L7

Euler's Formula

V = # vertices — 0-faces
E = # edges — 1-faces
F = # facets — 2-faces

in a polyhedron of genus 0

No "holes" unlike a donut

$V - E + F = 2$

proof (Eppstein lists 21 proofs – this is his favorite)

1. first flatten the polyhedron (by stretching one of its facets to be really flat) to get a planar graph G

2. Draw the dual graph G^* by drawing a vertex in each face and connecting the vertices from two adjacent faces by a curve e^* going through their shared edge e

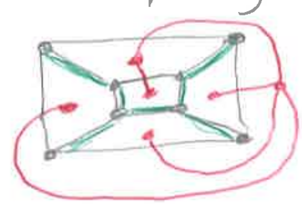
connectedness is dual to acyclicity

3. A cycle in G disconnects G^* (Jordan curve Thm) $\boxed{G^{**} = G}$

4. Any acyclic subgraph F of G is a forest and doesn't disconnect G^*

Choose any spanning tree T in G
The dual of it's compliment $(G \setminus T)^*$ is acyclic, connected subgraph of G^* (4) / is a spanning tree

The two trees together have $(V-1) + (F-1)$ edges



and they "count" all edges in G.

Giftwrapping

1970 Chand + Kapur did giftwrapping in d-dim.
[but try all vtes. to find next face]

Convex Hulls in 3D

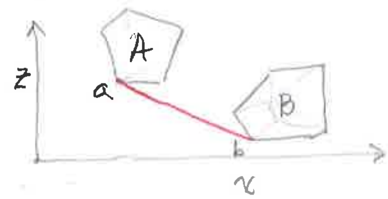
Divide and Conquer (Preparata + Hong 1977, Preparata + Shamos 1985)

(0) sort points P by x-coord (do this once)

CH(P) =
(1) P_1 = first half of points P_2 = second half by x-coord
(2) Find $A = CH(P_1)$ and $B = CH(P_2)$ recursively
(3) $\boxed{\text{Combine A and B into CH(P)}}$

Find lowest bridge in the xz-projection of A and B

Fold a sheet of metal along line $\overline{ab}$ until it bumps into another vertex



that vertex will be nbr. of a in A or nbr of b in B

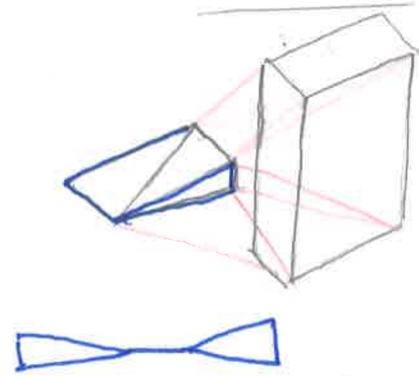
For each of A and B
keep track of best nbr of a
and best nbr of b (next to be touched)

Remove interior parts of A and B.

Combine time = sum of vertex degrees = O(n)

Total runtime: O(n log n)

O'Rourke's Example



not a simple cycle
[Edelsbrunner]

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(3)

Chan's $\boxed{\text{Kinetic}}$ Divide and Conquer CH algorithm

Idea: Represent a 3D (lower) Hull as a 2D movie

- an intial 2D hull
- a sequence of points added to / removed from the initial 2D hull over time.

Time determines a projection of the 3D points to 2D so that extreme points in 2D are extreme in 3D.

$P = p_1 p_2 \dots p_n$

for example $p_i(t) = (x_i, z_i - t y_i)$ (points move vertically in 2D at different speeds determined by y-coords)

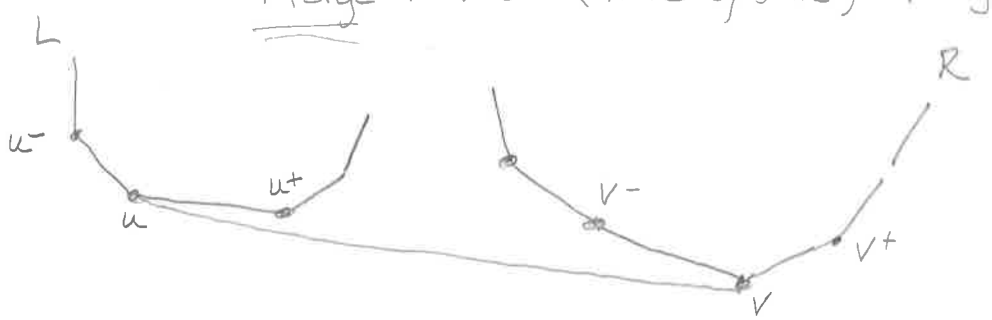
p_i is on 3D lower hull iff (def)

p_i lies on some plane $z = sx + ty + b$ and all other points lie above this plane iff

$p_i(t)$ lies on some line $y' = sx + b$ and all other points $p_j(t)$ lie above this line for some t iff

$p_i(t)$ is on 2D lower hull for some t

- Sort points by x-coord
- To calc. movie:
  - Split points into L (left) and R (right) halves
  - Recursively find movie(L) and movie(R)
  - Merge movies (frame by frame) to get hull H



What event causes next change in set of points on hull?

1. L undergoes an insertion or deletion event
   $\Rightarrow$ H under goes the same event if the point is left of u
2. R undergoes an insertion or deletion event
   $\Rightarrow$ H undergoes the same event if point is right of v
3. $u^- u v$ turns clockwise ($u^- v$ becomes bridge)
   $\Rightarrow$ H undergoes delete u event
4. $u u^+ v$ turns ccw ($u^+ v$ becomes bridge)
   $\Rightarrow$ H undergoes insert u^+ (between u and v)
5. $u v v^+$ turns cw ($u v^+$ becomes bridge)
   $\Rightarrow$ H undergoes delete v
6. $u v^- v$ turns ccw ($u v^-$ becomes bridge)
   $\Rightarrow$ H undergoes insert v^- (between u and v)

Calc. time of each of these events and pick earliest

Repeat

Running time is $O(n \log n)$

Chan's 3D CH

Same strategy as Chan's 2D CH

- Partition points into $\lceil \frac{n}{m} \rceil$ subsets of $\leq m$ points
- Use Div & Conquer (The Movie) to find CH of each subset | runtime $= {}^{n}/_{m}\, m \log m = O(n \log m)$
- Use Gift-Wrapping to find CH of CH's of subsets
  Use hierarchical representation of Dobkin and Kirkpatrick for these CHs to find faces adjacent to known CH face f.

$O(\log m)$ time per subset. $= O({}^{m}/_{n} \log m)$

Stop if # discovered faces $> 2m - 4$

How many faces in 3D (convex) poly with m vertices? $3F = 2E$ + Euler