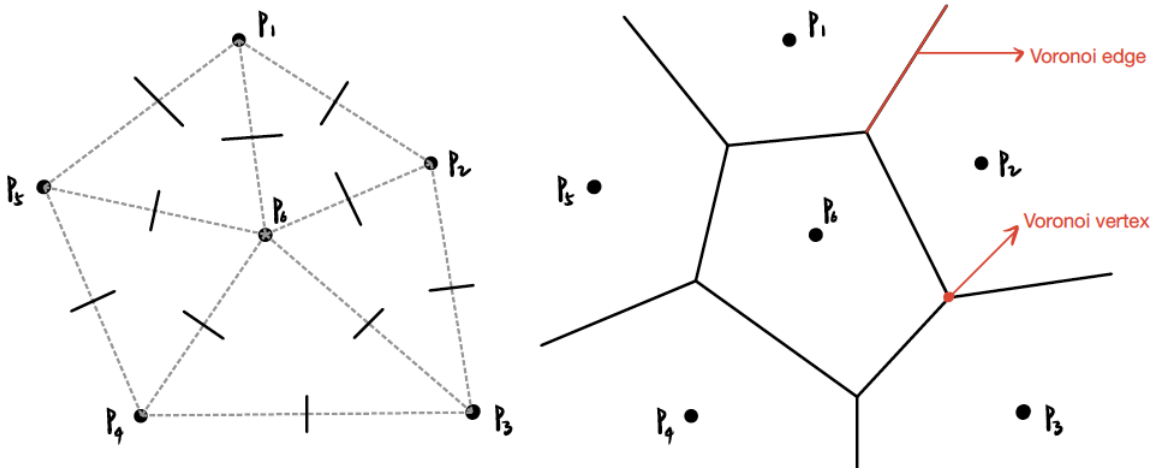
1 Voronoi Diagram

- Post office problem: given a point x, find the closest post offices.
- Voronoi diagram: post offices as "sites" $P = \{p_1, p_2, ..., p_n\}$ in $\mathbb{R}^2$.

Suppose in 1D, three sites of color blue, red and green are positioned as above, the Voronoi diagram is constructed by adding midpoints between two adjacent sites.

1.1 Construct 2D Voronoi Diagrams

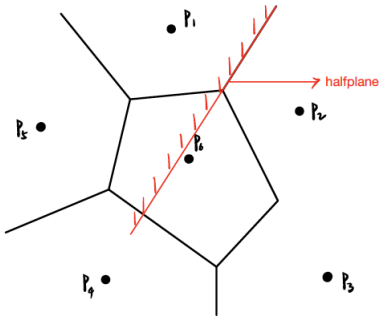
- For n sites, firstly find bisectors between each pair of two adjacent sites, then connect to get boundaries.



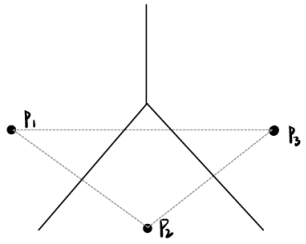
- Region for site p_i: $V(p_i) = \{x \in \mathbb{R}^2 | d(x, p_i) \leq d(x, p_j), \forall j\}$
- Voronoi diagram of all sites: $V(P) = V(p_1) \cup V(p_2) \cup ... \cup V(p_n)$

1.2 Facts of Voronoi Diagram

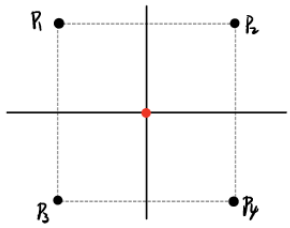
1. Every Voronoi region is convex.
   proof: The intersections of half planes that defines the points closer to p_i than to p_j is convex.



question: Does connecting two sites forms a line that must cross the bisector of the two sites? No.

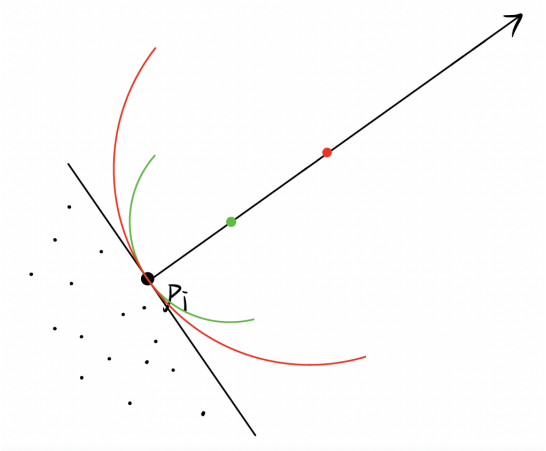


question: Do Voronoi vertices must have degree of 3? No.



2. Voronoi vertices have degree of 3, if non-degeneracy point configuration(no 4 circular points).

3. A Voronoi regions is unbounded iff the site is on the convex hull of all sites.
proof: let site p_i be such a point on the convex hull, all points on the below ray are closer to p_i than any other site.

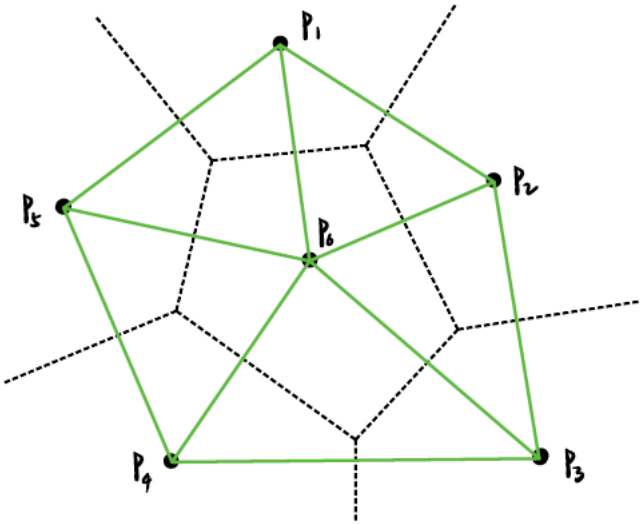


4. A Voronoi vertex is the center of an empty circle that go through 3(or more) sites, if $n \geq 3$.

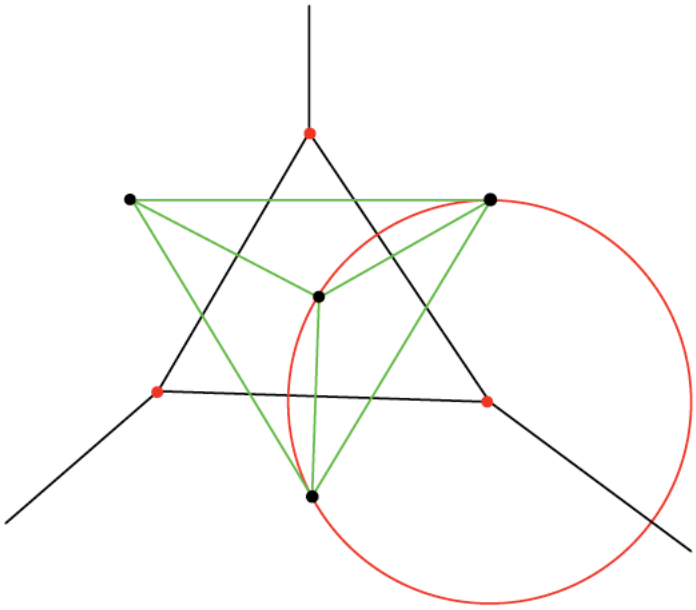
5. # of Voronoi vertices $< 2(n-2)$, # of Voronoi edges $< 3(n-2)$
proof: Given Euler's formula: $|V|-|E|+|F| = 2$, we also added a infinite vertex which is connected to all unbounded edges. $|F|$ =# of sites= n.
By fact 4., $2|E| = \sum_{v \in V} deg(v) \geq 3|V|$
$|V| - |E| + |F| \leq |V| - \frac{3}{2}|V| + |F|$, $|V| < 2(n-2), |E| < 3(n-2)$

6. All Voronoi vertices are of degree ≥ 3

2 Dual of Voronoi Diagram - Delaunay Triangulation



- Delaunay triangulation $D(P)$ is constructed by connecting two sites iff they share a Voronoi edge. Delaunay is a triangulation because we assume no 4 circular sites.



- Delaunay edge definition: p_ip_j is a Delaunay edge shared by $V(p_i), V(p_j)$ iff $\exists$ point x that is not a site that is closest to/ and equal distance from p_i and p_j than any other sites iff circle C centered at x goes through p_i and p_j is empty of sites.