**Delaunay Triangulation**

Connect two sites iff they share a Voronoi Edge

i.e. \( D(P) \) is the planar dual of \( V(P) \)

**Properties**

1. \( p_i \) and \( p_j \) is Delaunay edge iff (by def)
   \[ V(p_i) \text{ and } V(p_j) \text{ share Vor. Edge iff} \]
   
   \exists \text{ point } x \text{ (not a site) that is closer, (and equal distance) to } p_i \text{ and } p_j \text{ than any other site} \text{ iff} \]

   \[ \exists \text{ Circle } C \text{ (with center } x \text{) through } p_i \text{ and } p_j \text{ that is empty of other sites} \]

   Note: \( \overline{x}p_i \) is in \( V(p_i) \), \( \overline{x}p_j \) is in \( V(p_j) \)

2. No Delaunay Edges cross.

Suppose \( p_i, p_j \) and \( p_k, p_l \) cross

\( p_k \) and \( p_l \) must be outside circle C

\[ \Rightarrow p_k \text{ and } p_l \text{ outside } \triangle p_i p_j \]

\[ \Rightarrow p_k p_l \text{ intersects } \overline{x}p_i \text{ or } \overline{x}p_j \]

Similarly \( \overline{p_i p_j} \) intersects \( \overline{y}p_k \) or \( \overline{y}p_l \)

\[ \Rightarrow \text{ one of } \overline{x}p_i \text{ or } \overline{x}p_j \text{ intersects } \overline{y}p_k \text{ or } \overline{y}p_l \] (for empty circle with center through \( p_i \) and \( p_j \))

\[ \Rightarrow \text{ say... } \overline{x}p_i \in V(p_i) \text{ and } \overline{y}p_k \in V(p_k) \]

\[ \Rightarrow \text{ Hard to draw b/c impossible.} \]
Sites \( P_i, P_j, P_k \) form a face in Del. Triangulation iff there is an empty circle through them.

\[ \iff \] empty circle implies \( \overline{P_iP_j}, \overline{P_jP_k}, \overline{P_kP_i} \) are Del. Edges and nothing can cross them so it's a face.

\[ \Rightarrow \] if \( P_iP_jP_k \) is a face then \( \bigcirc \) \( P_iP_jP_k \) is empty since no sites in \( \Delta P_iP_jP_k \) and no site in \( P_i \).

Otherwise \( P_iP_j \) would not be Del. Edge.

Alternative definition of Delaunay triangulation:
All triangles* whose bounding circle contains no other sites

\* assuming no 4 colinear points.

Fortune's Sweep  Parabolic Front

Incremental Delaunay