Line Arrangements and Duality

Point-Line Duality

\[ P = (x, y) \quad \iff \quad P^* = y = m \cdot x - b \]
\[ l = (m, b) \quad \iff \quad l^* = y = m \cdot x + b \] (non-vertical)

For \( p \) in plane and nonvertical line \( l \) in plane, duality transform \( * \) is

1. incidence preserving: \( p \in l \iff l^* \in p^* \)
2. order preserving: \( p \) above \( l \iff l^* \) above \( p^* \)

For point \( p \) on parabola \( y = \frac{x^2}{2} \)
\( p^* \) is tangent at \( p \).

For point \( q \) not on parabola
- every point with \( x \)-coord = \( q \)
- has same slope as \( q \)

What is dual of a line segment?
A simple arrangement of $n$ lines has

- $(\binom{n}{2})$ vertices
- $(\binom{n}{2}) + n + 1$ faces
- $n^2$ edges

**Proof**

- Every pair of lines intersects in a distinct point $\Rightarrow (\binom{n}{2})$ vertices

- If we add a line $L$ to an arrangement of $n-1$ lines
  - we split $n-1$ edges into two $+ (n-1)$
  - we introduce $n$ new edges along $L + \binom{n}{2} + 2n-1$

\[
\frac{(n-1)^2 + 2n-1}{\text{Ind. hyp. for } n-1} = \frac{n^2}{n^2}
\]

- Use Euler's formula for faces
  \[
f = 2 + e - v = 2 + n^2 - \binom{n}{2} + 1
  = \binom{n}{2} + n + 1
\]
Construct a line arrangement in \( O(n^2) \) time.

What does this mean?

Incremental method

Given a \((i-1)\)-line arrangement, add a new line, \( l_i \)

1. Find leftmost face that contains \( l_i \) [takes \( O(i) \) time.]

2. Walk boundary of this face until reach \( l_i \)'s exit

3. Total length of all boundaries walked is certainly \( O(i^2) \)

Zone Theorem: The total number of edges in all faces whose closures intersect \( l \) in an arrangement of \( n \) lines is \( \ll 10n \)

Proof: Rotate the plane so that \( l \) is horizontal.

Each edge in \( A(l) \) bounds two faces

- one to its left and one to its right
- [for horizontal edges, the face above is to its left]

So each face \( F \) has left bounding edges (those with \( F \) to right) and right bounding edges.