**Simplex range query**

Given a set $A$ of points in $\mathbb{R}^d$, build a query structure that, given a query simplex $S$, returns $S \cap A$.

For query half-spaces $A$, rather than simplices, one can build a

**Partition tree** recursively:

1. Create a root
2. Choose a random subset $N$ of $A$ of size $V$.
3. Form the line arrangement of lines through pairs of points in $N$.
4. For each cell of this arrangement with $> V$ points of $A$ (recursively)
   - Form the partition tree for points of $A$ in the cell.
   - Make the tree a child of the root.

To search: visit child cells intersected by $A$.

To perform efficiently:

1. Total number of points in cells intersected by any line is $< \varepsilon m$ where $m$ is the # points in the current cell.
2. Total number of cells intersected by any line is reasonably small (it's $O(V^2)$ in any case)

Choose $V = \lceil C (\log(1/\varepsilon) + \log(1/\delta)) \rceil$ for constant $C$ yields 1 with prob $> 1-\delta$.
A range space is a pair \((X, R)\) where \(X\) is a set (of points) and \(R\) is a set of ranges \(= \text{subsets of } X\).

For example, \(X = \mathbb{R}\) and \(R = H\) (half-lines), \(X = \mathbb{R}\) and \(R = I\) (intervals), \(X = \mathbb{R}^d\) and \(R = H_d\) (half-spaces in \(\mathbb{R}^d\) bounded by hyperplanes).

An \(\varepsilon\)-net with respect to range space \((X, R)\) for a finite point set \(A \subseteq X\) is a set of points \(N \subseteq A\) such that \(N\) contains a point in \(r\) for every \(r \in R\) with \(\left\lvert \text{An} \right\rvert > \varepsilon \cdot \left\lvert A \right\rvert\).

i.e., if a range contains a large \((> \varepsilon)\) fraction of the points in \(A\) then the range must contain a point in the \(\varepsilon\)-net \(N\).

Example: For half-spaces in \(\mathbb{R}^d\), the smallest \(0\)-net for \(A\) is the set of extreme points in \(A\).

But for \(\varepsilon > 0\) \(\exists\) \(\varepsilon\)-nets for \(A\) of size \(\left\lceil \frac{8(d+1)\log \frac{8(d+1)}{\varepsilon}}{\varepsilon} \right\rceil\).
\( \varepsilon \)-nets and Vapnik-Chervonenkis dimension

VC-dimension of a range space \((X, R)\) is the size of the largest subset \(A \subseteq X\) that can be shattered by \(R\) (or \(\infty\) if no largest exists)

\[ |\mathcal{A} \cap r : r \in R| = 2^{|A|} \]

Ex 1 \((R, H_1)\) has VC-dim = 2

since for any set \(A\) with 3 points in \(\mathbb{R}\), \(a < b < c\)
no \(r \in R\) can produce the subset \(\{b\}\) of \(A\)

Ex 2 \((\mathbb{R}^2, H_2)\) has VC-dim = 3

any set \(A\) with 4 points in \(\mathbb{R}^2\)
- either has one inside convex hull
  no \(r \in R\) can produce \(\{d\}\)
- or has 4 on convex hull
  no \(r \in R\) can produce \(\{ac, bd\}\)

Ex 3 \((\mathbb{R}^2, B_2)\) has VC-dim = 3

any set \(A\) with 4 points in \(\mathbb{R}^2\)
- either has one inside convex hull
  no \(r \in R\) can produce \(\{d\}\)
- or has 4 on convex hull
  at least one of \(\{ac\}\) and \(\{bd\}\)
  cannot be produced by \(r \in R\)

why?
two circles would have to intersect 4 times.

use lifting map to relate \(\text{VC}(\mathbb{R}^2, B_2)\) to \(\text{VC}(\mathbb{R}^2, H_2)\)
Ex 4 \((\mathbb{R}^2, C_2)\) has \(VC\text{-dim} = \infty\)

For any \(n\), a set \(A\) with \(n\) points in convex position can be shattered: For any subset \(B \subseteq A\)

1. \(\text{CH}(B) \subseteq C_2\)
2. \(\text{CH}(B) \cap A = B\)