Column Planarity and Partial Simultaneous Geometric Embedding

William Evans[1], Vincent Kusters[2], Maria Saumell[3], and Bettina Speckmann[4]

[1] University of British Columbia. will@cs.ubc.ca
[2] Department of Computer Science, ETH Zürich. vincent.kusters@inf.ethz.ch
[3] Department of Mathematics and European Centre of Excellence NTIS, University of West Bohemia. saumell@kma.zcu.cz
[4] Technical University Eindhoven. b.speckmann@tue.nl

Abstract. We introduce the notion of *column planarity* of a subset R of the vertices of a graph G. Informally, we say that R is column planar in G if we can assign x-coordinates to the vertices in R such that any assignment of y-coordinates to them produces a partial embedding that can be completed to a plane straight-line drawing of G. Column planarity is both a relaxation and a strengthening of unlabeled level planarity. We prove near tight bounds for column planar subsets of trees: any tree on n vertices contains a column planar set of size at least $14n/17$ and for any $\epsilon > 0$ and any sufficiently large n, there exists an n-vertex tree in which every column planar subset has size at most $(5/6 + \epsilon)n$.
We also consider a relaxation of simultaneous geometric embedding (SGE), which we call partial SGE (PSGE). A PSGE of two graphs G_1 and G_2 allows some of their vertices to map to two different points in the plane. We show how to use column planar subsets to construct k-PSGEs in which k vertices are still mapped to the same point. In particular, we show that any two trees on n vertices admit an $11n/17$-PSGE, two outerpaths admit an $n/4$-PSGE, and an outerpath and a tree admit a $11n/34$-PSGE.

1 Introduction

A graph $G = (V, E)$ on n vertices is *unlabeled level planar (ULP)* if for all injections $\gamma : V \to \mathbb{R}$, there exists an injection $\rho : V \to \mathbb{R}$, so that embedding each $v \in V$ at $(\rho(v), \gamma(v))$ results in a plane straight-line embedding of G. Estrella-Balderrama, Fowler and Kobourov [9] originally introduced ULP graphs and characterized ULP trees in terms of forbidden subgraphs. Fowler and Kobourov [11] extended this characterization to general ULP graphs. ULP

W. Evans is supported by an NSERC Discovery Grant. V. Kusters is partially supported by the ESF EUROCORES programme EuroGIGA, CRP GraDR and the Swiss National Science Foundation, SNF Project 20GG21-134306. M. Saumell is supported by the project NEXLIZ CZ.1.07/2.3.00/30.0038, which is co-financed by the European Social Fund and the state budget of the Czech Republic. B. Speckmann is supported by the Netherlands Organisation for Scientific Research (NWO) under project no. 639.023.208.

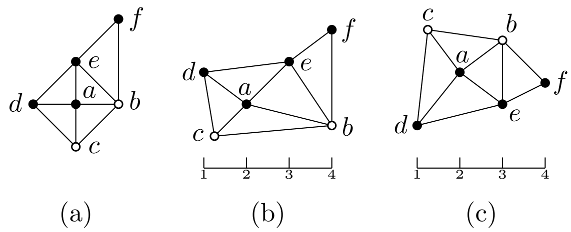

Fig. 1. (a) A graph $G = (V, E)$ with $R = \{a, d, e, f\}$ which is ρ-column planar for $\rho = \{d \mapsto 1, a \mapsto 2, e \mapsto 3, f \mapsto 4\}$. (b-c) Two assignments of y-coordinates to the vertices R and corresponding plane straight-line completions of G.

graphs are exactly the graphs that admit a simultaneous geometric embedding with a monotone path: this was the original motivation for studying them.

In this paper we introduce the notion of *column planarity* of a subset R of the vertices V of a graph $G = (V, E)$. Informally, we say that R is column planar in G if we can assign x-coordinates to the vertices in R such that any assignment of y-coordinates to them produces a partial embedding that can be completed to a plane straight-line drawing of G. Column planarity is both a relaxation and a strengthening of unlabeled level planarity. It is a relaxation since it applies only to a subset R of the vertices and a strengthening since the requirements on R are more strict than in the case of unlabeled level planarity.

More formally, for $R \subseteq V$, we say that R is *column planar in* $G = (V, E)$ if there exists an injection $\rho : R \to \mathbb{R}$ such that for all ρ*-compatible* injections $\gamma : R \to \mathbb{R}$, there exists a plane straight-line embedding of G where each $v \in R$ is embedded at $(\rho(v), \gamma(v))$. Injection γ is ρ-compatible if the combination of ρ and γ does not embed three vertices on a line. Clearly, if R is column planar in G then any subset of R is also column planar in G. We say that R is ρ*-column planar* when we need to emphasize the injection ρ (see Fig. 1 for an example). If $R = V$ is column planar in G then G is ULP since column planarity implies the existence of one assignment of x-coordinates to vertices that will produce a planar embedding for all assignments of y-coordinates, while to be a ULP graph the x-coordinate assignment may depend on the y-coordinate assignment. In this sense, column planarity of V is strictly more restrictive than unlabeled level planarity of G.

As mentioned above, the study of ULP was originally motivated by simultaneous geometric embedding, a concept introduced by Brass et al. [4]. Formally, given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of n vertices, they defined a *simultaneous geometric embedding (SGE)* of G_1 and G_2 as an injection $\varphi : V \to \mathbb{R}^2$ such that the straight-line drawings of G_1 and G_2 induced by φ are both plane. With slight abuse of notation, we refer to these drawings as $\varphi(G_1)$ and $\varphi(G_2)$. Fig. 2c depicts an SGE of the graphs in Fig. 2a and Fig. 2b. SGE has been studied in several subsequent papers. Bläsius et al. [2] give an excellent survey of the area with a comprehensive list of results. On the positive side, Brass et al. [4] prove that two paths, cycles or caterpillars always admit

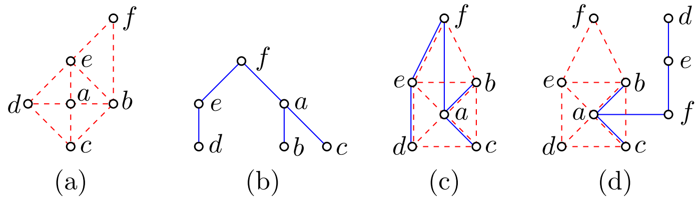

Fig. 2. (a-b) Two graphs on the same vertex set. (c) An SGE of these graphs. (d) A 3-PSGE of these graphs.

an SGE. Cabello et al. [5] prove that a matching and a tree or outerpath (a type of outerplanar graph) always admit an SGE. On the negative side, Brass et al. [4] prove that three paths sometimes do not admit an SGE. Erten and Kobourov [8] prove that a planar graph and a path may not admit an SGE. Frati, Kaufmann and Kobourov [12] strengthen this result to the case where the planar graph and the path do not share any edges. Geyer, Kaufmann and Kobourov [13] describe two trees that do not admit an SGE. Angelini et al. [1] close a long-standing open question by describing a tree and a path that admit no SGE. Finally, Estrella-Balderrama et al. [10] show that the decision problem for SGE is NP-hard.

In light of the restrictiveness of simultaneous geometric embedding, several other variations on the abstract problem have been studied. Cappos et al. [6] consider a version of SGE where edges are embedded as circular arcs or with bends. Di Giacomo et al. [7] consider *matched drawings*: a version of SGE where the location of a vertex in the drawing of G_1 need only have the same y-coordinate as its location in the drawing of G_2.

In this paper we consider a variant on SGE which we call *partial simultaneous geometric embedding* (PSGE). We do not require *every* vertex to map to a single point in the plane. Instead, some vertices can have a "split personality" and map to two different locations, one associated with G_1 and one associated with G_2. Specifically, given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of n vertices, a *k-partial simultaneous geometric embedding (k-PSGE)* of G_1 and G_2 is a pair of injections $\varphi_1 : V \to \mathbb{R}^2$ and $\varphi_2 : V \to \mathbb{R}^2$ such that (i) the straight-line drawings $\varphi_1(G_1)$ and $\varphi_2(G_2)$ are both plane; (ii) if $\varphi_1(v_1) = \varphi_2(v_2)$ then $v_1 = v_2$ and; (iii) $\varphi_1(v) = \varphi_2(v)$ for at least k vertices $v \in V$. An n-PSGE is simply an SGE. Fig. 2d depicts a 3-PSGE of the graphs in Fig. 2a and Fig. 2b.

PSGE is related to the notion of *planar untangling*: Given a straight-line drawing of a planar graph, change the embedding of as few vertices as possible in order to obtain a plane drawing. Goaoc et al. [14] describe an improvement of a result by Bose et al. [3] to show that $\sqrt[4]{(n+1)/2}$ vertices can always be kept in their original positions. Since we can simply take any plane embedding of G_1, use the same embedding for G_2 and then untangle G_2, it immediately follows that every two planar graphs on n vertices admit a $\sqrt[4]{(n+1)/2}$-PSGE.

Results and Organization. In Section 2, we study column planarity for subsets of trees. We prove that every tree on n vertices contains a column planar subset of size $14n/17$ and we show that there exist trees where every column planar subset has size at most $5n/6$. In Section 3, we establish the relation between column planarity and PSGE. We show that every two trees admit an $11n/17$-PSGE, that every tree and ULP graph admit a $14n/17$-PSGE, that every two outerpaths admit an $n/4$-PSGE, and that every outerpath and a tree admit an $11n/34$-PSGE.

2 Column planar sets in trees

In this section, we show how to find large column planar sets in trees. Let $p(v)$ be the parent of vertex v in a rooted tree T, and let $r(T)$ be the root of T. Given a subset R of the vertices of T, let $C_R(v)$ be the non-leaf children of v in R and let $C_R^+(v)$ be those vertices in $C_R(v)$ with at least one child in R. We first prove that subsets of T satisfying certain conditions are always column planar and next that every tree contains a large such subset.

Lemma 1. *For a rooted tree T, R is column planar in T if for all $v \in R$, either (1) $p(v) \in R$, the number of non-leaf children of v in R is at most two, and at most one of these children has a child in R (i.e. $C_R(v) \leq 2$ and $C_R^+(v) \leq 1$); or (2) $p(v) \notin R$, the number of non-leaf children of v in R is at most four, and at most two of these children have a child in R (i.e. $C_R(v) \leq 4$ and $C_R^+(v) \leq 2$).*

Proof. We will embed T recursively. The x-coordinates of V will be fixed in such a way that any assignment $\gamma : R \to \mathbb{R}$ of y-coordinates to R can be accommodated by embedding the vertices of $V \setminus R$ with y-coordinates much larger than $\max \gamma$ or much smaller than $\min \gamma$. Thus, the edges between $V \setminus R$ and R are embedded as near-vertical line segments. In the figures that accompany this proof, such edges will be drawn as curves.

For a subtree T' of T, let $p(T')$ be the parent of $r(T')$. If $r(T')$ is the root of T then $p(T')$, though it does not exist, is viewed as not in R. Our embedding will have the following properties for each subtree T': (i) if $r(T') \notin R$ or $\{r(T'), p(T')\} \subseteq R$, then $r(T')$ has either the smallest or largest x-coordinate among all vertices in T'; (ii) if $r(T') \notin R$, then $r(T')$ has either the smallest or largest y-coordinate among all vertices in T'; and (iii) no almost-vertical ray from $r(T')$ intersects any edge from T'.

Let T be the rooted tree we want to embed. Let $r = r(T)$. If $r \in R$, then recursively generate embeddings of all non-leaf children of r. Scale each such embedding horizontally to width 1. Suppose first that $p(T) \in R$. See Fig. 3a.

Embed r at $x = 1$ and its ℓ leaf children at $x = 2, \ldots, \ell + 1$. (Their y-coordinates are determined by γ.) Suppose $C_R(v) \subseteq \{r_1, s_1\}$ and $C_R^+(v) \subseteq \{r_1\}$. Embed r_1 and its subtree recursively and scale its x-coordinates to lie in $[\ell+3, \ell+4]$. By (i), and possibly after mirroring the embedding of the subtree rooted at r_1 horizontally, the edge $\{r, r_1\}$ does not cross edges in the subtree rooted at r_1.

Embed s_1 at $x = \ell + 2$. Let $T_1, \ldots, T_k$ be the child subtrees of s_1. Embed T_i recursively and scale its x-coordinates to lie in $[\ell + 3 + 2i, \ell + 4 + 2i]$ for all $1 \leq i \leq k$. Vertex s_1 will be above $\{r, r_1\}$ for some γ and below $\{r, r_1\}$ for other

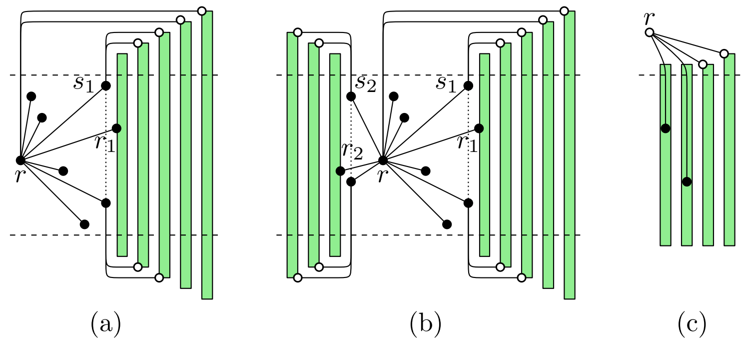

Fig. 3. Embedding a tree with a column planar set. The column planar vertices are black .

γ. If it is above, let $r(T_1), \ldots, r(T_k)$ have progressively larger y-coordinates (by scaling up and mirroring vertically if necessary). If it is below, let them have progressively smaller y-coordinates. Then none of the edges $\{s_1, r(T_i)\}$ cross $\{r, r_1\}$ and the edge $\{s_1, r(T_i)\}$ does not cross any edges in T_i by (i) and (ii).

Recursively, embed the remaining child subtrees $T'_1, \ldots, T'_t$ (none of whose roots are in R) with x-coordinates in $[\ell + 3 + 2k + 2i, \ell + 4 + 2k + 2i]$ for all $1 \leq i \leq t$ such that $r(T'_1), \ldots, r(T'_t)$ have progressively larger y-coordinates. The edge $\{r, r(T'_i)\}$ does not cross any edges in T'_i by (ii). In the completed drawing, note that r has the lowest x-coordinate, and thus (i) is satisfied. Properties (ii) and (iii) are trivially satisfied.

Suppose that $p(T) \notin R$. Proceed first as in the previous case. Suppose $C_R(v) \subseteq \{r_1, r_2, s_1, s_2\}$ and $C^+_R(v) \subseteq \{r_1, r_2\}$. Mirror the recursive embedding of the subtree rooted at r_2 horizontally and scale it to have x-coordinates in $[-3, -2]$. Embed the subtree rooted at s_1 as in the previous case. For s_2, proceed similarly but embed s_2 and its subtree to the left of r. See Fig. 3b. Properties (i)-(iii) are trivially satisfied.

Finally, suppose that $r = r(T) \notin R$. Embed its child subtrees $T_1, \ldots, T_t$ to have x-coordinates in $[2i, 2i + 1]$ for all $1 \leq i \leq t$, starting with the ones rooted at a vertex in R. Embed r sufficiently high on the line $x = 1$. For subtrees T_i with $r(T_i) \in R$, note that the edge $\{r, r(T_i)\}$ does not cross any edges of T_i due to (iii). For the other ones, $\{r, r(T_i)\}$ does not cross edges of T_i due to (i) and (ii). See Fig. 3c. Properties (i-iii) are satisfied. □

It remains to show that every tree contains a subset that satisfies the conditions imposed by Lemma 1. We show that every tree on n vertices contains such a subset of size at least $14n/17$ and that there are trees with no column planar subset of size larger than $5n/6$. Note that $14/17 \approx 5/6 - 0.01$, and thus our results are almost tight.

Lemma 2. *Let T be a tree on n vertices rooted at any vertex $r(T)$. Let c_i be the number of vertices with exactly i children. Then $c_0 = (n+1+\sum_{i=1}^{n-1}(i-2)c_i)/2$.*

Proof. The number of edges in T is $n-1$ and also equals the degree sum divided by two. Thus, $\sum_{i=0}^{n-1} c_i(i+1) = 2(n-1)+1 = 2n-1$. Since $\sum_{i=0}^{n-1} c_i = n$, $\sum_{i=0}^{n-1} c_i(i-2)+3n = 2n-1$, and $-2c_0 = -n-1-\sum_{i=1}^{n-1} c_i(i-2)$. The lemma follows. □

Theorem 1. *A tree T on n vertices contains a column planar set of size at least $14n/17$.*

Proof. Root T at an arbitrary non-leaf vertex $r(T)$. Orient every edge towards the root and topologically sort T to obtain an order $v_1, \dots, v_n$. We will greedily add vertices to R in this order. More precisely, let $R_0 = \emptyset$ and let $R_i := R_{i-1} \cup \{v_i\}$ if $R_{i-1} \cup \{v_i\}$ satisfies Lemma 1 and let $R_i := R_{i-1}$ otherwise. Let $R = R_n$ be our final subset of T.

We say that a vertex is *marked* if it is in R. Consider a vertex $v = v_i \notin R$. The reason that v is not in R is that $R_{i-1} \cup \{v\}$ does not satisfy the condition in Lemma 1 for v or a child u of v (or both). More precisely, v is contained in exactly one of the following sets:

$$\begin{aligned}
X_a &= \{v \in T \setminus R : && |C_R^+(v)| > 2\}\\
X_b &= \{v \in T \setminus R \setminus X_a : && |C_R(v)| > 4\}\\
X_c &= \{v \in T \setminus R \setminus X_a \setminus X_b : && |C_R^+(u)| > 1\}\\
X_d &= \{v \in T \setminus R \setminus X_a \setminus X_b \setminus X_c : && |C_R(u)| > 2\}.
\end{aligned}$$

We associate with each such v a witness tree $W(v)$ as follows (see Fig. 4). If $v \in X_a$, then let $W(v)$ be v, three vertices of $C_R^+(v)$ and a marked child of each of them (which must exist by definition of $C_R^+(v)$). If $v \in X_b$, then let $W(v)$ be v and five marked children of v. If $v \in X_c$, then let $W(v)$ be v, u, two vertices of $C_R^+(u)$ and a marked child of each of them. If $v \in X_d$, let $W(v)$ be v, u and three marked children of u. Note that $W(v)$ and $W(v')$ are disjoint for $v, v' \in T \setminus R$ with $v \neq v'$. We have

$$|X_a| + |X_b| + |X_c| + |X_d| + |R| = n. \tag{1}$$

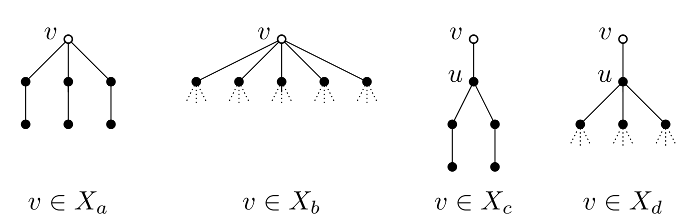

Fig. 4. The witness tree $W(v)$ when v is in X_a, X_b, X_c or X_d. The marked vertices are black. Dotted line segments indicate that a vertex has at least one child.

Let L_t and I_t be the set of marked vertices of $\bigcup_{v \in X_t} W(v)$ that are leaves and internal vertices in T, respectively, for $t = a, b, c, d$. We have

$$|I_a| + |L_a| = 6|X_a| \qquad |L_a| \le 3|X_a| \tag{2}$$
$$|I_b| + |L_b| = 5|X_b| \qquad |L_b| = 0 \tag{3}$$
$$|I_c| + |L_c| = 5|X_c| \qquad |L_c| \le 2|X_c| \tag{4}$$
$$|I_d| + |L_d| = 4|X_d| \qquad |L_d| = 0 \tag{5}$$

Since R always contains all leaves of T, we have

$$|R| \ge c_0 + |I_a| + |I_b| + |I_c| + |I_d|, \tag{6}$$

where c_i is the number of vertices with exactly i children in T. Note that $W(v)$ contains a vertex with at least three children if $v \in X_a \cup X_b \cup X_d$. Hence, by Lemma 2,

$$c_0 > \frac{n - c_1 + \sum_{i=3}^{n-1} c_i}{2} \ge \frac{n - c_1 + |X_a| + |X_b| + |X_d|}{2}. \tag{7}$$

In addition, we have

$$c_0 \ge |L_a| + |L_b| + |L_c| + |L_d|. \tag{8}$$

Before we bound $|R|$, consider the set S formed by all leaves and all vertices with one child. Then S is column planar by Lemma 1 and $|S| = c_0 + c_1$. Whenever the greedily chosen R has size less than $c_0 + c_1$, we choose $R = S$ instead. Thus, we may assume

$$|R| \ge c_0 + c_1. \tag{9}$$

Equations (7) and (9) yield

$$|R| > n - c_0 + |X_a| + |X_b| + |X_d|; \tag{10}$$

equations (2) and (8) yield

$$c_0 \ge 6|X_a| - |I_a| + |L_c|; \tag{11}$$

and equations (3), (4), (5), and (6) yield

$$|R| \ge c_0 + 5|X_b| + 5|X_c| + 4|X_d| - |L_c| + |I_a|. \tag{12}$$

To eliminate c_0, we combine equation (10) with two times (11) and three times (12) to obtain $4|R| > n + 13|X_a| + 16|X_b| + 15|X_c| + 13|X_d| - |L_c| + |I_a|$. With equation (4), this gives $4|R| > n + 13|X_a| + 16|X_b| + 13|X_c| + 13|X_d| + |I_a| \ge n + 13(|X_a| + |X_b| + |X_c| + |X_d|)$. Together with equation (1), this yields the desired bound of $|R| > 14n/17$. □

The greedy algorithm achieves exactly this amount on the tree depicted in Fig. 5. Note that also $|S| = c_0 + c_1 = 14n/17$ in this tree. In general, Theorem 1 is close to best possible:

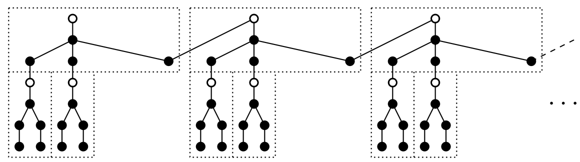

Fig. 5. A tree for which $|R| = |S| = 14n/17$. The set R is colored black.

Theorem 2. *For any $\epsilon > 0$ and any $n > 2/\epsilon + 5$, there exists a tree T with n vertices in which every column planar subset in T has at most $(5/6+\epsilon)n$ vertices.*

Proof. Let $p = \lfloor n/6 \rfloor$. Let T be p copies, $T_1, T_2, \ldots, T_p$, of the tree shown in Fig. 6a in which the root of T_{i+1} is made a child of the rightmost leaf of T_i, for $i = 1, \ldots, p-1$. Suppose there is a column planar set R of marked vertices in T with $|R|/n > 5/6 + \epsilon$. Then in some sequence of at most $k = \lceil 1/(3\epsilon) \rceil$ subtrees $T_i, T_{i+1}, \ldots, T_j$ there must be at least two trees with 6 marked vertices and the other trees with 5 marked vertices. If not, since each subtree has 6 vertices, the average fraction of marked vertices per tree is less than $\frac{5k+2}{6k} < 5/6 + \epsilon$.

Let $T_i, T_{i+1}, \ldots, T_j$ be such a sequence. By possibly deleting a prefix of the sequence, we can assume that T_i has 6 marked vertices. Let $\ell > i$ be the smallest index such that the root of T_ℓ is marked. Since $T_i, T_{i+1}, \ldots, T_j$ contains at least two trees with 6 marked vertices, T_ℓ exists. Let H be the subtree induced by the root of T_ℓ and the vertices in $T_i \cup T_{i+1} \cup \cdots \cup T_{\ell-1}$. By definition, the unmarked vertices in H are exactly the roots of the subtrees $T_{i+1}, T_{i+2}, \ldots, T_{\ell-1}$. We claim that the marked vertices are not column planar in H.

To simplify notation, let $H_1, H_2, \ldots, H_{q-1}$ be the sequence of subtrees in H and let r_q be the (marked) root of T_ℓ. Label the vertices of H_i as in Fig. 6a subscripted by i. See Fig. 6b. Let R' be the marked vertices in H and suppose R' is ρ-column planar in H. For an edge $\{a, b\}$ in H with $a, b \in R'$, let $\rho(a, b) = [\rho(a), \rho(b)]$ be the x-interval of edge $\{a, b\}$. For two edges $\{a, b\}$ and $\{c, d\}$ in H where a, b, c, and d are distinct vertices in R', $\rho(a, b) \cap \rho(c, d) = \emptyset$: otherwise, by choosing γ appropriately we can cause the edges to intersect within their shared x-interval. This implies, for example, that the x-interval spanned by marked vertices in one subtree does not intersect that of a different subtree.

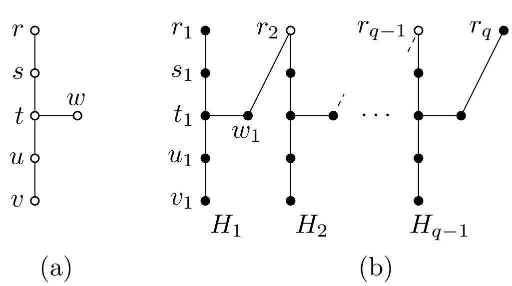

Fig. 6. (a) The tree T_i and (b) H used in the proof of Theorem 2.

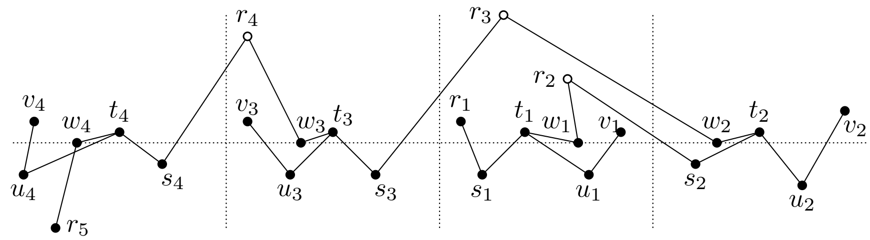

Fig. 7. An example of how γ is chosen in the proof of Theorem 2 where $q = 5$. Note that forcing r_5 (bottom left) below the x-axis causes the edge $\{w_4, r_5\}$ to intersect another edge.

For H_1, since $\rho(s_1, t_1) \cap \rho(u_1, v_1) = \emptyset$ and $\rho(t_1, u_1) \cap \rho(r_1, s_1) = \emptyset$, $\rho(t_1)$ is *between* $\rho(r_1, s_1)$ and $\rho(u_1, v_1)$ (meaning either $\rho(r_1, s_1) < \rho(t_1) < \rho(u_1, v_1)$ or $\rho(u_1, v_1) < \rho(t_1) < \rho(r_1, s_1)$, where $A < B$ if for all $a \in A$ and $b \in B$, $a < b$). By similar reasoning, $\rho(w_1)$ is between $\rho(t_1)$ and $\rho(u_1, v_1)$ or between $\rho(t_1)$ and $\rho(r_1, s_1)$. Let us assume, by renaming vertices if necessary, that $\rho(w_1)$ is between $\rho(t_1)$ and $\rho(u_1, v_1)$. See Fig. 7.

The basic idea is to choose γ so that vertices in R are close to the x-axis (with $\gamma(u_i) < \gamma(s_i) < 0 = \gamma(w_i) < \gamma(t_i) < \gamma(v_i)$ for all i except when mentioned otherwise) and so that unmarked vertices are forced to be above the x-axis. We set $\gamma(u_1)$ to be negative and $\gamma(v_1)$ to be positive (so w_1 lies in the triangle $t_1u_1v_1$). This, together with the fact that r_2 is connected to s_2, forces the edge from w_1 to r_2 to be upward and thus r_2 to be above the x-axis.

Consider the order of $\rho(s_2)$, $\rho(t_2)$ and $\rho(u_2, v_2)$. If $\rho(s_2)$ is between $\rho(t_2)$ and $\rho(u_2, v_2)$, then setting γ so that the path t_2, u_2, v_2 is above s_2 ($\gamma(t_2) < \gamma(v_2) < 0 < \gamma(s_2) < \gamma(u_2)$) causes the path to intersect $\{r_2, s_2\}$. Note that $\rho(u_2, v_2)$ cannot be between $\rho(t_2)$ and $\rho(s_2)$ since $\rho(u_2, v_2) \cap \rho(s_2, t_2) = \emptyset$. Hence, $\rho(t_2)$ is between $\rho(s_2)$ and $\rho(u_2, v_2)$. Now let us consider the possible positions of $\rho(w_2)$. If $\rho(s_2)$ is between $\rho(w_2)$ and $\rho(t_2)$, then setting γ so that the path u_2, t_2, w_2 is above s_2 ($\gamma(w_2) < \gamma(u_2) < 0 < \gamma(s_2) < \gamma(t_2)$) causes the path to intersect $\{r_2, s_2\}$. Note that $\rho(u_2, v_2)$ cannot be between $\rho(w_2)$ and $\rho(t_2)$ since $\rho(u_2, v_2) \cap \rho(t_2, w_2) = \emptyset$. Hence, $\rho(w_2)$ is between $\rho(s_2)$ and $\rho(t_2)$ or between $\rho(t_2)$ and $\rho(u_2, v_2)$. In the first case, we set $\gamma(s_2) < 0 = \gamma(w_2) < \gamma(t_2)$ so the edge from w_2 to r_3 is forced upward to avoid intersecting path r_2, s_2, t_2. In the second case, we set γ so that the path t_2, u_2, v_2 is below w_2 ($\gamma(u_2) < 0 = \gamma(w_2) < \gamma(t_2) < \gamma(v_2)$) and the edge from w_2 to r_3 is forced upward. By repeating this argument, we force all the unmarked vertices as well as r_q to be above the x-axis. Since r_q is marked, we derive a contradiction by setting $\gamma(r_q) < 0$. □

3 Partial simultaneous geometric embedding

The relation between column planarity and PSGE is expressed by the following theorem, which relates the size of column planar sets to PSGE.

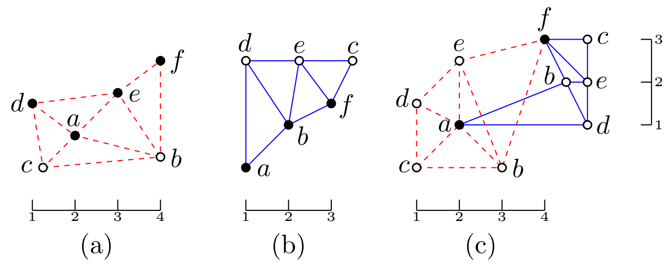

Fig. 8. (a) Graph G_1 with $R_1 = \{a, d, e, f\}$ and $\rho_1 = \{d \mapsto 1, a \mapsto 2, e \mapsto 3, f \mapsto 4\}$. (b) Graph G_2 with $R_2 = \{a, b, f\}$ and $\rho_2 = \{a \mapsto 1, b \mapsto 2, f \mapsto 3\}$. (c) A 2-PSGE of G_1 and G_2 where vertex set $R = R_1 \cap R_2 = \{a, f\}$ is shared.

Theorem 3. *Consider planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on n vertices. If R_1 is column planar in G_1, R_2 is column planar in G_2 and $|R_1| + |R_2| > n$, then G_1 and G_2 admit a $(|R_1| + |R_2| - n)$-PSGE.*

Proof. Fig. 8 illustrates the construction. The set $R = R_1 \cap R_2$ has size at least $|R_1| + |R_2| - n > 0$ and is column planar in both G_1 and G_2. More specifically, there exist injections $\rho_1 : R \to \mathbb{R}$ and $\rho_2 : R \to \mathbb{R}$ such that R is ρ_1-column planar in G_1 and ρ_2-column planar in G_2. By exchanging the roles of the x- and y-coordinates in the definition of column planar in G_2, we see that for all injections $\gamma : R \to \mathbb{R}$, there exists a plane straight-line embedding of G_2 that embeds each $v \in R$ at $(\gamma(v), \rho_2(v))$. In particular, we may choose $\gamma = \rho_1$. □

Two trees. Combining Theorem 3 and Theorem 1 immediately yields the following lower bound on the size of a PSGE of two trees.

Corollary 1. *Every two trees on a set of n vertices admit an $11n/17$-PSGE.*

There are two trees T_1 and T_2 on 226 vertices that do not admit an SGE [13]. Thus, an upper bound on the size of the common set in a PSGE of T_1 and T_2 is 225. Root T_1 arbitrarily and let T_1^k be the result of taking k copies of T_1 and connecting their roots with a path. Define T_2^k similarly. Then an upper bound on the size of the common set in a PSGE of T_1^k and T_2^k is $225k$. It follows that there exist two trees on a set of n vertices that admit no k-PSGE for $k > 225n/226$.

Tree and ULP graph. If one of the two graphs in our PSGE is ULP, then the size of the common set depends only on how large a column planar set we can find in the other graph:

Lemma 3. *Consider a planar graph $G_1 = (V, E_1)$ and a ULP graph $G_2 = (V, E_2)$ on n vertices. If R is column planar in G_1, then G_1 and G_2 admit a $|R|$-PSGE.*

Proof. By exchanging the roles of x- and y-coordinates in the definition of column planar, we see that for all injections $\gamma : R \to \mathbb{R}$, there exists a plane straight-line embedding of G_1 with $v \in R$ at $(\gamma(v), \rho(v))$. Since G_2 is a ULP graph, for all injections $y : V \to \mathbb{R}$, there exists an injection $x : V \to \mathbb{R}$ such that placing $v \in V$ at $(x(v), y(v))$ results in a straight-line embedding of G_2. Thus, placing the vertices $v \in R$ at $(x(v), \rho(v))$ permits both a straight-line embedding of G_1 and G_2. □

Combining this with Theorem 1 yields

Corollary 2. *A tree and a ULP graph admit a* $14n/17$*-PSGE.*

Two outerpaths & outerpath and tree. An *outerplanar* graph is a planar graph that admits an embedding (called the *outerplane* embedding) that places all its vertices on the unbounded face. An *outerpath* is an outerplanar graph whose *weak dual* (the graph obtained from the dual graph by deleting the vertex corresponding to the unbounded face) is a path. A maximal outerpath has exactly two vertices of degree two: these vertices are on the faces that correspond to the terminal vertices of the dual path. Consider a maximal outerpath $G = (V, E)$. The *outer cycle* of G is the Hamiltonian cycle of G that bounds the unbounded face in the outerplane embedding of G. Denote by $C(G)$ the vertices of degree two in G. Deleting $C(G)$ from G partitions the outer cycle of G into two connected components whose vertices we refer to as $A(G)$ and $B(G)$. Note that $A(G) \cup B(G) \cup C(G) = V$. It is easy to see that:

Lemma 4. *Given a maximal outerpath* $G = (V, E)$*, the subsets* $A(G) \cup C(G)$ *and* $B(G) \cup C(G)$ *are column planar.*

Unlike in the tree setting, Theorem 3 does not immediately give a lower bound on the size of a PSGE of two outerpaths, since we might have $|A(G)| = |B(G)| = n/2 - 1$. Fortunately, this is easily resolved:

Theorem 4. *Every two outerpaths on a set of* n *vertices admit an* $n/4$*-PSGE.*

Proof. Consider outerpaths $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Without loss of generality, G_1 and G_2 are maximal. Let $X_i^+ := X(G_i) \cup C(G_i)$ for $X = A, B$ and $i = 1, 2$. Then by Theorem 3 and Lemma 4, G_1 and G_2 admit a $\max\{|A_1^+ \cap A_2^+|, |A_1^+ \cap B_2^+|, |B_1^+ \cap A_2^+|, |B_1^+ \cap B_2^+|\}$-PSGE. Since the union of these four sets is again V, the maximum of their cardinalities must be at least $n/4$, which concludes the proof. □

Note that $\max\{|A(G) \cup C(G)|, |B(G) \cup C(G)|\} \geq n/2 + 1$. Hence, by Theorem 3 and Theorem 1 we get

Corollary 3. *An outerpath and a tree on* n *vertices admit a* $11n/34$*-PSGE.*

4 Discussion and Open Problems

Our results leave several directions for future research. The tree drawings produced by Theorem 1 may have exponential area. It would be interesting to see whether polynomial area is sufficient. Further research could be directed towards closing the gap between the lower and upper bound on the size of column planar sets for trees and on developing bounds for such sets in general planar graphs.

Acknowledgments. Research on the topic of this paper was initiated at the 1st International Workshop on Drawing Algorithms for Networks of Changing Entities (DANCE'2014) in Langbroek, The Netherlands, supported by the Netherlands Organisation for Scientific Research (NWO) under project no. 639.023.208. We wish to thank all participants, and in particular Csaba Tóth and Michael Hoffmann, for useful discussions on the topic of this paper.

References

1. Angelini, P., Geyer, M., Kaufmann, M., Neuwirth, D.: On a tree and a path with no geometric simultaneous embedding. In: Brandes, U., Cornelsen, S. (eds.) Graph Drawing, LNCS, vol. 6502, pp. 38–49. Springer (2011)
2. Bläsius, T., Kobourov, S.G., Rutter, I.: Simultaneous embedding of planar graphs. In: Tamassia, R. (ed.) Handbook of Graph Drawing and Visualization. Chapman and Hall/CRC (2013)
3. Bose, P., Dujmović, V., Hurtado, F., Langerman, S., Morin, P., Wood, D.R.: A polynomial bound for untangling geometric planar graphs. Disc. & Comp. Geom. 42(4), 570–585 (2009)
4. Brass, P., Cenek, E., Duncan, C.A., Efrat, A., Erten, C., Ismailescu, D.P., Kobourov, S.G., Lubiw, A., Mitchell, J.S.: On simultaneous planar graph embeddings. Comp. Geom. 36(2), 117–130 (2007)
5. Cabello, S., van Kreveld, M., Liotta, G., Meijer, H., Speckmann, B., Verbeek, K.: Geometric simultaneous embeddings of a graph and a matching. J. Graph Alg. Appl. 15(1), 79–96 (2011)
6. Cappos, J., Estrella-Balderrama, A., Fowler, J.J., Kobourov, S.G.: Simultaneous graph embedding with bends and circular arcs. Comp. Geom. 42(2), 173–182 (2009)
7. Di Giacomo, E., Didimo, W., van Kreveld, M., Liotta, G., Speckmann, B.: Matched drawings of planar graphs. In: Hong, S.H., Nishizeki, T., Quan, W. (eds.) Graph Drawing, LNCS, vol. 4875, pp. 183–194. Springer (2008)
8. Erten, C., Kobourov, S.G.: Simultaneous embedding of planar graphs with few bends. In: Pach, J. (ed.) Graph Drawing, LNCS, vol. 3383, pp. 195–205. Springer (2005)
9. Estrella-Balderrama, A., Fowler, J.J., Kobourov, S.G.: Characterization of unlabeled level planar trees. Comp. Geom. 42(6), 704–721 (2009)
10. Estrella-Balderrama, A., Gassner, E., Jünger, M., Percan, M., Schaefer, M., Schulz, M.: Simultaneous geometric graph embeddings. In: Hong, S.H., Nishizeki, T., Quan, W. (eds.) Graph Drawing, LNCS, vol. 4875, pp. 280–290. Springer (2008)
11. Fowler, J.J., Kobourov, S.G.: Characterization of unlabeled level planar graphs. In: Hong, S.H., Nishizeki, T., Quan, W. (eds.) Graph Drawing, LNCS, vol. 4875, pp. 37–49. Springer (2008)
12. Frati, F., Kaufmann, M., Kobourov, S.G.: Constrained simultaneous and near-simultaneous embeddings. In: Hong, S.H., Nishizeki, T., Quan, W. (eds.) Graph Drawing, LNCS, vol. 4875, pp. 268–279. Springer (2008)
13. Geyer, M., Kaufmann, M., Vrt'o, I.: Two trees which are self-intersecting when drawn simultaneously. Disc. Math. 309(7), 1909–1916 (2009)
14. Goaoc, X., Kratochvíl, J., Okamoto, Y., Shin, C.S., Spillner, A., Wolff, A.: Untangling a planar graph. Disc. & Comp. Geom. 42(4), 542–569 (2009)