a,d), (b,d), and (c,d) can be mapped to s_{aw}, s_{bw}, and s_{cw}, respectively. This results in a SefeNoMap of G_1 and G_2', where G_2' now includes vertex d and edges (a,d), (b,d), and (c,d).

Repeating this operation for every internal vertex of G_2 eventually results in a SefeNoMap of G_1 and G_2. This completes the proof of Theorem 3.

4 Conclusions

In this paper we studied the problem of determining the largest $k_1 \leq n$ such that every n-vertex planar graph and every k_1-vertex planar graph admit a SefeNoMap. We proved that $k_1 \geq n/2$. No upper bound smaller than n is known. Hence, tightening this bound (and in particular proving whether $k_1 = n$ or not) is a natural research direction.

To achieve the above result, we proved that every n-vertex plane graph has an $(n/2)$-vertex induced outerplane graph, a result related to a famous conjecture stating that every planar graph contains an induced forest with half of its vertices [2]. A suitable triangulation of a set of nested 4-cycles shows that $n/2$ is a tight bound for our algorithm, up to an additive constant. However, we have no example of an n-vertex plane graph whose largest induced outerplane graph has less than $2n/3$ vertices (a triangulation of a set of nested 3-cycles shows that $2n/3$ is an upper bound). The following question arises: What are the largest k_2 and k_3 such that every n-vertex plane graph has an induced outerplane graph with k_2 vertices and an induced outerplanar graph with k_3 vertices? Any bound $k_2 > n/2$ would improve our bound for the SefeNoMap problem, while any bound $k_3 > 3n/5$ would improve the best known bound for Conjecture 1, via the results in [12].

A different technique to prove that every n-vertex planar graph and every k_4-vertex planar graph have a SefeNoMap is to ensure that a mapping between their vertex sets exists that generates no shared edge. Thus, we ask: What is the largest $k_4 \leq n$ such that an injective mapping exists from the vertex set of any k_4-vertex planar graph to the vertex set of any n-vertex planar graph generating no shared edge? It is easy to see that $k_4 \geq n/4$ (a consequence of the four color theorem [14,15]) and that $k_4 \leq n - 5$ (an n-vertex planar graph with minimum degree 5 does not admit such a mapping with an $(n-4)$-vertex planar graph having a vertex of degree $n-5$).

Finally, it would be interesting to study the geometric version of our problem. That is: What is the largest $k_5 \leq n$ such that every n-vertex planar graph and every k_5-vertex planar graph admit a geometric simultaneous embedding without mapping? Surprisingly, we are not aware of any super-constant lower bound for the value of k_5.

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