CPSC 516   22 Sept 2022   1

P = ~~set of~~ points
n = # points in P
h = # vertices of $CH(P)$

Chan's Algorithm 1996

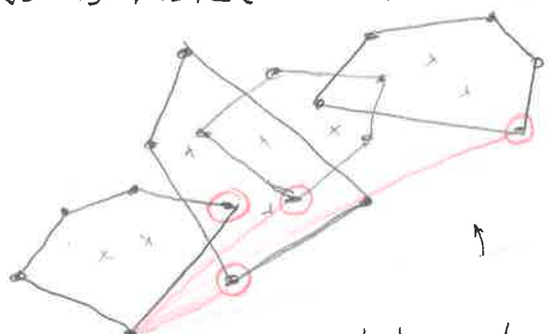
Suppose we know h <What can we do?>

$\boxed{\text{Let } m = h}$

1. Partition points into $\lceil \frac{n}{m} \rceil$ subsets of $\leq m$ points
2. Use Graham Scan to find CH of each subset

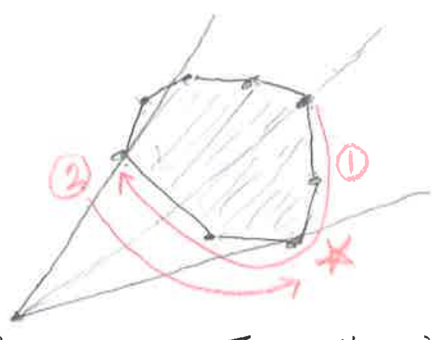
   runtime $= n/m \cdot m \log m = O(n \log m)$

3. Use Jarvis March to find CH of CH's of subsets.



p_i = last point known to be on $CH(P)$

3.1 Use binary search on each subset CH to find "minimum" (i.e. clockwise-most w.r.t. p_i) in each subset in time $O(\log m)$



p_i

Find all minima, [and take the smallest] in $O(n/m \log m)$ time

<How many times do we need to perform such a wrap step?>

After h wrap steps we get the convex hull

Jarvis March time $= O(h \, n/m \log m)$

Total time $\underbrace{O(n \log m)}_{\text{Graham}} + \underbrace{O(h \, \tfrac{n}{m} \log m)}_{\text{Jarvis}} = O(n \log h)$ if $m = h$

Yay!!

But we don't know h so we'll guess different values for m.

Try small values of m first.

If after $\boxed{m}$ wrap steps we do not find CH (come back to first point)

then STOP [which takes $O(n \log m) + O(m \, \tfrac{n}{m} \log m) = O(n \log m)$ time]

and increase m.

Repeat

$\boxed{\text{Double } m}$

$m = 1\ 2\ 3\ 4\ 5\ 6\ 7 \quad 151 \quad \overset{h}{152} \quad \ldots \quad n$

N N N N ... N Y Y Y ... Y

GS too expensive

$m = 2, 4, 8, 16, \ldots$

$$\text{total time} = \sum_{i=1}^{\log h} n \log 2^i = n \sum_{i=1}^{\lg h} i = n (\lg h)^2 \quad \text{Too BIG}$$

$2^i \geq h \Rightarrow i \geq \lg h$

$\boxed{\text{Square } m}$

$m = 2, 4, 16, 256,$

$$\text{total time} = \sum_{i=0}^{\log\log h} n \log\left(2^{2^i}\right) = n \sum_{i=0}^{\log\log h} 2^i = n\, 2^{\log\log h} = n \log h$$

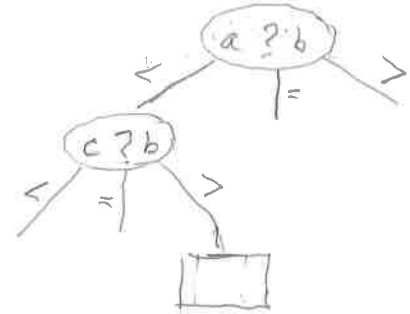
Just Right

$2^{2^i} \geq h \Rightarrow i \geq \lg\lg h$

Algebraic decision tree Lower Bound for CH

Decision Tree is a model of computation that focuses on decision points in an algorithm.

Decision tree for all inputs of size n (fixed)



Every path is a possible execution trace of an algorithm on some input of size n

Leaves are outputs

Algebraic decision trees permit a, b, c, to be algebraic expressions of degree $\boxed{d}$ (for fixed d)

- The variables in the expressions are real numbers (like the coordinates of input points for CH)
- The set of inputs that lead to a leaf is a set of vectors (in $\mathbb{R}^{2n}$ for CH of n points in 2D)

Thm [Ben-Or 1983] Let $W \in \mathbb{R}^n$ be any set and let T be any d^{th} order algebraic decision tree that decides membership in W. If W has m disjoint connected components then T has height $\Omega(\log m - n)$.

Problem Multiset size verification: Given multiset $Z = \{z_1 z_2 \dots z_n\}$ where $z_i \in \mathbb{R}$ and integer k does Z have exactly k distinct elements?

$$M_k = \{ (z_1 \dots z_n) \in \mathbb{R}^n \mid |\{z_1 \dots z_n\}| = k \} \text{ has } \geq k!\, k^{n-k}$$
disjoint components

Thus by Ben Or MSV requires d^{th} order alg. decision tree height
$$\Omega(\log(k!\, k^{n-k})) = \Omega(n \lg k)$$

Thm: CH size verification requires $\Omega(n \log h)$ steps in the worst case using any d^{th} order decision tree alg.

Proof: Construct points from instance $Z = \{z_1, z_2 \dots z_n\}$ of MSV problem.

$$p_i = (z_i, z_i^2)$$

Then $\{p_1, p_2 \dots p_n\}$ has k hull points iff Z has k distinct elements.

$\boxtimes$

Kirkpatrick & Seidel

Go on to show that even if we assume that all points are distinct, CH size verification still requires $\Omega(n \log h)$ steps.