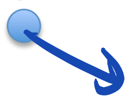
$s_t,\ a_t \rightarrow \boxed{\text{env}} \rightarrow s_{t+1}$

$x,\ u \rightarrow \boxed{\text{plant}} \rightarrow \dot{x} \quad \frac{dx}{dt}$

Rigid Body Dynamics

- from particles to rigid bodies…

state s $= \begin{bmatrix} \vec{p} \\ \vec{v} \end{bmatrix}$ position velocity



$\vec{f}$

Newton's equations of motion

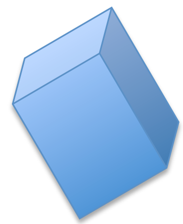
$$\sum \vec{F} = M\vec{a}$$

$$\begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \Sigma F \end{bmatrix}$$

while $(t<T)$ {

1. $F = mg + f$ // compute sum of forces
2. solve $M \cdot \vec{a} = \vec{F}$ for $\vec{a}$
3. $\left.\begin{array}{l} \vec{p} = \vec{p} + \vec{v} \cdot \Delta t \\ \vec{v} = \vec{v} + \vec{a} \cdot \Delta t \end{array}\right\}$ explicit Euler integration
4. $t = t + \Delta t$

state: $\vec{p}$ position
$\vec{v}$ velocity
q or R orientation
$\vec{\omega}, \vec{\dot{q}}, \dot{R}$ angular velocity



Newton-Euler equations of motion

$$\begin{bmatrix} \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix} & \\ & \begin{bmatrix} I \end{bmatrix} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \\ \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \Sigma F \\ \Sigma \tau - \vec{\omega} \times I\vec{\omega} \end{bmatrix}$$

inertia tensor

Coriolis term

Preliminaries

- cross product via a matrix multiply

$$\tilde{a} = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

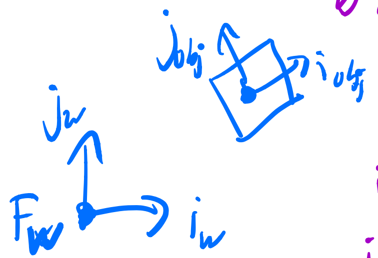
$\vec{a} \times \vec{b} = \tilde{a} b$

$-\vec{b} \times \vec{a} = \vec{a} \times \vec{b}$

$-\tilde{b} \vec{a} = \vec{a} \times \vec{b}$

- rotation matrix

$R = \begin{bmatrix} i_{obj} & j_{obj} & k_{obj} \end{bmatrix}$



$\|i\| = \|j\| = \|k\| = 1$

$i \cdot j = j \cdot k = i \cdot k = 0$

$i \times j = k$

or $\det(R) = +1$

$SO(3)$ "special orthogonal"

- unit quaternion

$q = [w \; x \; y \; z] \in \mathbb{R}^4$

$\|q\| = 1$

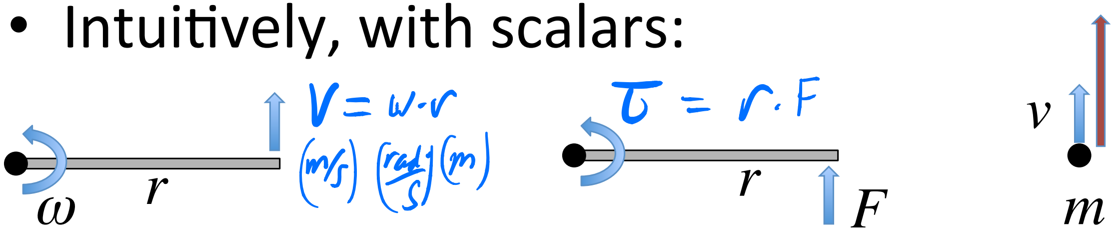
Complex number representation of a 3D orientation

unique, except that q and $-q$ are the same orientation

Kinematics of Rotation

linear momentum

- Intuitively, with scalars:

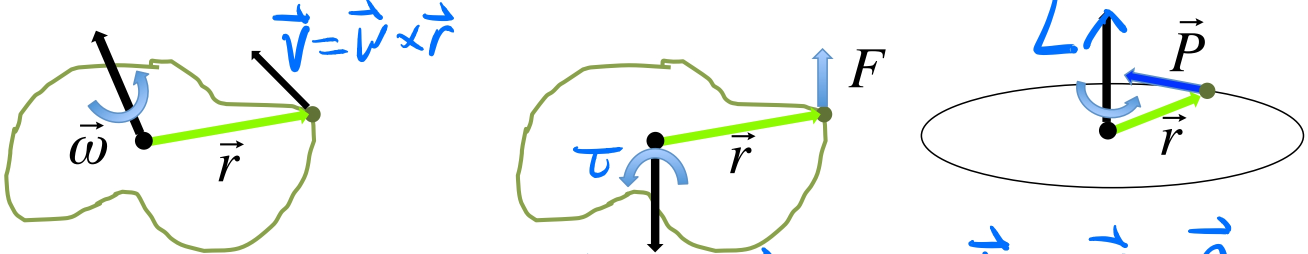


$V = \omega \cdot r$ $\left(m/s\right)$ $\left(\frac{rad}{s}\right)(m)$

$\tau = r \cdot F$

$\vec{p} = m \cdot \vec{v}$

- More generally:



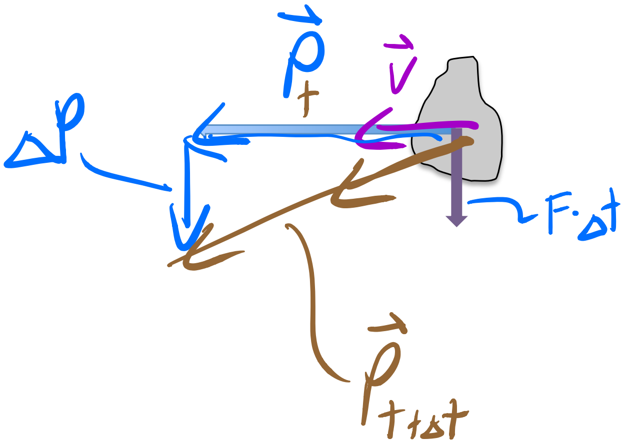
$\vec{v} = \vec{\omega} \times \vec{r}$

$\vec{\tau} = \vec{r} \times \vec{F}$

$\vec{L} = \vec{r} \times \vec{p}$

angular momentum

Newton's Law



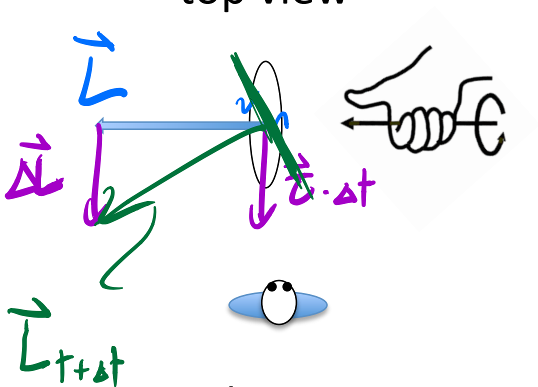
$$\vec{p} = m\vec{v}$$

$$\boxed{\frac{d\vec{p}}{dt} = \sum_i F_i}$$

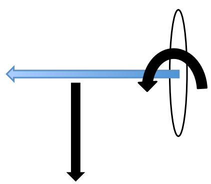
$$\Delta p = F \Delta t$$

Euler's Law

top view



side view



$$\vec{L} = I\,\vec{\omega}$$

I: inertia tensor, 3x3 matrix

$\vec{\omega}$: angular velocity

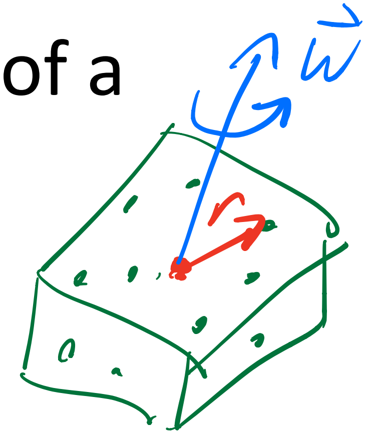
$\|\vec{\omega}\|$ = angular speed

$\frac{\vec{\omega}}{\|\vec{\omega}\|}$ = axis of rotation

$$\frac{d\vec{L}}{dt} = \sum_i \tau_i$$

$$\Delta\vec{L} = \vec{\tau}\cdot\Delta t$$

Angular Momentum of a Set of Particles



$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

$$= \sum_i \vec{r}_i \times m_i \vec{v}_i$$

$$= \sum_i r_i \times m_i (\vec{\omega} \times \vec{r}_i)$$

$$= -\sum_i \tilde{r}_i \, m_i \, \tilde{r}_i \, \vec{\omega}$$

$$= \underbrace{-\sum_i m_i \tilde{r}_i \tilde{r}_i}_{I:\ 3\times 3 \text{ inertia tensor}} \vec{\omega}$$

$$\vec{L} = I \cdot \vec{\omega} \qquad \omega = I^{-1} \vec{L}$$

In~~t~~ertia Tensor

$$\vec{L} = -\sum_i m_i \tilde{r}_i \hat{r}_i \cdot \vec{\omega}$$

$$= -\sum_i m_i \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \vec{\omega}$$

$$= \underbrace{-\sum_i m_i \begin{bmatrix} z_i^2 + y_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & z_i^2 + x_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{bmatrix}}_{I} \vec{\omega}$$

For an object with x, y, z symmetries
e.g. rectangular block

$$I = \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix}$$

Newton-Euler Equations of Motion

Newton

$$\sum F = \frac{dp}{dt} = \frac{d(m \cdot v)}{dt}$$

$$= \underbrace{\cancel{\dot{m} \cdot v}}_{0} + m \cdot \dot{v}$$

$$= m \cdot \dot{v}$$

Euler

$$\sum \tau = \frac{dL}{dt} = \frac{d(I\vec{\omega})}{dt}$$

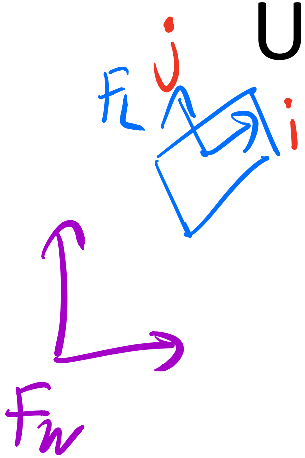
$$= \dot{I}\vec{\omega} + I\dot{\vec{\omega}}$$

$$= \underbrace{\vec{\omega} \times I\vec{\omega}}_{\text{Coriolis term}} + I\dot{\vec{\omega}}$$

These eqns only hold in an inertial frame

→ non-accelerating

Updating the Inertia Tensor



Need to compute I in the world frame

solve for this

4 equations

known

(1) $\vec{L}_W = I_W \cdot \vec{\omega}_W$

(2) $\vec{L}_L = I_L \cdot \vec{\omega}_L$

(3) $\vec{L}_W = R\,\vec{L}_L$

(4) $\vec{\omega}_W = R\,\vec{\omega}_L \Rightarrow \omega_L = \underbrace{R^{-1}\omega_W}$

(2) into (3) $\vec{L}_W = R\,I_L\,\vec{\omega}_L$

$= \underbrace{R\,I_L\,R^{-1}}_{I_W}\,\omega_W$ compare to (1)

$I_W = R \cdot I_L \cdot R^{-1}$

Simulation Loop

state: linear position linear velocity angular orientation angular velocity	x v R rotation matrix $\vec{\omega}$	x v q quaternion L angular momentum.
for each timestep		
setup	– compute applied forces & torques, in world frame $I_W = R\, I_L\, R^{-1}$	– compute forces & torques
solve eqns of motion world frame	$\sum F = m \cdot \dot{v}$ $\sum \tau = I \dot{\omega} + \boxed{\omega \times I \omega}$ Coriolis	$\sum F = m \cdot \dot{v}$ $\sum \tau = \dot{L}$
integrate	$x = x + v \Delta t$ $v = v + \dot{v} \Delta t$ $R = R + \dot{R} \Delta t$ where $\dot{R} = \tilde{\omega} R$ $\omega = \omega + \dot{\omega} \Delta t$	$x = x + v \Delta t$ $v = v + \dot{v} \Delta t$ $q = q + \dot{q} \Delta t$ $L = L + \dot{L} \Delta t$

$\dot{q} = 0.5 \cdot \hat{\omega} \otimes q$ where $\omega = I^{-1} L$