Euler's Formula

\[ V - E + F = 2 \]

Proof (Eppstein lists 21 proofs - this is his favorite)

1. First flatten the polyhedron to get a planar graph \( G \).

2. Draw the dual graph \( G^* \) by drawing a vertex in each face and connecting the vertices from two adjacent faces by a curve \( e^* \) going through their shared edge \( e \).

3. A cycle in \( G \) disconnects \( G^* \) (Jordan curve).

4. Any acyclic subgraph \( F \) of \( G \) is a forest and doesn't disconnect \( G^* \).

Choose any spanning tree \( T \) in \( G \). The dual of its compliment \( (G \setminus T)^* \) is an acyclic, connected subgraph of \( G^* \). The two trees together have \((V-1) + (F-1)\) edges and they "count" all edges in \( G \).
Convex Hulls in 3D

Divide and Conquer (Preparata & Hong 1977, Preparata & Shamos 1985)

1. Sort points $P$ by $x$-coord (do this once)

$\text{CH}(P) = \begin{cases} 
1 & P_1 = \text{first half of points}, \; P_2 = \text{second half by } x\text{-coord} \\
2 & \text{Find } A = \text{CH}(P_1) \text{ and } B = \text{CH}(P_2) \text{ recursively} \\
3 & \text{Combine } A \text{ and } B \text{ into } \text{CH}(P) 
\end{cases}$

Find lowest bridge in the $xz$-projection of $A$ and $B$

Fold a sheet of metal along line $\overline{AB}$ until it bumps into another vertex

For each of $A$ and $B$

- keep track of best nbr of $a$ in $A$
- keep track of best nbr of $b$ in $B$

that vertex will be nbr of $a$ in $A$ or nbr of $b$ in $B$

Remove interior parts of $A$ and $B$

Combine time $= \text{sum of vertex degrees} = O(n)$

Total runtime $= O(n \log n)$

O'Rourke's Example

Not a simple cycle [Edelsbrunner]
Idea: Represent a 3D (lower) hull as a 2D movie.

- an initial 2D hull
- a sequence of points added to/removed from the initial 2D hull over time

Time determines a projection of the 3D points to 2D, so that extreme points in 2D are extreme in 3D.

For example: \( p_i(t) = (x_i, z_i - ty_i) \) (points move vertically in 2D at different speeds, determined by y-coords).

\( p_i \) is on 3D lower hull iff (def)

\( p_i \) lies on some plane \( z = sx + ty + b \) and all other points lie above this plane iff

\( p_i(t) \) lies on some line \( y' = sx + b \) and all other points \( p_j(t) \) lie above this line for some \( t \)

\( p_i(t) \) is on 2D lower hull for some \( t \)

Sort points by \( x \)-coord.

To calc. movie:
- Split points into \( L \) (left) and \( R \) (right) halves.
- Recursively find movie(\( L \)) and movie(\( R \)).
- Merge movies (frame by frame) to get hull \( H \)

What event causes next change in set of points on hull?
1. L undergoes an insertion or deletion event
   \[ \Rightarrow H \text{ undergoes the same event if} \]
   the point is left of u

2. R undergoes an insertion or deletion event
   \[ \Rightarrow H \text{ undergoes the same event if} \]
   point is right of v

3. \( u^- u v \) turns clockwise \( (u^- v \text{ becomes bridge}) \)
   \[ \Rightarrow H \text{ undergoes delete } u \text{ event} \]

4. \( u u^+ v \) turns ccw \( (u^+ v \text{ becomes bridge}) \)
   \[ \Rightarrow H \text{ undergoes insert } u^+ \text{ (between } u \text{ and } v) \]

5. \( u v v^+ \) turns cw \( (u v^+ \text{ becomes bridge}) \)
   \[ \Rightarrow H \text{ undergoes delete } v \]

6. \( u v^- v \) turns ccw \( (uv^- \text{ becomes bridge}) \)
   \[ \Rightarrow H \text{ undergoes insert } v^- \text{ (between } u \text{ and } v) \]

Calc. time of each of these events and pick earliest

Repeat

Running time is \( O(n \log n) \)

Chan's 3D CH

Same strategy as Chan's 2D CH
- Partition points into \( \left\lceil \frac{n}{m} \right\rceil \) subsets of \( \leq m \) points
- Use Divide and Conquer to find CH of each subset
  \( \text{Runtime} = \frac{n}{m} m \log m = O(n \log m) \)
- Use Gift-Wrapping to find CH of CH's of subsets
  Use hierarchical representation of Dobkin and Kirkpatrick for these CH's to find
  \[ O(\log m) \text{ time per subset. } = O(\frac{n}{m} \log m) \]

Stop if \# discovered faces > \( 2m - 4 \)