1 Overview

In the first lecture we were reminded of properties of Laplacians and eigenvectors and eigenvalues and exploring Hall’s method. We ended by stating Kirchoff’s Theorem on the number of spanning trees in a graph. In the second lecture we finished the proof of Kirchoff’s theorem and ended with the Wikipedia proof of Kirchoff’s theorem.

2 Review

2.1 Laplacians

Let $L$ denote the Laplacian of a matrix.
The unit eigenvectors of $L$ are $e_1, e_2, ..., e_n$.
The eigenvalues of $L$ are $\lambda_1, \lambda_2, ..., \lambda_n$.
The property of eigenvalues and eigenvectors is that:

$$\forall i, Le_i = \lambda_i e_i$$

By definitions of matrix multiplication, it holds that:

$$x^T L x = \sum_{ij \in E} w_{ij} (x_i - x_j)^2$$

2.2 Raleigh’s Optimization Theorem

$$\lambda_i = \min_{x \perp e_1 e_2 ... e_{i-1}} x^T L x$$

$$e_i = \arg \min_{x \perp e_1 e_2 ... e_{i-1}} x^T L x$$

3 Hall’s Method

3.1 Ways to avoid degenerate drawings

1. Fix positions of a subset of vertices [Tutte]
2. Repulsive forces between pairs of vertices [Eades, Fruchterman and Reingold]
3. Multidimensional Scaling : Euclidean distance $\approx$ graph distance (shortest path [Kamada-Kawai])
4. Exclude degenerate solutions from space we choose to minimize over [Hall 1970]
3.2 Eigenvectors of Laplacian

Smallest eigenvector for the Laplacian $L$ for any graph is the all ones vector. Consider the path graph on 20 vertices. We can plot some of the eigenvectors and observe some periodic behaviour.

![Graph with eigenvectors](image1.png)

Figure 1: Second and third eigenvalues of path graph on 20 vertices.[2]

![Graph with eigenvectors](image2.png)

Figure 2: Last eigenvalue of path graph on 20 vertices.[2]

If we look at smallest two eigenvalues (other than the all ones vector) we can decompose the graph into these eigenvalues and draw that. ($e_2$ gives $x$ values, $e_3$ gives $y$ values.

$$e_2 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad e_3 = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

We can plot a graph with these as its coordinates.

2
Figure 3: Plot of airfoil graph. [2]

Hall’s method

\[ x^T Lx = \sum_{ij \in E} w_{ij} (x_i - x_j)^2 \]

Choose \( x \) to minimize \( x^T Lx \), but the all zero vector minimizes it.
So subject it to

1. \( \|x\|^2 = x^T x = 1 \) to fix scale of embedding. (but \( x_i = 1/\sqrt{n} \) for all \( i \) is optimal but degenerate.

2. \( 1^T x = 0; \sum x_i = 0 \) the sum of coordinates of the vector is zero. Fix the shift of the data points on the line so that the average is centered at zero.

This only places vertices on the x axis, we want to minimize:

\[
\sum_{ij \in E} \left\| \begin{pmatrix} x_i \\ y_i \end{pmatrix} - \begin{pmatrix} x_j \\ y_j \end{pmatrix} \right\|^2 = \sum_{ij} (x_i - x_j)^2 + (y_i - y_j)^2 = x^T Lx + y^T Ly
\]

This would allow \( x = y = e_2 \) as a possible solution.

3. \( y \perp x \) so solution is \( e_2 \) and \( e_3 \).

If you do this in 3d for any graph of the Platonic solid you get the Platonoid solid, and will work for any one-skeleton of a convex polytope.

Figure 4: Plot of first three nontrivial eigenvectors of a dodecahedron. [2]
4 Kirchoff’s Theorem

4.1 Kirchoff’s Theorem (1870s) [Tutte 1948]

(Notes by David Perkinson in Reed College)

Let $K$ be the complete directed graph on $n+1$ vertices $\{0, 1, \ldots, n\}$.

Let $w_{ij}$ be the (symbolic) weight of edge $ij$.

The weighted Laplacian has:

$$L_{ij} = \begin{cases} 
\sum_{k \neq i} w_{ik} & \text{if } i = j \\
-w_{ij} & \text{otherwise}
\end{cases}$$

Let $T_k$ be the set of directed spanning trees in $K$ directed towards the root $k$.

For $\tau \in T_k$ let $w(\tau) = \prod_{ij \in \tau} w_{ij}$.

Let $L^{(k)}$ be the $n \times n$ matrix obtained from $L$ by deleting row $k$ and column $k$.

**Theorem 1. Kirchoff’s theorem**

$$\det L^{(k)} = \sum_{\tau \in T_k} w(\tau)$$

**Proof**

We only need to prove this for $k = 0$, since we can just relabel vertices.
\[ \det L^{(0)} = \sum_{\pi \in S_n} \text{sgn}(\pi) L^{(0)}_{1,\pi(1)} L^{(0)}_{2,\pi(2)} \cdots L^{(0)}_{n,\pi(n)} \]

Where \( S_n \) is the set of all permutations of \( \{1, 2, \ldots, n\} \), and \( \text{sgn}(\pi) \) is +1 if the number of swaps to sort \( \pi \) is even, and -1 if odd.

If we replace \( \text{sgn} \) by 1, then this is the permanent which is \( \#P \) complete (Valiant).

For fixed points \( i \) in the permutation, \( L_{i\pi(i)} \) is the sum of the weights, otherwise negative of one weight.

Let \( \Upsilon \) be the set of subgraphs of \( K \) with outdegree(0) = 0 and outdegree(\( i \)) = 1 for all \( i \neq 0 \). Each \( \gamma \in \Upsilon \) corresponds to a selection of edges from \( K \).

Let \( W(\gamma) = w_{\gamma(1)} w_{\gamma(2)} \cdots w_{\gamma(n)} \) where \( \gamma(i) \) is the destination of the out edge from vertex \( i \).

When \( \gamma \) is a tree \( \tau \in T_0 \), then \( w(\gamma) = w(\tau) \).

Every \( \gamma \in \Upsilon \) that is not in \( T_0 \) contains a cycle. \( \gamma \) is not a permutation. It is possible that \( \gamma(i) = 0 \) or that \( \gamma(i) = \gamma(j) \).

After fully expanding \( \det L^{(0)} \), each term in the expansion is \( \pm w(\gamma) \) for some \( \gamma \in \Upsilon \).

Every permutation \( \pi \) contributes \( \text{sgn}(\pi) w(\gamma) \) to \( \det L^{(0)} \) for every \( \gamma \) that contains all non-trivial cycles of \( \pi \). For example, the identity permutation contributes +\( w(\gamma) \) to \( \det L^{(0)} \) for all \( \gamma \in \Upsilon \) since it has no non-trivial cycles.

If \( \gamma \) contains \( s > 0 \) disjoint cycles, then those \( \pi \) that contain exactly one of those cycles, say \( C = i_1 i_2 \cdots i_k \), contribute \( \text{sgn}(C) (-1)^k w(\gamma) = -w(\gamma) \) to \( L^{(k)} \). This is because odd length cycles have \( \text{sgn} = +1 \) and even length cycles have \( \text{sgn} = -1 \).

In general, those \( \pi \) that contain exactly \( t \) cycles of \( \gamma \) contributes \( (-1)^t w(\gamma) \) to \( \det L^{(0)} \). Total contribution of \( w(\gamma) \) to \( \det L' \) is:

\[ \sum_{t=0}^{s} \binom{s}{t} (-1)^t = (1 - 1)^s = 0 \text{ if } s > 0 \]

When \( s = 0 \) there is a contribution of 1, so we are done.

### 4.2 Wikipedia proof of Kirchoff’s theorem [1]

For a graph, \( G \) the incidence matrix \( E \) is a \(|V(G)| \times |E(G)|\) graph with 1 and -1 at end points of vertices incident to edges. \( EE^T = L \).

(example with square with one line through and incident matrix)

\[ F = E \text{ with first row deleted} \]

Then \( FF^T = L^{(1)} \). By the Cauchy-Binet theorem,

\[ \det L^{(1)} = \sum_S \det(F_S) \det(F_S^T) = \sum_S \det(F_S)^2 \]

Where \( S \) is ranging over all subsets of edges 1, 2, \ldots\( m \) of size \( n - 1 \). \( F_S \) is the submatrix of \( F \) with only the columns of \( S \).

Each \( S \) that corresponds to a tree contributes 1 to the sum, and 0 otherwise.
References
