L24

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Tutte's *How to draw a graph* 1963

sources: Spielman's notes Spectral Graph Theory Lec 9
✱ Lovasz Geometric Representation of Graphs Chap 4

First a characterization of 3-connected planar graphs

why 3-connected is important

Def A cycle C in a graph G is separating if $G \setminus V(C)$ has ≥ 2 connected components (where a chord of C is counted as a component)

Prop In a 3-connected planar graph, a cycle bounds a face iff it is non-separating. <no proof>

so faces are not defined by an embedding in a 3-connected graph.

they are defined by being non-separating.

Cor The set of cycles that bound faces is determined in a 3-connected planar graph. [Will: To specify planar embedding only the outerface needs to be identified in 3-connected planar graphs]

Thm (Tutte) Let G be a 3-connected graph. Then every edge of G is contained in at least two non-separating cycles.
G is planar iff every edge is contained in exactly two non-separating cycles.

Thm (Steinitz's Theorem)
A simple graph is isomorphic with the skeleton of a 3 polytope if and only if it is 3-connected and planar.
<no proof>

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Now the drawing theorem.

Def. Rubber band representation of G in $\mathbb{R}^d$ given a nailed set $S \subseteq V$

[idea: replace edges with rubber bands, fix vertices in S at their assigned positions, let go]

vertex positions are defined by a function $u^0 : S \to \mathbb{R}^d$

use p rather than u for "position"

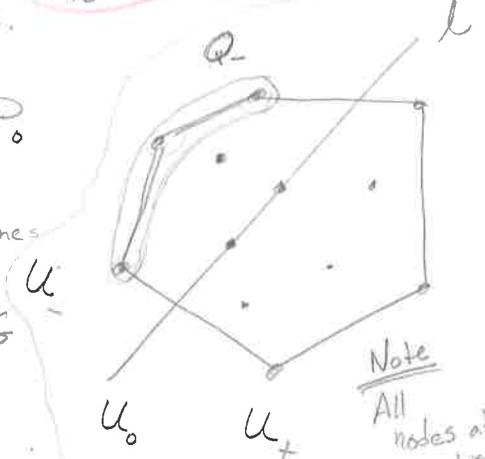
is a function $u : V \to \mathbb{R}^d$ that agrees with u^0 on S and minimizes $\mathcal{E}(u) = \sum_{ij \in E} |u_i - u_j|^2 = \sum_{ij} \sum_{k=1}^{d} (u_{ik} - u_{jk})^2$

Fact $\mathcal{E}(u)$ is convex
and minimizing u is unique. (provided $S \neq \emptyset$)
and for minimizing u for all $i \in V$ $\boxed{\sum_{j : ij \in E} (u_i - u_j) = 0} \equiv \boxed{u_i = \frac{1}{d(i)} \sum_{ij \in E} u_j}$

(the function u is harmonic at $i \notin S$)

Thm If G is a simple 3-connected planar graph then every rubber band rep of G in $\mathbb{R}^2$ w/ S = boundary of a face nailed to convex poly P_0 gives a planar drawing of G with each face convex. define

Proof Let u be such a rubber band rep.
Let ℓ be a line intersecting interior of P_0
Let U_0 be nodes on ℓ
U_- and U_+ be on two (open) half planes defined by ℓ.



Claim (1) U_- (and U_+) induce connected subgraphs of G

proof Since P_0 is convex, (the nodes on) its vertices in U_- form a (non empty) path Q_-

Note All nodes at max distance from ℓ in U_- are in Q_-

We show $a \in U_- \setminus Q_-$ has a path in U_- to Q_-
Since u_a is the average of u_b for $b \in N(a)$ there is a nbr b with u_b at least as far away as u_a from ℓ (so $b \in U_-$)
If $d(u_b, \ell) > d(u_a, \ell)$ either $b \in Q_-$ or repeat with $a = b$
If $d(u_b, \ell) \leq d(u_a, \ell)$ for all $b \in N(a)$ then $d(u_b, \ell) = d(u_a, \ell)$ $\forall b \in N(a)$

Let H be connected component of a with vertices $b : d(u_b, \ell) = d(u_a, \ell)$
If H contains a vertex in Q_- then done.
Otherwise, $\exists$ vertex $b \in H$ with edge to a node outside H. (G is connected)
Since u_b is average of its nbrs, there must be c s.t. $d(u_c, \ell) > d(u_b, \ell) = d(u_a, \ell)$ (so $c \in U_-$)

Either c is in Q_- or repeat with $a = c$,

<u>Claim 2</u> Every $a \in U_0$ has nbrs in both U_- and U_+

<u>proof</u> trivial if $a \in P_0$ so suppose a is a free (non-nailed) node

If a has nbr in U_- then (since u_a is average of its nbrs)

a has nbr in U_+.

So we only need to show that

T = the set of nodes a with $N(a) \subseteq U_0$

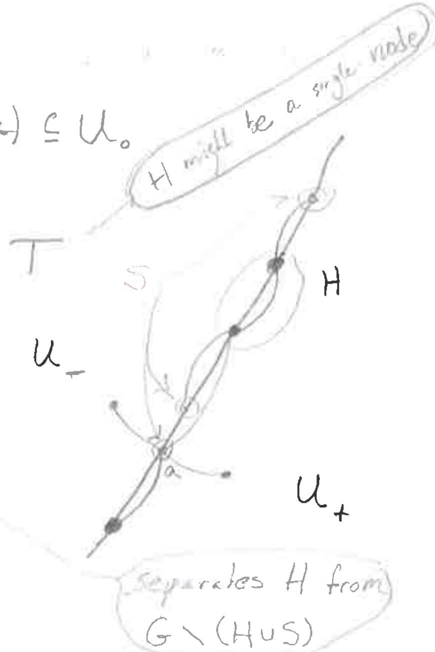
is empty.

Let H be a connected component of T

Let S be nbrs of H outside H.

Since $H \cup S \subseteq U_0$ and doesn't contain

all nodes in G, S is a cut set.



3-connected $\Rightarrow$ $|S| \geq 3$

$s_1\ s_2\ s_3 \in S$

contains $K_{3,3}$ as minor

$\Rightarrow\Leftarrow$

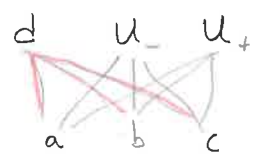
<u>Claim 3</u> Only two vertices on a face can be in U_0

<u>proof</u> Suppose $a, b, c \in U_0$ are on a common face f.

f is not P_0.

Create node d and connect it to a, b, c The resulting graph is still planar.

However



contains K_{33} as a minor. $\Rightarrow\Leftarrow$

<u>Claim 4</u> For any two faces p and q sharing an edge ab (where $a, b \in U_0$) $p \subseteq U_+$ and $q \subseteq U_-$ (or vice-versa)

<u>proof</u> suppose p has $c \neq a, b$ and q has $d \neq a, b$

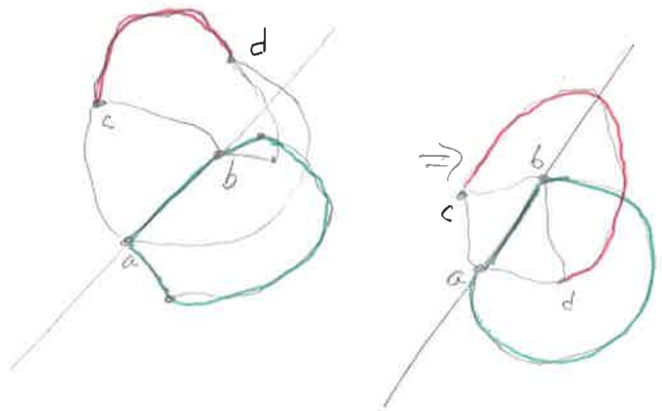
and both c and d are in U_- ($c, d \notin U_0$ by Claim 3)

Claim 1 $\Rightarrow$ path P_{cd} in U_- connecting c, d.

Claim 2 $\Rightarrow$ a and b have nbrs in U_+ that are connected in U_+ (by Claim 1) by path P_{ab}

P_{cd} and P'_{ab} are node-disjoint. $ab \cup P_{ab}$ is closed curve that separates c and d

Why ?



attempt at planar drawing

red path and green cycle share no vertex.

Claim 5 The boundary of every face is mapped to a convex polygon.

proof By Claim 4 no edge of a face extended to a line can intersect the interior of the face.

Claim 6 Interiors of face polygons (except P_0) are disjoint

proof Let x be a point in nailed face P_0 interior to a face polygon. We may assume x is not on an edge. — why?
Draw a line l through x that does not contain a node.
Walk on l from outside P_0 to x. When we enter P_0, coverage = 1.
Claim 4 implies when l crosses an edge, coverage remains 1.
(shared by two interior faces)

Finally: Suppose two edges cross. Then two of the faces incident to them share a common point contradicting Claim 6.