Graphs admitting d-realizers: tree-decompositions and box-representations

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Abstract

A d-realizer is a collection $R = \{\pi_1, \ldots, \pi_d\}$ of d permutations of a set V representing an antichain in $\mathbb{R}^d$. We use R to define a graph G_R on the suspended set $V^+ = V \cup \{s_1, \ldots, s_d\}$. It turns out that G_R has $dn + \binom{d}{2}$ edges ($n = |V|$), among them the edges of the outer clique on $\{s_1, \ldots, s_d\}$. The inner edges of G_R can be partitioned into d trees such that T_i spans $V + s_i$. In the case $d = 3$ the graph G_R is a planar triangulation and T_1, T_2, T_3 is a Schnyder wood on G_R. The following two results show that d-realizers resemble Schnyder woods in several aspects:

- Complete point-face contact systems of homothetic simplices in $\mathbb{R}^{d-1}$ induce a d-realizer.
- Any spanning subgraph of a graph G with a d-realizer has a d-dimensional proper touching box representation.

We think that d-realizers will prove to be valuable generalization of Schnyder woods to higher dimensions.

Mathematics Subject Classifications (2010) 05C10, 05C62, 06A07,

1 Introduction

We consider $\mathbb{R}^d$ equipped with the *dominance order*, i.e., for $x, y \in \mathbb{R}^d$ we have $x \leq_{\mathsf{dom}} y$ if and only if $x_i \leq y_i$ for $i = 1, \ldots, d$. A set $P \subset \mathbb{R}^d$ is in *general position* if no two points of P share a coordinate. If no two points of a set P are in the dominance relation $\leq_{\mathsf{dom}}$, then we call P an *antichain.* If P is in general position, then the projection to the ith coordinate yields a permutation π_i of P. In compliance with the previous definition, we call a family of permutations $\pi_1, \ldots, \pi_d$ of V an *antichain* if for all $x, y \in V$ there are indices i and j such that x precedes y in π_i and y precedes x in π_j. We use the notation $x \prec_i y$ to denote that x precedes y in π_i.

An antichain V in $\mathbb{R}^d$ is *suspended* if V contains a suspension vertex for each i, i.e., a vertex $s_i = (0, \ldots, 0, M_i, 0, \ldots, 0)$ and $0 \leq v_i < M_i$ for all $v \in V \backslash s_i$. Similarly s_i is an i-suspension for $\pi_1, \ldots, \pi_d$ if s_i is the last element of π_i and among the first $d-1$ elements in π_j for $j \neq i$. The family $\pi_1, \ldots, \pi_d$ is suspended if it has an i-suspension for each $i \in [d]$.

Definition 1 *A* d-realizer *is a suspended antichain* $\pi_1, \ldots, \pi_d$ *of permutations of* V^+ *where* $V^+ = V \cup S$ *and* $S = \{s_1, \ldots, s_d\}$ *is the set of suspensions.*

Definition 2 *The* graph of a d-realizer $(\pi_1, \ldots, \pi_d)$ *is the graph* $G_R = (V^+, E^+)$ *with* $E^+ = E_R \cup E_S$ *where* E_S *is the set of edges of a clique on* S *and pairs* x, y *are edges in* E_R *if they satisfy two properties:*

(x, y) **is a candidate pair** *: for all* $z \neq x, y$ *there is an* i *with* $z \prec_i x$ *and* $z \prec_i y$.

(x, y) **has the 1-of-d-property** *: there is a unique* $i \in [d]$ *with* $x \prec_i y$, *i.e.,* $y \prec_j x$ *for all* $j \neq i$.

The definition of Schnyder woods was originally motivated by the study of the order dimension of incidence posets of graphs. In this line of research the following definition was proposed in [5]:

The dimension *of* $G = (V, E)$ *is at most* k *if there are permutations* $\pi_1, \ldots, \pi_k$ *of* V *such that each edge* $(x, y) \in E$ *is a candidate pair.*

If G is two-connected, then it follows that $\pi_1, \ldots, \pi_k$ is a antichain. The following are known:

- $\dim(G) \leq 3$ iff G is planar (Schnyder [9]).

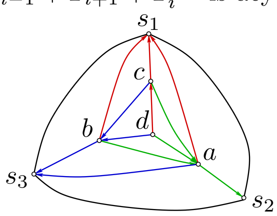
*Research supported in part by NSERC of Canada
†Partially supported by DFG grant FE-340/7-2 and ESF EuroGIGA project GraDR.
‡Research funded in part by NSF grants CCF-0545743 and CCF-1115971

- $\dim(G) \leq 4 \implies G$ has at most $3/8n^2$ edges.
- Exact values of $\dim(K_n)$ are known for $n < 10^{40}$.

The 1-of-d-property naturally leads to a coloring and an orientation of the edges of G_R: The orientation is $x \to y$ if x precedes y only in a single π_i. The color of $x \to y$ is the index i with $x \prec_i y$. Let T_i be the set of edges of color i.

Note that in the case $d = 3$ the 1:2 property is fulfilled by all candidate edges; this is where Schnyder's coloring and orientation of edges comes from. Schnyder found that for all i the following two properties hold:

(a) T_i is an in-arborescense with root s_i.
(b) $T_{i-1} + T_{i+1} + T_i^{-1}$ is acyclic.



$\pi_1 = 1\ 2\ 4\ 3\ s_1$
$\pi_2 = 2\ 3\ 4\ 1\ s_2$
$\pi_3 = 1\ 3\ 4\ 2\ s_3$

Figure 1: An example of a 3-realizer and its graph.

In the next section we show that this also holds in the case of a d-realizer. In Section 3 we continue to show how d-realizer can be used to construct proper touching box representations; the $d = 3$ case of this result was obtained in [1]. In Section 4 we connect d-realizers to orthogonal surfaces and show how they arise from touching simplices. We conclude with examples and some open problems.

2 Tree-decompositions

Proposition 1 *If G_R is defined by a d-realizer $(\pi_1, \dots, \pi_d)$ and T_i is the set of edges of color i, then T_i is an in-arborescense with root s_i.*

Proof. We first show that each $v \in V$ has a unique out-edge in T_i.

Let $H_i(x)$ be the set of all y with $x \prec_i y$ and $y \prec_j x$ for all $j \neq i$, i.e., the set of all y such that (x, y) has the 1-of-d-property. Let $p_i(x)$ be the first element of $H_i(x)$ with respect to π_i, i.e., $p_i(x)$ is the least element of π_i such that $(x, p_i(x))$ has the 1-of-d-property.

Claim a. $(x, p_i(x))$ is a candidate.

Consider $z \neq x, p_i(x)$. Since a d-realizer is an antichain there is some j with $x \prec_j z$. If $j \neq i$, then $p_i(x) \prec_j x$ and by transitivity $p_i(x) \prec_j z$. If the only choice for j is i, then $z \in H_i(x)$ and $p_i(x) \prec_j z$ follows from the choice of $p_i(x)$. △

From Claim a. it follows that $(x, p_i(x)) \in T_i$.

Claim b. If (x, y) is a candidate with $y \in H_i(x)$, then $y = p_i(x)$.

Indeed if $y \neq p_i(x)$ then there is no π_j where x and y precede $p_i(x)$. In π_i we have $x \prec_i p_i(x) \prec_i y$ and if $j \neq i$, then $p_i(x) \prec_j x$. △

Hence $(x, p_i(x))$ is the only out-edge of x in T_i. Therefore the number of edges of T_i is $|V|$. Since T_i is spanning $V + s_i$ it only remains to show that T_i is connected. For $x \in V$ define $x_0 = x$ and for $k \geq 0$ let $x_{k+1} = p_i(x_k)$. This defines a path that moves to the right on π_i; hence it must reach s_i. □

Corollary 1 *A graph G_R defined by a d-realizer on a vertex set V^+ with $|V^+| = n + d$ has $dn + \binom{d}{2}$ edges.*

Proposition 2 *If G_R is defined by a d-realizer , then $T_i^{-1} + \sum_{j \neq i} T_j$ is acyclic.*

Proof. From the 1-of-d-property it follows that directed edges from T_j with $j \neq i$ point to the left in the order of vertices given by π_i. The same is true if we revert the direction of the edges of T_i, i.e, for the directed edges of T_i^{-1}. □

3 Box-representations

Theorem 1 *Any spanning subgraph H of a graph G with a d-realizer has a d-dimensional proper touching box representation.*

Proof. Let $(\pi_1, \dots, \pi_d)$ be the d-realizer for G. We assume that the order of the first $d-1$ elements in π_i (these are suspensions) is $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_d)$. This has the advantage that for $i < j$ the pair (s_i, s_j) has the 1-of-d-property. The pair is a candidate so we can treat it as a regular edge in T_i.

With $\mathsf{rank}_i(x)$ we denote the position of x in π_i, i.e., if we think of π_i as a bijective map $\pi_i : [n+d] \to V^+$, then $\mathsf{rank}_i(x) = \pi_i^{-1}(x)$. For each x and i we need $\mathsf{rank}_i(p_i(x))$. For a suspension s_i and all $j \leq i$ we assume the value $n + d + 1$ for the strictly speaking undefined expression $\mathsf{rank}_j(p_j(s_i))$.

We first show how to represent G. The box for vertex x in G is $B(x) = \prod_{i=1}^{d} [\mathsf{rank}_i(x), \mathsf{rank}_i(p_i(x))]$.

We need to show proper contact between the box $B(x)$ and the box $B(p_i(x))$ for all i. Let $y = p_i(x)$. Since the projection to $B(x)$ and $B(y)$ to dimension i share the point $\mathsf{rank}_i(y)$, it suffices to show that $\mathsf{rank}_j(x) \in (\mathsf{rank}_j(y), \mathsf{rank}_j(p_j(y)))$ for all $j \neq i$. By the 1-of-d-property, $\mathsf{rank}_j(y) < \mathsf{rank}_j(x)$ for all $j \neq i$. So it suffices to check that $\mathsf{rank}_j(x) < \mathsf{rank}_j(p_j(y))$ for all $j \neq i$.

Let $z = p_j(y)$ and suppose $z \prec_j x$. By the 1-of-d-property, $z \prec_k y$ for all $k \neq j$. Since $y \prec_k x$ for all $k \neq i$ transitivity implies that $z \prec_k x$ for all $k \neq i, j$ and by supposition also for $k = j$. Since a d-realizer is an antichain we can conclude that $x \prec_i z$.

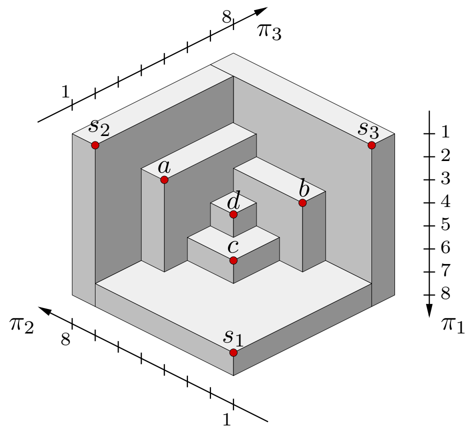


Figure 2: The proper touching box representation of the graph from Fig. 1 obtained with our method.

It now happens that (x, z) and (x, y) both have the 1-of-d-property and $x \prec_i z \prec_i y$. This however contradicts the choice of $y = p_i(x)$ as the least element of π_i such that (x, y) has the 1-of-d-property. Therefore $x \prec_j z$ as needed for the box contact.

To represent a subgraph of G, remove unneeded boxes and edges from the box representation. To get rid of an edge $(x, p_i(x))$ change the extent of $B(x)$ in dimension i to $[\mathsf{rank}_i(x), \mathsf{rank}_i(p_i(x)) - \varepsilon]$. $\square$

4 Orthogonal surfaces and simplizes

In this section we take a more geometric look at the graphs of d-realizers.

With a point $p \in \mathbb{R}^d$ we associate its *cone* $C(p) = \{q \in \mathbb{R}^d : p \leq_{\mathsf{dom}} q\}$. The *filter* $\langle V \rangle$ generated by V is the union of all cones $C(v)$ for $v \in V$. The *orthogonal surface* S_V generated by V is the boundary of $\langle V \rangle$. A point $p \in \mathbb{R}^d$ belongs to S_V if and only if p shares a coordinate with all $v \leq_{\mathsf{dom}} p$, $v \in V$. The generating set V is an antichain if and only if all elements of V appear as minima on S_V.

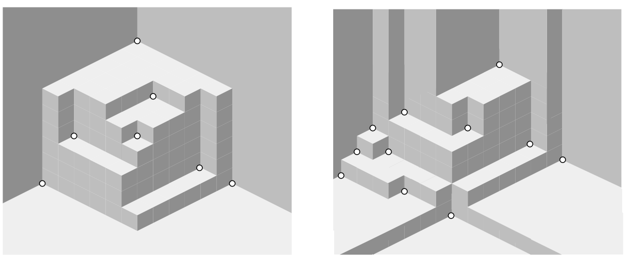


Figure 3: Two orthogonal surfaces in $\mathbb{R}^3$: the left one is generated by a suspended antichain in general position; the antichain generating the right one is neither suspended nor in general position.

Miller [7] observed the connection between Schnyder woods and orthogonal surfaces in $\mathbb{R}^3$. He and subsequently others [3, 6] used orthogonal surfaces to give new proofs for the Brightwell-Trotter theorem about the order dimension of face lattices of 3-polytopes [2]. In fact the dominance order of critical points (maxima, minima, and saddle points) of a 3-dimensional orthogonal surface that is generated by a suspended antichain is the truncated face lattice of a 3-polytope with one facet removed. The converse also holds: every 3-polytope with a facet, selected for removal, has a corresponding orthogonal surface.

The Brightwell-Trotter theorem is an important generalization of Schnyder's dimension theorem. Since orthogonal surfaces can be considered in arbitrary dimensions they provide a direction for generalizing Schnyder structures to higher dimensions. This approach has been taken in [4]. The strongest result in the area is a theorem of Scarf [8] that can be restated as follows: the dominance order of critical points of a d-dimensional orthogonal surface that is generated by a suspended antichain in general position is the truncated face lattice of a simplicial d-polytope with one facet removed. However, the general situation is not nearly as nice as in 3-dimensions. There are simplicial d-polytopes that do not have a corresponding orthogonal surface and if we allow general position the dominance order of critical points need not even be a truncated lattice [4].

The orthogonal surface view for graphs given by a d-realizer R is as follows: Embed vertex v at the point p_v whose coordinates are the ranks of v in the realizer. The out-neighbor of v in color i is the vertex w whose cone $C(p_w)$ is first hit by the ray leaving p_v in the ith coordinate direction.

In the 3-dimensional case we can embed every triangulation (graph with a 3-realizer) on an orthogonal surface S_V with a coplanar V, i.e., all $p \in V$ lie in a plane h with normal $\mathbb{1} = (1, 1, 1)$. Identifying h with $\mathbb{R}^2$ we can find the three edges of a vertex v by growing homothetic equilateral triangles with a corner in v until they hit another vertex; Fig. 4 shows an example.

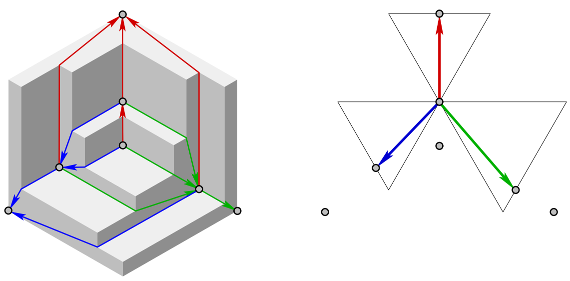


Figure 4: The graph from Fig. 1 on a coplanar orthogonal surface and a sketch illustrating how to recover the out-edges of a vertex from the generating set of points in the plane.

In the same way we may use a set of points in d-space and the homothets of a d-simplex to build a graph from the class defined by $(d+1)$-realizers. The details are as follows: Let Δ be a fixed d-simplex in $\mathbb{R}^d$

and let P be a set of points such that no hyperplane parallel to a facet of Δ contains more than one point (this is the appropriate general position assumption). Let S be the set of corners of a homothet of the dual of Δ that contains P, this is the set of suspensions. Now, for each point $p \in P$ and each corner x of Δ find the unique point q such that there is a homothety that maps Δ to Δ' such that (1) Δ' has no point of P in the interior (2) x is mapped to p and (3) q is on the boundary of Δ'. This condition characterizes the edges $x \to y$ of color x in the graph $G_\Delta(P)$.

Problem 1 *Let G be the graph of a d-realizer. Is it always possible to find a point set P in $\mathbb{R}^{d-1}$ such that $G = G_\Delta(P)$?*

There is one class of graphs where we know that the answer to the problem is yes. These are the skeleton graphs of d-dimensional stacked polytopes, also known as simple d-trees. For these graphs the point set P can be constructed along the stacking (construction) sequence.

In fact, besides this class we know only a few examples of graphs that have a d-realizer with $d > 3$. We know that unlike in the $d = 3$ case we also have non-simple d-trees in the class: Consider a simple d-tree with realizer $(\pi_1, \ldots, \pi_d)$ and let x be vertex with $\mathsf{deg}(x) = d$, for example the last vertex of the construction sequence has this property. Add a new vertex x' by placing it immediately before x in π_1 and π_2 and immediately after x in all the other π_j. It is easily seen that x and x' have the same nighbors in the same colors, in particular they are stacked over the same clique.

Problem 2 *Characterize the d-trees that have a d-realizer.*

Problem 3 *Find meaningful examples and families of graphs that have a d-realizer.*

Regarding the recognition of graphs that have a d-realizer, we have the criterion that to qualify, a graph G must contain a d-clique of suspensions such that there is an orientation of the edges of G with $\mathsf{out\text{-}deg}(x) = d$ for all non-suspensions x.

Problem 4 *Find additional obstructions against having a d-realizer.*

Another situation where induced subgraphs of graphs with a d-realizer appear is given by families of interiorly disjoint pairwise homothetic d-simplices in d-space with vertex-facet incidences. To produce a d-realizer for a supergraph add a small tetrahedron over each vertex that does not take part in a vertex-facet contact and then use the directions of outward pointing normals of the facets to list the tetrahedra.

Problem 5 *Is it possible to realize every simple d-tree as vertex-facet contact graph of homothetic simplices in $\mathbb{R}^d$?*

Acknowledgments

We started the research reported in this paper during the *Bertinoro Workshop on Graph Drawing 2013*.

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