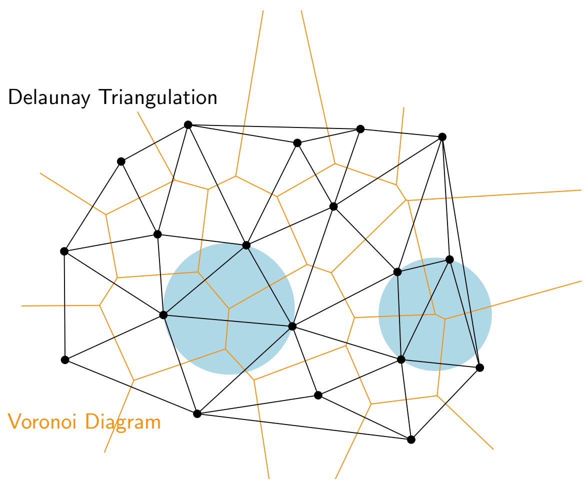
CPSC 516 - 20.10.22
Crust

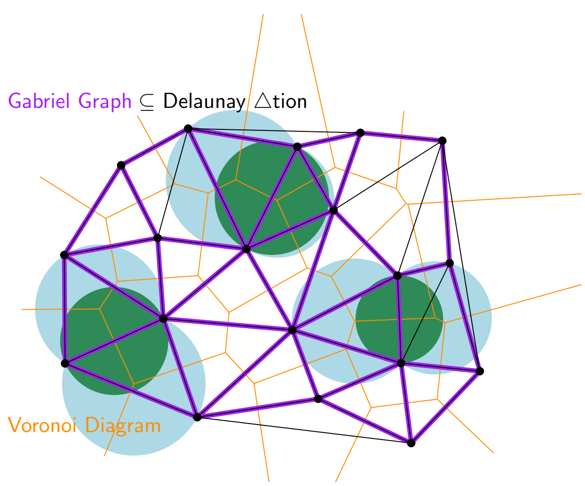
Vertex Connecting Hierarchy

We first define a bunch of different rules for connecting 2 vertices, and then show that each is a subset of the previous.

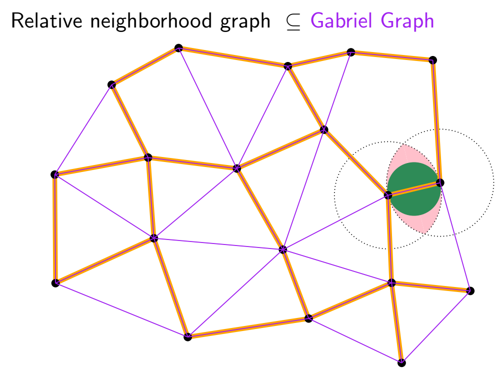
1. **Delaunay Triangulation**, where we connect three points pqr in a triangle if no other points lie in their circumcircle. Or equivalently for two points, if we can find any disk with pq on the perimeter and no other points inside.



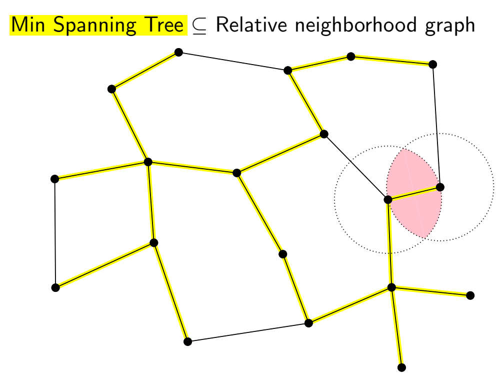
2. **Gabriel Graph**, where we connect two points pq if the circle with diameter $\overline{pq}$ contains no other points. This is a subset of Delaunay Triangulation by the second Delaunay definition we gave, because the Gabriel circle satisfies it.



3. **Relative Neighbourhood Graph**, where we connect two points pq if no other point is closer to both of them. Or equivalently, if there are no points in the intersection of the p-centered disk of radius $\overline{pq}$ and the q-centered disk of radius $\overline{pq}$. This is a subset of Gabriel Graph because the Gabriel circle is clearly contained the intersection of these two disks.



4. **(Euclidean) Minimum Spanning Tree**, where we aim to connect all vertices such that the total edge length is minimized. The subset is a bit more subtle here, but essentially we can take it as a contrapositive. If there is a point r which is closer to both p and q (a.k.a. no edge $\overline{pq}$ in the RNG), then it is cheaper in the MST to connect them both to r, and so the edge $\overline{pq}$ is not in the MST.



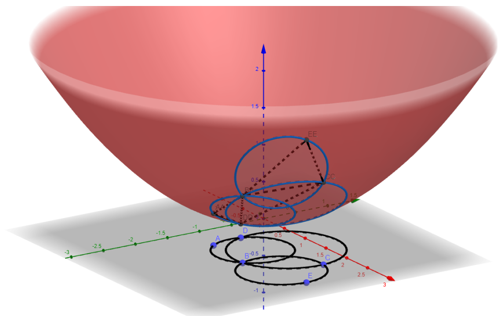
This gives us a final hierarchy of

$$MST(P) \subseteq RNG(P) \subseteq GG(P) \subseteq DT(P)$$

1 Delaunay Triangulation via 3D Convex Hull

An alternative method for computing the Delaunay Triangulation of a set of points is to lift them into 3D. For a given point (x, y), place it at $(x, y, x^2 + y^2)$. This essentially puts all the points onto a hyperparaboloid centered at zero and pointing up the z axis. Then take the bottom convex hull and delete the z coordinate.

Delaunay Triangulation from 3D Convex hull

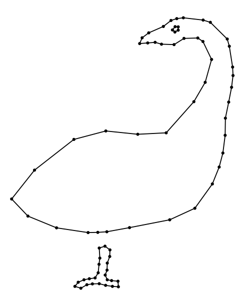


2 The Crust and the β-skeleton

The crust is defined as follows:

Definition 1. Connect $p, q \in S$ if $\overline{pq} \in DT(S \cup V_{VD}(S))$, that is to say if the edge is in the Delaunay triangulation of the vertices combined with the Voronoi vertices.

The Crust and the β-skeleton [Amenta, Bern & Eppstein 98]



What follows is essentially a walk through the paper *The Crust and the β-Skeleton: Combinatorial Curve Reconstruction* by Amenta, Bern, and Eppstein.

Definition 2. We then define the *medial axis* of F to be the set of points with ≥ 2 closest points on the curve, and the *feature distance* for $p \in F$ is the distance from p to F's medial axis.

Definition 3. We can then define the *r-sampling condition* to be where the distance from an arbitrary $p \in F$ to the nearest sample $s \in S$ is bounded above by r times its feature distance.

Theorem 1. *r-sampling with $r < .4$ implies that the crust of S is connected.*

Lemma 1. *Any Euclidean disk B containing ≥ 2 points of a smooth curve F either intersects the curve in a single segment or contains a medial axis point.*

Lemma 2. *Any Voronoi disk B of finite $S \subseteq F$ a smooth curve contains a point of the medial axis.*

Definition 4. The *local feature size* at a point $p \in F$ is the distance from p to the nearest point on the medial axis of F.

Corollary. *A disk at $p \in F$ of radius less than the local feature size of p contains one segment.*

Lemma 3. *Let F be an r-sampled smooth curve in the plane with $r \leq 1$. There is a disk touching each adjacent pair from S which is empty of points.*

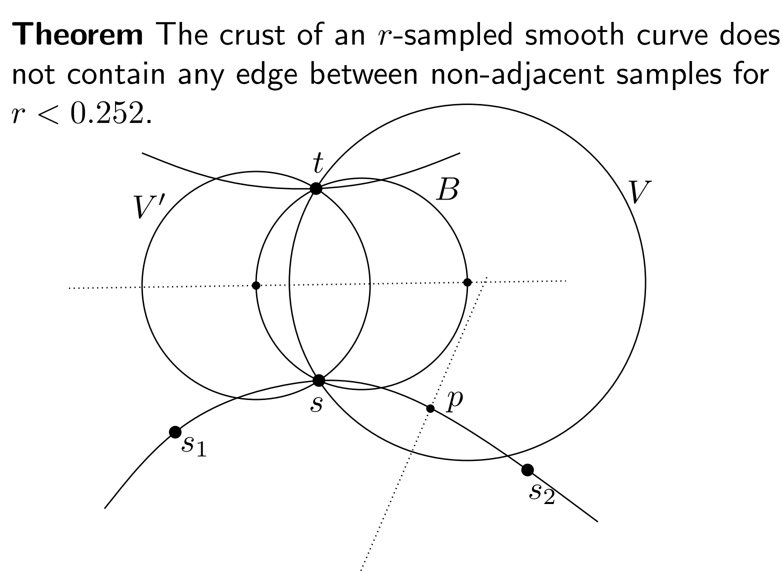
Corollary. *Adjacent edges have a Delaunay edge in $DT(S)$.*

Theorem 2. *When $r < .4$, the crust contains an edge between every pair of adjacent samples.*

Proof. Take a point $p \in F$, assume WLOG that $LFS(p) = 1$, and let B be a Voronoi disk centered at p. If there is a Voronoi vertex $v \in B$, then the radius R of its Voronoi disk V is at most the distance to he nearer of p's two samples. By 2 C contains a point of the medial axis, but the disk B' around p with radius 1 cannot contain the medial axis by the definition of local feature size. *Roughly* we then proceed to force $r + R \leq 1$, so B' contains C which gives us a contradiction with the whole "contains the medial axis" thing. □

Theorem 3. *When $r < .252$, the crust does not contain any other edges.*

Proof. Proof by picture:



$\square$

3 Briefly, Applications

From 1977-2008 there was a search for a solution to minimum triangulation (in 2008, Mulzer & Rote proved it to be NP-hard), and this was one considered option. They first tried Delaunay, but it is not always minimal, it simply maximizes the minimum angle. They also tried the β-skeleton, but it need not be connected, so that didn't do it either.