Vertex Connecting Hierarchy

We first define a bunch of different rules for connecting 2 vertices, and then show that each is a subset of the previous.

1. **Delaunay Triangulation**, where we connect three points $pqr$ in a triangle if no other points lie in their circumcircle. Or equivalently for two points, if we can find any disk with $pq$ on the perimeter and no other points inside.

2. **Gabriel Graph**, where we connect two points $pq$ if the circle with diameter $pq$ contains no other points. This is a subset of Delaunay Triangulation by the second Delaunay definition we gave, because the Gabriel circle satisfies it.
3. **Relative Neighbourhood Graph**, where we connect two points $pq$ if no other point is closer to both of them. Or equivalently, if there are no points in the intersection of the $p$-centered disk of radius $pq$ and the $q$-centered disk of radius $pq$. This is a subset of Gabriel Graph because the Gabriel circle is clearly contained the intersection of these two disks.

![Relative neighborhood graph ⊆ Gabriel Graph](image)

4. **(Euclidean) Minimum Spanning Tree**, where we aim to connect all vertices such that the total edge length is minimized. The subset is a bit more subtle here, but essentially we can take it as a contrapositive. If there is a point $r$ which is closer to both $p$ and $q$ (a.k.a. no edge $pq$ in the RNG), then it is cheaper in the MST to connect them both to $r$, and so the edge $pq$ is not in the MST.

![Min Spanning Tree ⊆ Relative neighborhood graph](image)

This gives us a final hierarchy of

$$MST(P) \subseteq RNG(P) \subseteq GG(P) \subseteq DT(P)$$

### 1 Delaunay Triangulation via 3D Convex Hull

An alternative method for computing the Delaunay Triangulation of a set of points is to lift them into 3D. For a given point $(x, y)$, place it at $(x, y, x^2 + y^2)$. This essentially puts all the points onto
a hyperparaboloid centered at zero and pointing up the $z$ axis. Then take the bottom convex hull and delete the $z$ coordinate.

\[ \text{Delaunay Triangulation from 3D Convex hull} \]

2 The Crust and the $\beta$-skeleton

The crust is defined as follows:

**Definition 1.** Connect $p, q \in S$ if $pq \in DT(S \cup V_D(S))$, that is to say if the edge is in the Delaunay triangulation of the vertices combined with the Voronoi vertices.

What follows is essentially a walk through the paper *The Crust and the $\beta$-Skeleton: Combinatorial Curve Reconstruction* by Amenta, Bern, and Eppstein.
Definition 2. We then define the medial axis of $F$ to be the set of points with $\geq 2$ closest points on the curve, and the feature distance for $p \in F$ is the distance from $p$ to $F$’s medial axis.

Definition 3. We can then define the $r$-sampling condition to be where the distance from an arbitrary $p \in F$ to the nearest sample $s \in S$ is bounded above by $r$ times its feature distance.

Theorem 1. $r$-sampling with $r < 0.4$ implies that the crust of $S$ is connected.

Lemma 1. Any Euclidean disk $B$ containing $\geq 2$ points of a smooth curve $F$ either intersects the curve in a single segment or contains a medial axis point.

Lemma 2. Any Voronoi disk $B$ of finite $S \subseteq F$ a smooth curve contains a point of the medial axis.

Definition 4. The local feature size at a point $p \in F$ is the distance from $p$ to the nearest point on the medial axis of $F$.

Corollary. A disk at $p \in F$ of radius less than the local feature size of $p$ contains one segment.

Lemma 3. Let $F$ be an $r$-sampled smooth curve in the plane with $r \leq 1$. There is a disk touching each adjacent pair from $S$ which is empty of points.

Corollary. Adjacent edges have a Delaunay edge in $DT(S)$.

Theorem 2. When $r < 0.4$, the crust contains an edge between every pair of adjacent samples.

Proof. Take a point $p \in F$, assume WLOG that $LFS(p) = 1$, and let $B$ be a Voronoi disk centered at $p$. If there is a Voronoi vertex $v \in B$, then the radius $R$ of its Voronoi disk $V$ is at most the distance to he nearer of $p$’s two samples. By 2 $C$ contains a point of the medial axis, but the disk $B'$ around $p$ with radius 1 cannot contain the medial axis by the definition of local feature size. Roughly we then proceed to force $r + R \leq 1$, so $B'$ contains $C$ which gives us a contradiction with the whole ”contains the medial axis” thing.  

Theorem 3. When $r < 0.252$, the crust does not contain any other edges.

Proof. Proof by picture:
3 Briefly, Applications

From 1977-2008 there was a search for a solution to minimum triangulation (in 2008, Mulzer & Rote proved it to be NP-hard), and this was one considered option. They first tried Delaunay, but it is not always minimal, it simply maximizes the minimum angle. They also tried the $\beta$-skeleton, but it need not be connected, so that didn’t do it either.