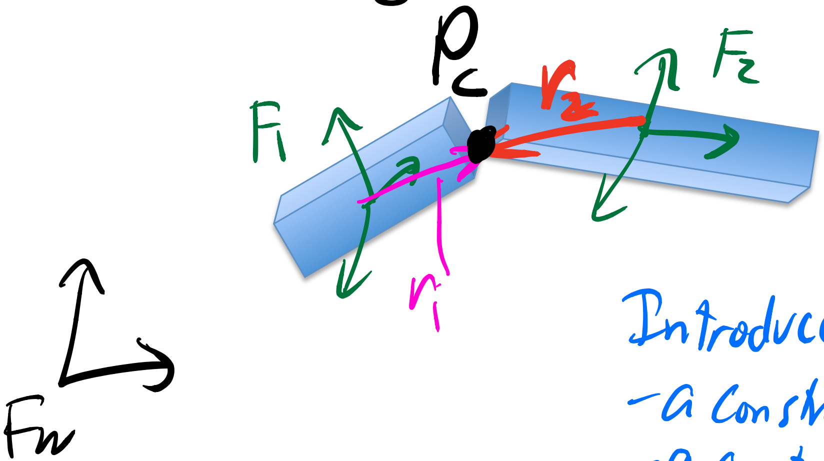
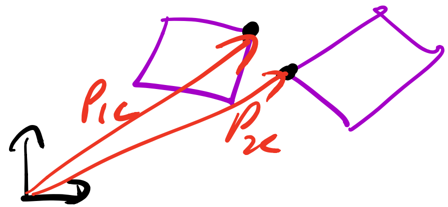
Two rigid bodies with a constraint





Introduce:

- a constraint force, F_c (3 extra unknowns)
- a constraint equation (3 extra equations)

$$p_{1c} - p_{2c} = 0$$

$$\dot{p}_{1c} - \dot{p}_{2c} = 0$$

$$\boxed{\ddot{p}_{1c} - \ddot{p}_{2c} = 0}$$

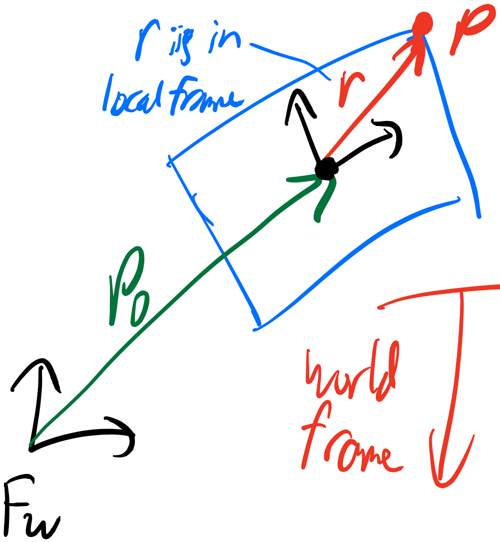
$$C(q) = 0$$

$$C(q, \dot{q}) = 0$$

$$C(q, \dot{q}, \ddot{q}) = 0$$

where q is a vector that describes all the ~~DOF~~ degrees of freedom

Acceleration of a point on a rigid body



rotation matrix

r in local frame

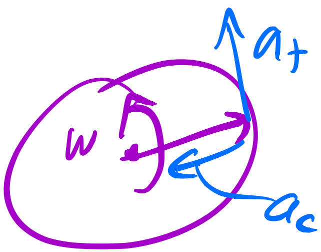
$$p = p_0 + \underbrace{Rr}_{r \text{ in world frame}}$$

$$\dot p = \dot p_0 + \dot R r + R\,\cancel{\dot r}\,0 \quad \text{because } \dot r = 0$$

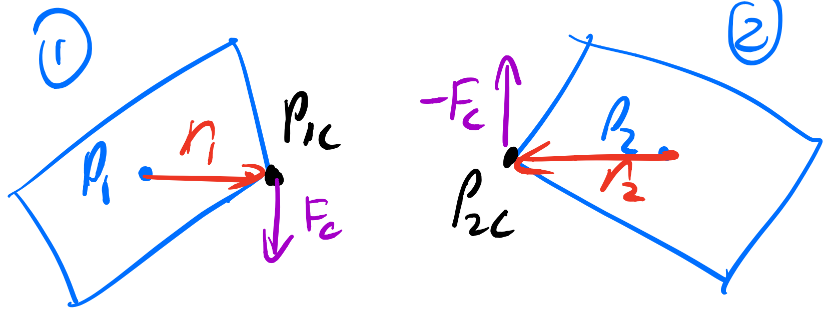
$$= \dot p_0 + \omega \times r \quad \text{or } \tilde\omega r$$

$$\omega \times (R r_L) \qquad r_w$$

$$\ddot p = \underbrace{\ddot p_0}_{\text{linear acc}} + \underbrace{\dot\omega \times r}_{\text{tangential accel, } a_t} + \underbrace{\omega \times (\omega \times r)}_{\text{centripetal accel, } a_c}$$



Newton-Euler EOM for two rigid bodies + pt-to-pt constraint



$$\vec{g} = \begin{bmatrix} 0 \\ -9.8 \\ 0 \end{bmatrix}$$

Block ① Newton

$$\sum F = M_1 \ddot{p}_1$$

$$F_c + M_1 \vec{g} = M_1 \ddot{p}_1$$

$$M_1 \ddot{p}_1 - F_c = M_1 g$$

Block ②

Similar, but F_c is negative

Block ① Euler

$$r_1 \times F_c \quad \sum \tau = I_1 \dot{\omega}_1 + \omega_1 \times I \omega_1$$

$$I_1 \dot{\omega}_1 - \tilde{r}_1 F_c = -\omega_1 \times I \omega_1$$

(continued)

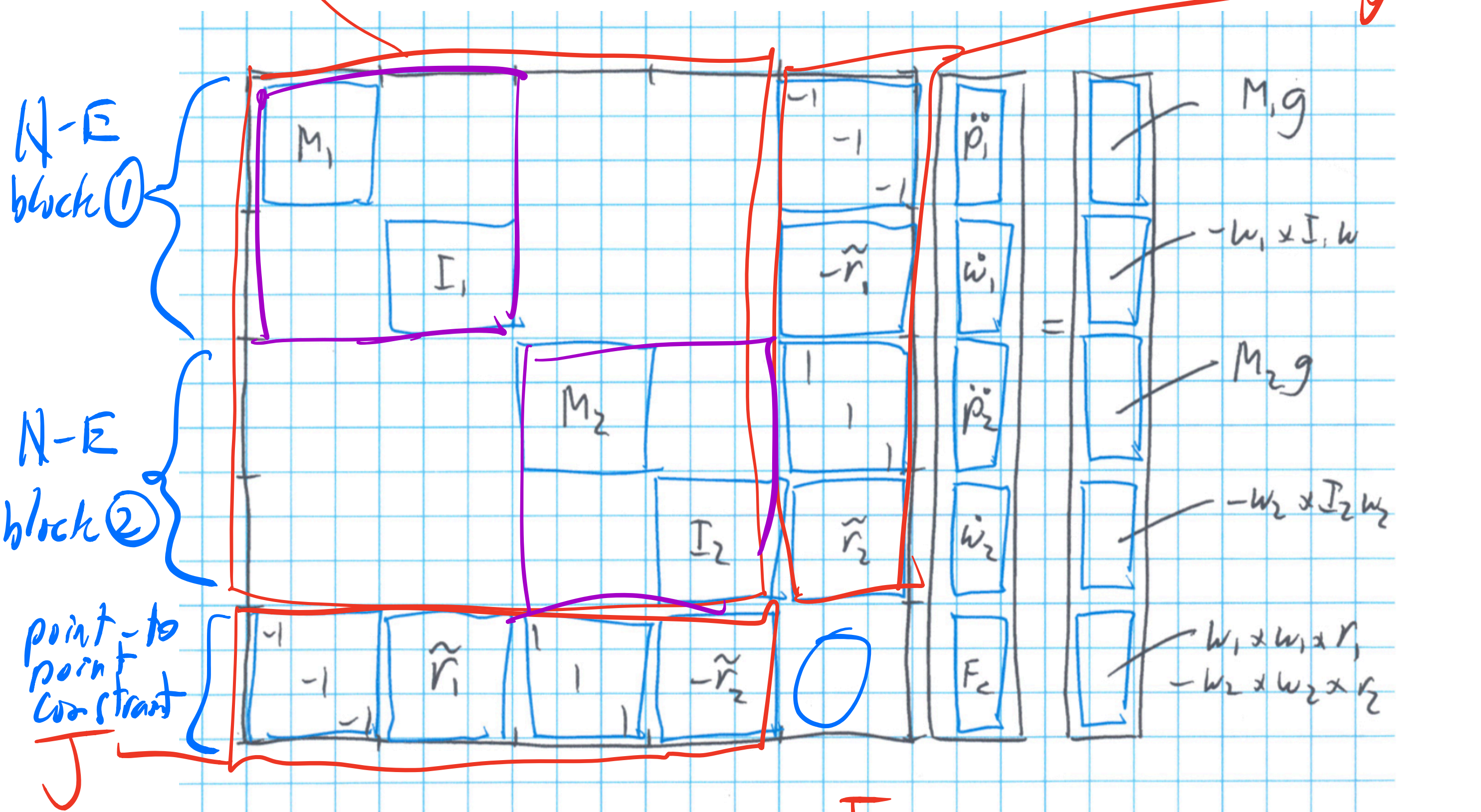
Constraint equation

$$\ddot{p}_{1c} = \ddot{p}_{2c}$$

$$\ddot{p}_1 + \dot{\omega}_1 \times r_1 + \omega_1 \times (\omega_1 \times r_1) = \ddot{p}_2 + \dot{\omega}_2 \times r_2 + \omega_2 \times (\omega_2 \times r_2)$$

$$\ddot{p}_1 - \tilde{r}_1 \dot{\omega}_1 - \ddot{p}_2 + \tilde{r}_1 \dot{\omega}_2 = \underbrace{-\omega_1 \times (\omega_1 \times r_1) + \omega_2 \times (\omega_2 \times r_2)}_{\text{Constants}}$$

Equations in Matrix Form



Note: $\tilde{r}_1^T = -\tilde{r}_1$ etc.

Constrained rigid body dynamics – general form (maximal coords)

$$M\ddot{q} + J^T F_c = F_{ext}$$

$$J\ddot{q} \; (+ \, 0 F_c) = c$$

$$\underbrace{\begin{bmatrix} M & J^T \\ J & 0 \end{bmatrix}}_{\text{KKT matrix}} \begin{bmatrix} \ddot{q} \\ F_c \end{bmatrix} = \begin{bmatrix} F_{ext} \\ c \end{bmatrix}$$

constraint force acts as Lagrange multiplier.

Canonical equations of motion in robotics:

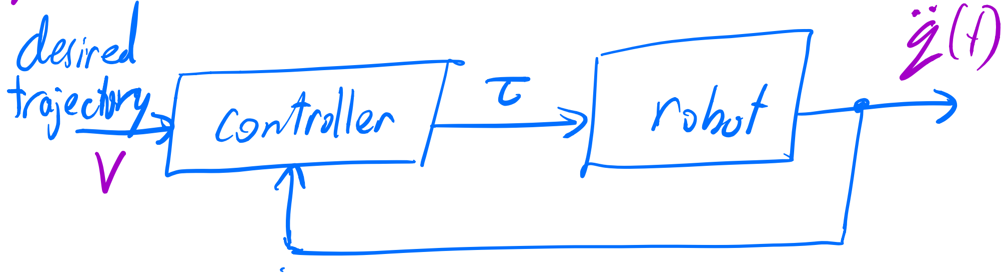
$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \tau_g(q) = \tau$$

Coriolis term

gravitational torques

actuator torques

Computed Torque Control (via inverse dynamics)



$$\tau = M(q)v + C(q,\dot{q})\dot{q} + \tau_g(q)$$

where $v = \ddot{q}_d$, i.e., specify desired joint accelerations

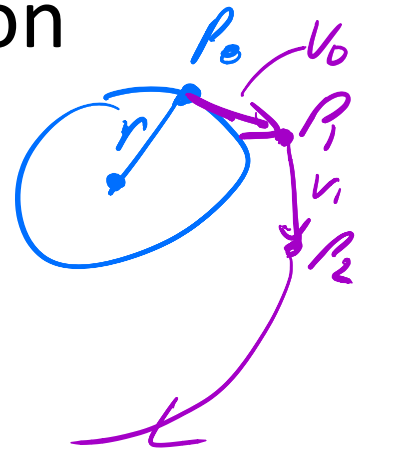
In practice, there will be errors, so add feedback:

$$v = \ddot{q}_d + K_d \dot{e} + K_p e$$

where $e(t) = q_d(t) - q(t)$

This control only applies to fully actuated systems

Constraint stabilization



- Constraints can easily drift apart
  e.g. motion of a point a circle

$\ddot{p}_{1c} - \ddot{p}_{2c} = 0$ via eqns of motion

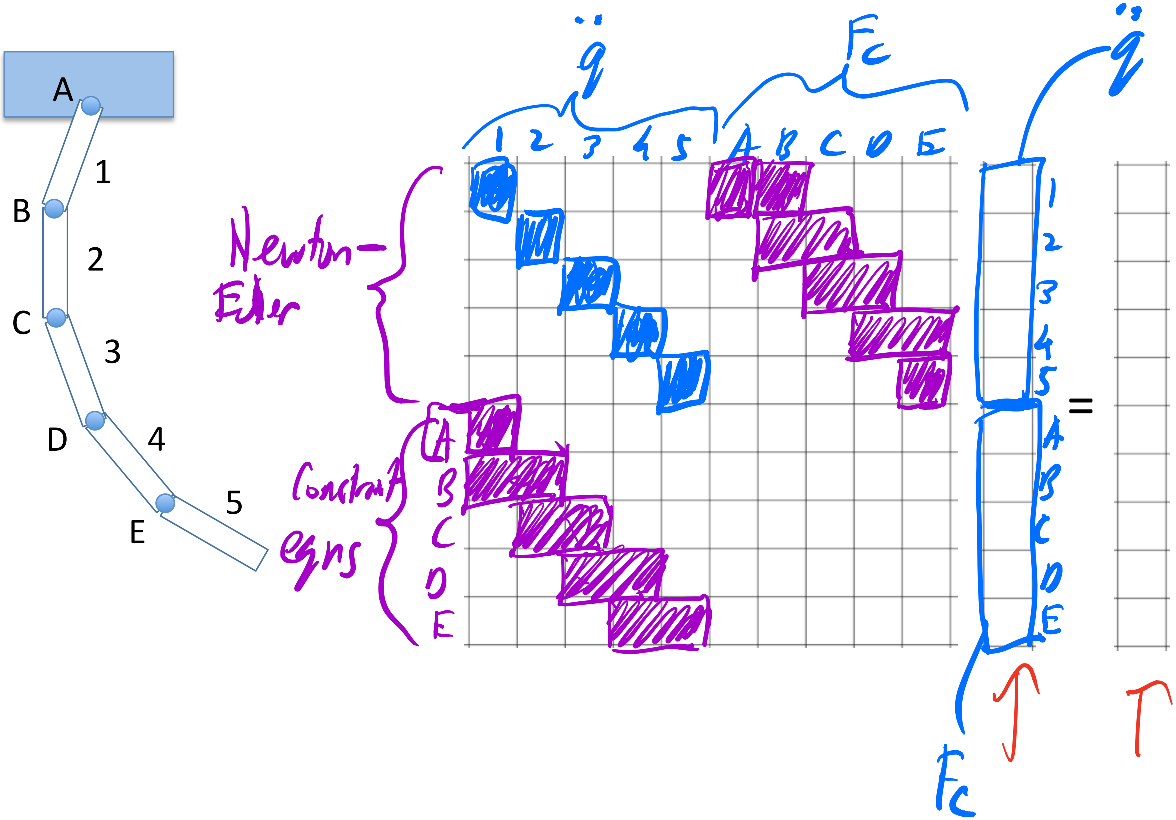
$$\left.\begin{aligned} \dot{p}_{1c} - \dot{p}_{2c} &= 0 \\ p_{1c} - p_{2c} &= 0 \end{aligned}\right\}$$

begin to be violated because of numerical integration over a time step

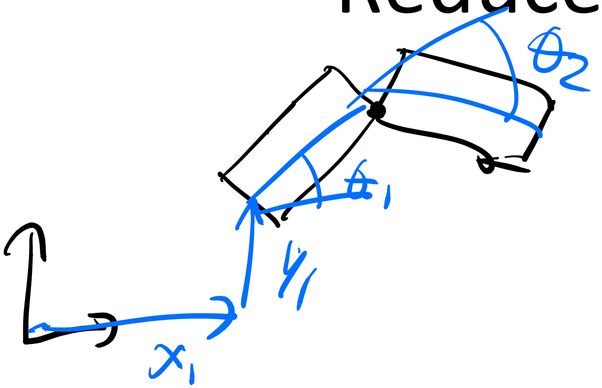
- constraint stabilization adds terms to help correct this drift
  see Baumgarte (1972) Ascher (1995)

$$\ddot{p}_{1c} - \ddot{p}_{2c} = \boxed{-k_p(p_{1c} - p_{2c}) - k_d(\dot{p}_{1c} - \dot{p}_{2c})}$$

Simulation of a 5-link chain



Reduced Coordinates



reduced coordinates (RC)

$$q = \begin{bmatrix} x_1 \\ y_1 \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

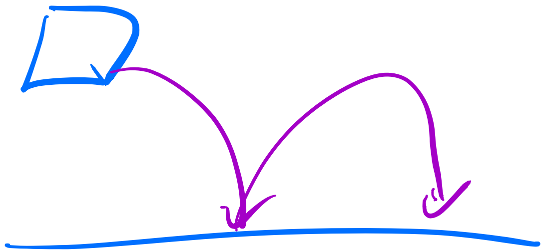
maximal coordinates (MC)

$$q = \begin{bmatrix} x_1 \\ y_1 \\ \theta_1 \\ x_2 \\ y_2 \\ \theta_2 \end{bmatrix}$$

RC: - no need to stabilize constraints
- equations: small-but-dense matrix

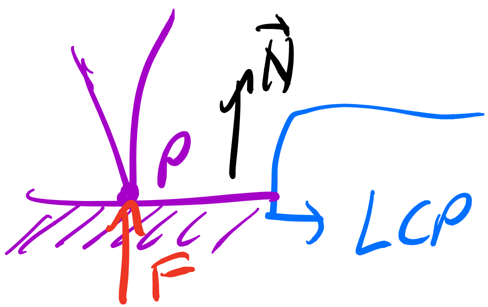
MC: - equations: large-but-sparse matrix

Collisions



Subproblems:

1. collision detection
   - broad phase: bounding spheres, clusters of objects
   - narrow phase → triangle – triangle.
2. collision resolution: force impulses
3. resting contact



→ LCP: Linear Complementarity Problem

we want: $F \cdot \vec{N} \geq 0$

$\ddot{p} \cdot \vec{N} \geq 0$

$F_N \cdot \ddot{p}_N = 0$