Schnyder labeling is angle labeling (colouring) such that:

(A) for all vertices.

(B) for all faces.

(C) for outer face

Angles at half-edges are

from: Geometric Graphs and Arrangements by Stefan Felsner
Lemma. The four angles of each edge contain all three labels.

(i.e., \( \begin{array}{c}
\begin{array}{c}
\star \\
\star
\end{array}
\end{array}
\) or \( \begin{array}{c}
\begin{array}{c}
\star \\
\star
\end{array}
\end{array}\) \( i = 0, 1, 2 \))

Proof.

Let \( d(v) \) be number of edges incident to \( v \) whose angles at \( v \) have distinct labels. \( d(v) = 3 \).

Let \( d(F) \) be number of edges bounding face \( F \) whose angles in \( F \) have distinct labels. \( d(F) = 3 \).

\[
S \triangleq \sum_v d(v) + \sum_F d(F) = 3n + 3f = 3|E| + 6
\]

Let’s count \( S \) another way...
Number colors 0, 1, 2.

For $\epsilon_i \in \{0, 1\}$ by properties of labeling.

Since $\sum_i \epsilon_i = 0 \pmod{3}$, sum must be 0 or 3.

Since $S = \sum_{e \in E} \sum_i \epsilon_i = 3|E| + 6$, the sum for every edge must be 3, which implies 3 different labels.

(The extra 6 comes from the three half-edges.) □

**Lemma** and face rule imply interior angles are red at $a_0$, green at $a_1$, and blue at $a_2$. 
**Schnyder wood** is edge orientation and labeling such that:

1. Edges are uni- or bi-directed and (resp.) uni- or bi-labeled
2. No face cycle in one color
3. No face cycle in one color
4. Half edges are directed out; red at $a_0$, green at $a_1$, and blue at $a_2$. 
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Schnyder labeling $\leftrightarrow$ Schnyder wood

$i = 0, 1, 2$
Let $T_0$, $T_1$, $T_2$ be digraphs of (resp.) red, green, blue edges.
Lemma. The digraph $D_i = T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ is acyclic.

Proof. If cycle exists it bounds a face.

cycle in $D_0$ around fewest faces. (Edges shown with original direction)

If $x$ in face, follow red and blue from $x$. Creates smaller cycle $\Rightarrow \Leftarrow$. 
Lemma. The digraph $D_i = T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ is acyclic.

Proof. If cycle exists it bounds a face.

but then no face angle is green (if cw) or blue (if ccw).
Consecutive edges on clockwise face boundary:
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Corollary. $T_i$ is a directed tree with root $a_i$. 
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Corollary. \( T_i \) is a directed tree with root \( a_i \).
Let $P_i(v)$ be the directed path from $v$ to $a_i$ in $T_i$.
Let $R_i(v)$ be the region bounded by $P_{i-1}(v)$ and $P_{i+1}(v)$. 
Lemma. If $u \in R_i(v)$ then $R_i(u) \subseteq R_i(v)$. If $u$ strictly inside $R_i(v)$ then $R_i(u) \subset R_i(v)$.

Proof Let $x$ be first vertex on $P_{i+1}(u)$ on boundary of $R_i(v)$.

Suppose $x \in P_{i-1}(v)$ (even $x = v$). Then $x$’s $(i-1)$-edge is out of order.
Lemma. If $u \in R_i(v)$ then $R_i(u) \subseteq R_i(v)$. If $u$ strictly inside $R_i(v)$ then $R_i(u) \subset R_i(v)$.

Proof. Let $x$ be first vertex on $P_{i+1}(u)$ on boundary of $R_i(v)$. $x$ is on $v$'s $(i+1)$-path.

$y$ is on $v$'s $(i−1)$-path.
Corollary.

\[ R_i(u) \subset R_i(v) \]
\[ R_{i-1}(u) \supset R_{i-1}(v) \]
\[ R_{i+1}(u) \supset R_{i+1}(v) \]

\[ R_i(u) \subset R_i(v) \]
\[ R_{i-1}(u) \supset R_{i-1}(v) \]
\[ R_{i+1}(u) = R_{i+1}(v) \]
Let $v_i = \# \text{ faces in } R_i(v)$.

$$(v_0, v_1, v_2) = (2, 5, 6)$$
\[ f(v) = \frac{\alpha v_0 + \beta v_1 + \gamma v_2}{f-1} \]
If $x_0 > u_0, v_0$ and $y_0 > u_0, v_0$:

$$\alpha x_0 + \beta t + \gamma (1 - x_0 - t)$$
\[ R_i(u) \subset R_i(v) \Rightarrow u_i < v_i \]
\[ R_{i-1}(u) \supset R_{i-1}(v) \Rightarrow u_{i-1} > v_{i-1} \]
\[ R_{i+1}(u) \supset R_{i+1}(v) \Rightarrow u_{i+1} > v_{i+1} \]

face angles \( \leq 180^\circ \)  
convex faces