Recall

**VC-dimension** of a range space \((X, \mathcal{R})\) is the size of the largest subset \(A \subseteq X\) that can be shattered by \(\mathcal{R}\) (or \(\infty\) if no largest exists).

\[\{\forall r \in \mathcal{R}_A^2 \} = 2^A\]

Intuition

Range spaces with finite VC-dimension are "simpler" than ones with infinite VC-dim.

Let \(R|_A = \{r \cap A \mid r \in \mathcal{R}_A^2\}\)

**Sauer's Lemma** If \((X, \mathcal{R})\) is a range space with finite VC-dim at most \(d\) then

\[|R|_A| \leq \phi_d(n) = \sum_{i=0}^{d} \binom{n}{i}\]

for all \(A \subseteq X\) with \(|A| = n\)

**Old proof**

First observe \(\phi_d(n) = \begin{cases} 
0 & \text{if } d = -1 \\
1 & \text{if } n = 0 \\
\phi_{d-1}(n) + \phi_{d}(n-1) & \text{otherwise}
\end{cases}\)

Second

\[\text{VC}(A, R|_A) \leq \text{VC}(X, \mathcal{R}) \leq d\]

Consider \((A \setminus \{x\}, R-x)\)

\[R-x = \{r \setminus \{x\} \mid r \in \mathcal{R}_A^2\}\]

Note:

\[|R|_A| = |R-x| + |R^{(x)}|\]

By induction

\[|R-x| \leq \phi_{d-1}(n-1)\]

If \(S \subseteq A \setminus \{x\}\) is shattered by \(R^{(x)}\) then \(S \cup \{x\}\) is shattered by \(R|_A\)

\[\Rightarrow \text{VC}(A \setminus \{x\}, R^{(x)}) \leq d-1 \text{ and } |R^{(x)}| \leq \phi_{d-1}(n-1)\]
Pajor's Theorem

Every finite set family $F$ shatters at least $|F|$ sets.

Proof (by induction)

Base: Every $F$ with $|F|=1$ shatters the empty set.

Step: If $|F| > 1$, let $x$ be an element in some but not all sets in $F$.

Let $F_x = \{S \in F \mid x \in S\}$ and $F_{\bar{x}} = \{S \in F \mid x \not\in S\}$.

By induction $F_x$ shatters at least $|F_x|$ sets and $F_{\bar{x}}$ shatters at least $|F_{\bar{x}}|$ sets.

None of the sets shattered by $F_x$ or $F_{\bar{x}}$ contain $x$.

If $S$ is shattered by both $F_x$ and $F_{\bar{x}}$, then both $S$ and $S \cup \{x\}$ are shattered by $F$.

If $S$ is shattered by one of $F_x$ or $F_{\bar{x}}$ then $S$ is shattered by $F$.

$\Rightarrow |F| \geq |F_x| + |F_{\bar{x}}| = |F|$ sets.

Pajor $\Rightarrow$ Sauer-Shelah $F$

If $|R|_A > \Phi_d(n) = \sum_{i=0}^{d} \binom{n}{i}$ then, by Pajor, there are more than $\Phi_d(n)$ sets shattered by $R|_A$.

But there are only $\Phi_d(n)$ subsets of $A$ of size $\leq d$.

So $R|_A$ must shatter some subset of $A$ of size $> d$.

$\Rightarrow \text{VC-dim } (X, R) > d \Rightarrow \epsilon$
How to find an $\varepsilon$-net for a set $A \subseteq X$

w.r.t. range space $(X, R)$

**Theorem** If $\text{VC}(X, R) \leq d$ and $\varepsilon \leq \frac{1}{2}$ then there exists an $\varepsilon$-net for $(A, R|_A)$ w.r.t. $\mu$

of size at most $\frac{c d \ln \frac{1}{\varepsilon}}{\varepsilon}$ for some constant $c$.

**Proof** Let $S = \frac{c d \ln \frac{1}{\varepsilon}}{\varepsilon}$. Assume this is an integer.

Let $N$ be a random sample of size $S$ drawn indep. w.r.t. $\mu$ from $A$.

To show: $N$ is $\varepsilon$-net with at least constant probability.

Assume all $S \in F$ have $\mu(S) \geq \varepsilon$ (smaller sets don't matter).

For each $S \in F$,

$\Pr[N \cap S = \emptyset] \leq (1-\varepsilon)^S \leq e^{-\varepsilon S}$

If $S \geq \frac{1}{\varepsilon} \ln(1/F_1+1)$,

so we're done.

**Tool** Let $X = X_1 + X_2 + \ldots + X_n$ where $X_i$ are indep random variables.

Then $\Pr[X \geq \frac{1}{2} np] \geq \frac{1}{2}$ (when $np \geq 8$)

$\Pr[X_i = 1] = p$

$\Pr[X_i = 0] = 1-p$

**Proof** (Chebyshev's Ineq.)

$\Pr[|X - E[X]| < t] \leq \frac{\text{Var}[X]}{t^2}$

$E[X] = np$, $\text{Var}[X] = \sum \text{Var}[X_i] \leq np$

$\Pr[X < \frac{1}{2} np] \leq \Pr[|X - E[X]| \geq \frac{1}{2} np] \leq \frac{\text{Var}[X]}{np} \leq \frac{1}{2}$ for $np \geq 8$.