DeLaunay Triangulation

- of convex polygon [Chew 1986]

Let $S = \text{ccw list of } n \text{ vertices of convex poly } P$

$DT(S)$

If $|S| = 3$ then return $\Delta$ with vertices $S$

Pick $q$ at random from $S$ (let $p$ and $r$ be its neighbors)

$T = DT(S \setminus \{q\}) + \Delta pqr$

return $\text{Flip}(T, q, rp)$

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if $\bar{rp}$ is bad

remove $\bar{rp}$ from $T$, add $\bar{q}r$ to $T$

$\text{Flip}(T, q, rx)$

$\text{Flip}(T, q, xp)$

What is number of bad $\Delta$'s in $T$?

- Proportional to the degree of $q$ in $DT(S)$.

So expected runtime for $\text{Flip}$ is $O(\text{average degree of vertex})$

$= \sum_{q \in DT(S)} \text{deg}(q) = 2 \left( \frac{\# \text{edges in } DT(S)}{n} \right)$

$= 2 \left( \frac{2n-3}{n} \right) = 4 - \frac{6}{n}$

Thus expected runtime is $O(n)$.
Incremental Delaunay Addition (of arbitrary 2D point set)

Same idea but...

Adding \( q \) to \( DT(S \setminus \{q\}) \) requires
Finding \( \Delta \) in \( DT(S \setminus \{q\}) \) that contains \( q \)

Option 1: Maintain a search structure for \( DT(S \setminus \{q\}) \)

Option 2: Rebucket remaining points to be added into
newly created \( \Delta \)s

Both add expected \( O(\log n) \) time to the cost to
add \( q \).

What is probability that a point \( \kappa \) is rebucketed
when \( |S| = i \)?

\[
\text{prob. } \Delta \text{ containing } \kappa \text{ in } DT(S) \text{ is created by adding } q = \frac{3}{n}
\]

\[
E[\# \text{ rebuckets for } \kappa] \leq \sum_{i=1}^{n} \frac{3}{i} = O(\log n)
\]
Relatives of Delaunay Triangulations.

1. Nearest Neighbor graph $NN(S)$ of $S$
   - Draw edge $x \rightarrow y$ if $y$ is closest to $x$, $x, y \in S$

   \[
   \text{Claim } NN(S) \subseteq DT(S) \\
   xy \in NN(S) \Rightarrow xy \in DT(S)
   \]

2. Minimum Spanning tree $MST(S)$ of $S$

   \[
   \text{Claim } MST(S) \subseteq DT(S)
   \]

3. Relative Neighborhood Graphs $RNG(S)$
   - Add $xy$ if lens is empty

4. Gabriel Graph $GG(S)$

NN(S) $\subseteq$ MST(S) $\subseteq$ RNG(S) $\subseteq$ GG(S) $\subseteq$ DT(S)