L9 Randomization + Backwards Analysis of small $d>3$ Conv. Hull

P_r is the conv hull of the first r points $p_1 p_2 \cdots p_r \equiv S_r$

(0.1) Randomly permute the points in S

(0.2) Form facet graph $G(P_{d+1})$

(complete graph on $d+1$ vertices which correspond to all d-tuples of S_{d+1})

(0.3) For $r = d+2$ to n
Insert Point into $G(P_{r-1})$ to get $G(P_r)$
using Increment Step.

Delaunay Δtion & Convex polygon [Chew 1986]

Expected value of $\deg(p_r, P_r)$??

Backwards analysis

With prob. $1/r$, point p_r was the last vertex added

$\Rightarrow$ Expected value of $\deg(p_r, P_r)$ over all permutations

is $\frac{1}{r} \sum_{p \in S_r} \deg(p, P_r)$

Since every facet contains d vertices this is $\frac{d\,F(P_r)}{r}$ ← # facets in P_r

Since P_r has at most r vertices $F(P_r) = O(r^{\lfloor d/2 \rfloor})$ Fact (A)

$\Rightarrow$ Expected value of $\deg(p_r, P_r) = O(r^{\lfloor d/2 \rfloor - 1})$

$\Rightarrow$ runtime for alg. (without step 1) is

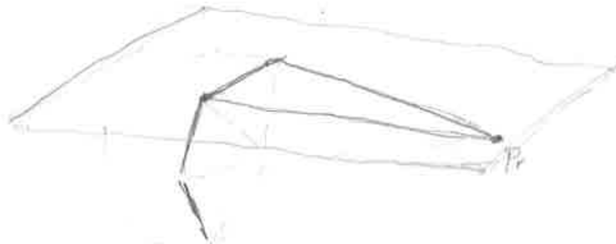
$$\sum_{d+1 < r \le n} O(r + r^{\lfloor d/2 \rfloor - 1}) = O(n^{\lfloor d/2 \rfloor}) \text{ for } d > 3$$

But we still need to solve step 1

→ Find the visible facets from p.

In fact, we only need to find one visible facet
then we can find the others by DFS in $G(P)$.

Finding one visible facet from p_r = find hyperplane through p_r that has all vertices of P_r below and is "lowest"



Linear programming with r constraints and d variables
That can be solved in $O(d!\, r)$ expected time
(similar analysis also by Seidel).

Summing over all n insertion steps yields
expected $O(d!\, n^2)$ time

For $d > 3$, this is $O(d!\, n^{\lfloor d/2 \rfloor})$

So the entire runtime is
$O(d!\, n^{\lfloor d/2 \rfloor})$.

Minimum enclosing ball for set T of n points in $\mathbb{R}^d$

<u>idea</u> Remove a point p chosen randomly from T

minball $(T \setminus \{p\})$ either has p inside (done)
or not (p must be <u>on</u> the enclosing ball)
so
minball$(T) \equiv$ minball $(T \setminus \{p\}$ with ball thru p)

known to be on boundary of minball $(T \cup C)$

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minball(T, C)
    if T = ∅ return ball(C)          ← O(d³) time
    Let T' = T \ {p}    p chosen randomly from T

    B' = minball(T', C)              ← O(d) exp time
    if p ∈ B' return B'
        else return minball(T', C ∪ {p})
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ball(C) returns smallest ball with all of C on boundary in $O(d^3)$ time.

Backwards Analysis

Probability that p in T is not in B' is $\frac{d+1-|C|}{|T|} = \frac{\delta}{n}$.

For a call to minball (T, C) where $|T| = n$ and $\delta = d+1-|C|$

Let $f(n,\delta) =$ exp # calls to minball (X, Y) with $X \neq \emptyset$ (each takes $O(d)$)

$g(n,\delta) =$ exp # calls to minball (X, Y) with $X = \emptyset$ (each takes $O(d^3)$)

ball(C)

p is added to C

$$f(n,\delta) = \begin{cases} 0 & \text{if } n = 0 \\ 1 + f(n-1,\delta) + \delta/n\, f(n-1,\delta-1) & \text{o.w.} \end{cases}$$

$$g(n,\delta) = \begin{cases} 1 & \text{if } n = 0 \\ g(n-1,\delta) + \delta/n\, g(n-1,\delta-1) & \text{o.w.} \end{cases}$$

$$f(n,\delta) \le \sum_{i=1}^{\delta} \frac{1}{i!}\,\delta!\, n = O(\delta!\, n)$$

$$g(n,\delta) \le d!\,(1 + H_n)^{\delta} = O(\delta! \log^{\delta} n)$$

Expected runtime $O(d\,(d+1)!\, n)$