Particle-Plane Collisions

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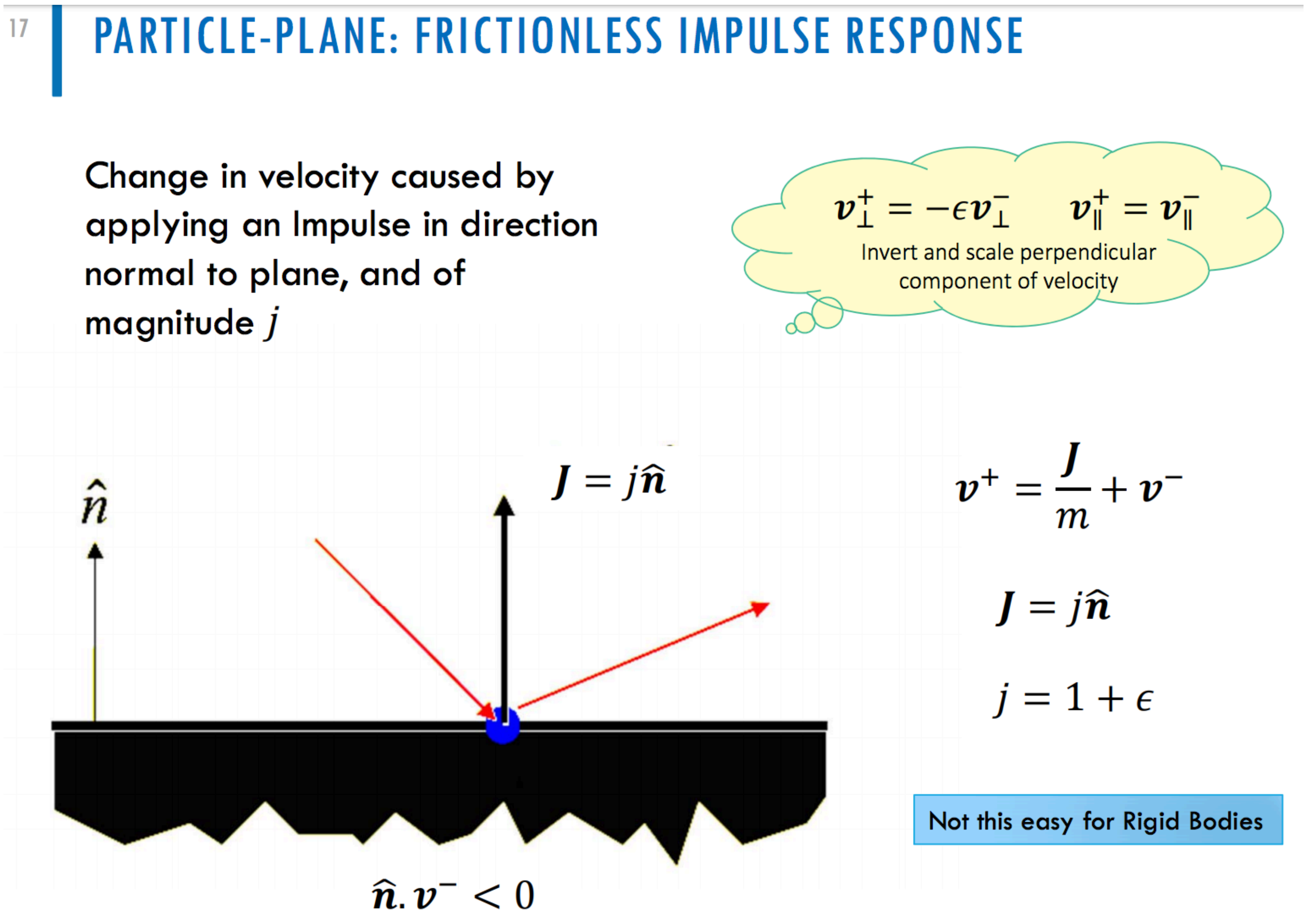


PARTICLE-PLANE: FRICTIONLESS IMPULSE RESPONSE

Change in velocity caused by applying an Impulse in direction normal to plane, and of magnitude j

$$\boldsymbol{v}_{\perp}^{+} = -\epsilon \boldsymbol{v}_{\perp}^{-} \qquad \boldsymbol{v}_{\parallel}^{+} = \boldsymbol{v}_{\parallel}^{-}$$

Invert and scale perpendicular component of velocity

$\hat{n}$

$\boldsymbol{J} = j\hat{\boldsymbol{n}}$

$\hat{\boldsymbol{n}}.\boldsymbol{v}^{-} < 0$

$$\boldsymbol{v}^{+} = \frac{\boldsymbol{J}}{m} + \boldsymbol{v}^{-}$$

$$\boldsymbol{J} = j\hat{\boldsymbol{n}}$$

$$j = 1 + \epsilon$$

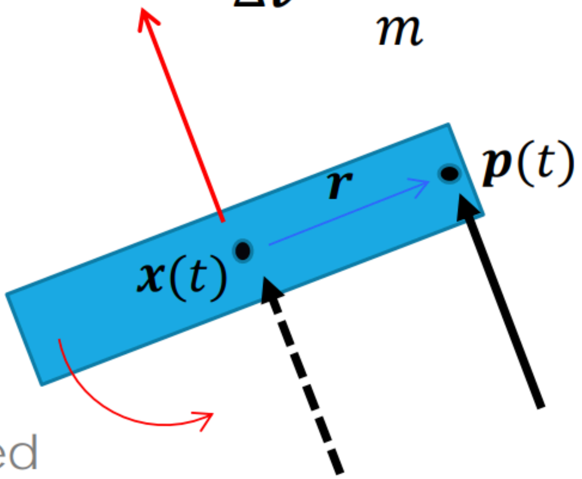
Not this easy for Rigid Bodies

Impulse causes change in Linear and Angular velocity

The effect of an impulse (or for that matter a force) on an object's linear and angular momentum are independent

- Linear component: similar to particles
  - Causes a change in velocity inversely proportional to mass
  - As if force was applied at c.o.m.
- Angular component: impulsive torque
  - Causes change in angular velocity inversely proportional to moment of inertia (determined from inertial tensor)
  - Dependent on position of impulse

$$\Delta \boldsymbol{v} = \frac{\boldsymbol{J}}{m}$$



$$\begin{aligned} \Delta \boldsymbol{\omega} &= \boldsymbol{I}^{-1}(\boldsymbol{r} \times \boldsymbol{J}) \\ &= \boldsymbol{I}^{-1}((\boldsymbol{p} - \boldsymbol{x}) \times \boldsymbol{J}) \end{aligned}$$

But what is the value of $\boldsymbol{J}$?

RIGID BODY COLLISION

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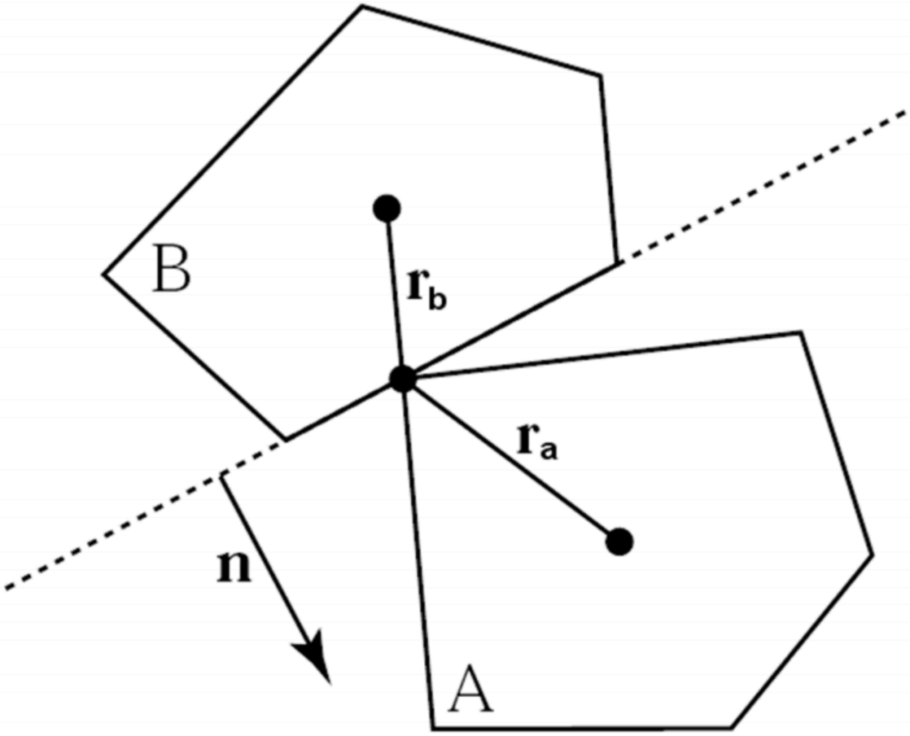
$$v_{rel}^{+} = -\epsilon v_{rel}^{-}$$

$$v_{rel} = \hat{\boldsymbol{n}}(\dot{\boldsymbol{p}}_A - \dot{\boldsymbol{p}}_B)$$



$$\begin{aligned}\dot{\boldsymbol{p}}_A &= \boldsymbol{v}_A + \boldsymbol{\omega}_A \times (\boldsymbol{p}_A - \boldsymbol{x}_A)\\ &= \boldsymbol{v}_A + \boldsymbol{\omega}_A \times \boldsymbol{r}_A\end{aligned}$$

$$\begin{aligned}\dot{\boldsymbol{p}}_A &= \boldsymbol{v}_B + \boldsymbol{\omega}_B \times (\boldsymbol{p}_B - \boldsymbol{x}_B)\\ &= \boldsymbol{v}_A + \boldsymbol{\omega}_A \times \boldsymbol{r}_A\end{aligned}$$

Linear velocity component

Angular component: Linear velocity of point p due to its rotation. See previous slide

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Post-collision velocity should be

$$\dot{\boldsymbol{p}}_A^+ = \boldsymbol{v}_A^+ + \boldsymbol{\omega}_A^+ \times \boldsymbol{r}_A$$

However we need to express this in terms of the previous state of the object (pre-collision values)

$$\boldsymbol{v}_A^+ = \quad \boldsymbol{v}_A^- + \Delta\boldsymbol{v} \quad = \boldsymbol{v}_A^- + \frac{j\hat{\boldsymbol{n}}}{m_A} \quad \leftarrow \quad \text{SEE SLIDE 19}$$

$$\boldsymbol{\omega}_A^+ = \quad \boldsymbol{\omega}_A^- + \Delta\boldsymbol{\omega} \quad = \boldsymbol{\omega}_A^- + \boldsymbol{I}_A^{-1}(\boldsymbol{r}_A \times j\hat{\boldsymbol{n}})$$

$$\dot{\boldsymbol{p}}_A^+ = \boldsymbol{v}_A^- + \frac{j\hat{\boldsymbol{n}}}{m_A} + (\boldsymbol{\omega}_A^- + \boldsymbol{I}_A^{-1}(\boldsymbol{r}_A \times j\hat{\boldsymbol{n}})\ ) \times \boldsymbol{r}_A$$

$$= \boxed{\boldsymbol{v}_A^- + \boldsymbol{\omega}_A^- \times \boldsymbol{r}_A} \quad + \quad j\left(\frac{\hat{\boldsymbol{n}}}{m_A} + \boldsymbol{I}_A^{-1}(\boldsymbol{r}_A \times \hat{\boldsymbol{n}})\ \right) \times \boldsymbol{r}_A$$

↓ PREVIOUS SLIDE

$$= \dot{\boldsymbol{p}}_A^- \qquad\qquad + \quad j\left(\frac{\hat{\boldsymbol{n}}}{m_A} + \boldsymbol{I}_A^{-1}(\boldsymbol{r}_A \times \hat{\boldsymbol{n}})\ \right) \times \boldsymbol{r}_A$$

Similarly (by Newton's 3rd Law: every reaction has equal and opposite reaction):

$$\dot{\boldsymbol{p}}_B^+ = \dot{\boldsymbol{p}}_B^- + j\left(\frac{\hat{\boldsymbol{n}}}{m_A} + \boldsymbol{I}_A^{-1}(\boldsymbol{r}_A \times \hat{\boldsymbol{n}})\ \right) \times \boldsymbol{r}_A$$

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Putting this in $v_{rel}^{+} = \hat{\boldsymbol{n}}(\dot{\boldsymbol{p}}_A^{+} - \dot{\boldsymbol{p}}_B^{+})$ *we get*

From previous slide

$$v_{rel}^{+} = \hat{\boldsymbol{n}}\left(\left(\boxed{\boldsymbol{p}_A^{-} + j\left(\frac{\hat{\boldsymbol{n}}}{m_A} + \boldsymbol{I}_A^{-1}(\boldsymbol{r}_A \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_A}\right) - \left(\boxed{\boldsymbol{p}_B^{-} + j\left(\frac{\hat{\boldsymbol{n}}}{m_B} + \boldsymbol{I}_B^{-1}(\boldsymbol{r}_B \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_B}\right)\right)$$

$$= \boxed{\hat{\boldsymbol{n}}(\dot{\boldsymbol{p}}_A^{-} - \dot{\boldsymbol{p}}_B^{-})} + j\left(\frac{1}{m_A} + \frac{1}{m_B} + \hat{\boldsymbol{n}}\left(\boldsymbol{I}_A^{-1}(\boldsymbol{r}_A \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_A + \hat{\boldsymbol{n}}\left(\boldsymbol{I}_B^{-1}(\boldsymbol{r}_B \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_B\right)$$

$$= v_{rel}^{-} + j\left(\frac{1}{m_A} + \frac{1}{m_B} + \hat{\boldsymbol{n}}\left(\boldsymbol{I}_A^{-1}(\boldsymbol{r}_A \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_A + \hat{\boldsymbol{n}}\left(\boldsymbol{I}_B^{-1}(\boldsymbol{r}_B \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_B\right)$$

And since $v_{rel}^{+} = -\epsilon v_{rel}^{-}$

$$-\epsilon v_{rel}^{-} = v_{rel}^{-} + j\left(\frac{1}{m_A} + \frac{1}{m_B} + \hat{\boldsymbol{n}}\left(\boldsymbol{I}_A^{-1}(\boldsymbol{r}_A \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_A + \hat{\boldsymbol{n}}\left(\boldsymbol{I}_B^{-1}(\boldsymbol{r}_B \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_B\right)$$

giving

$$j = \frac{-(1+\epsilon)\, v_{rel}^{-}}{m_A^{-1} + m_B^{-1} + \hat{\boldsymbol{n}}\left(\boldsymbol{I}_A^{-1}(\boldsymbol{r}_A \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_A + \hat{\boldsymbol{n}}\left(\boldsymbol{I}_B^{-1}(\boldsymbol{r}_B \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_B}$$

This gives us the impulse magnitude we were looking for

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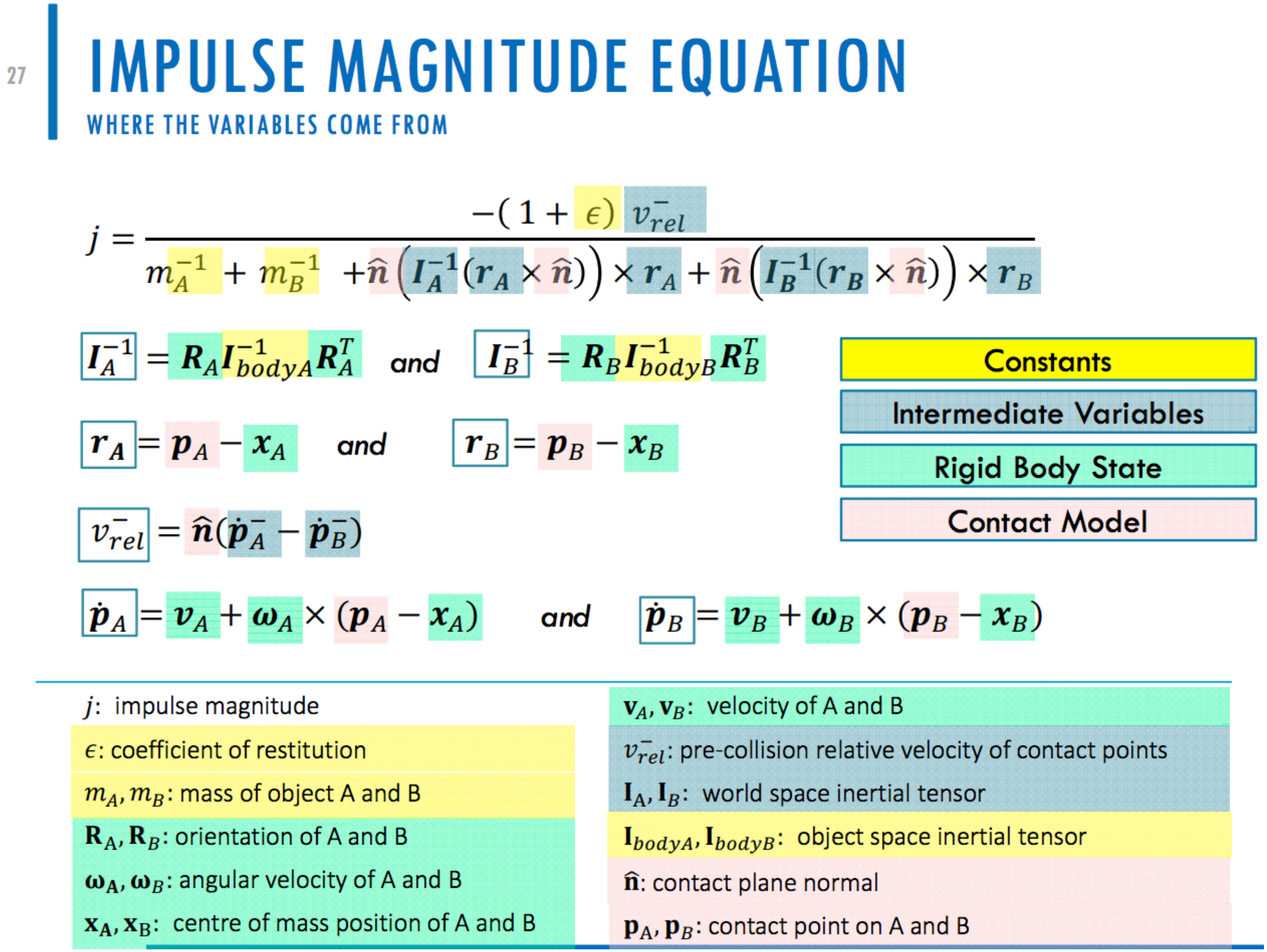

IMPULSE MAGNITUDE EQUATION

WHERE THE VARIABLES COME FROM

$$j = \frac{-(1+\epsilon)\, v_{rel}^{-}}{m_A^{-1} + m_B^{-1} + \hat{n}\left(I_A^{-1}(r_A \times \hat{n})\right) \times r_A + \hat{n}\left(I_B^{-1}(r_B \times \hat{n})\right) \times r_B}$$

$$\boxed{I_A^{-1}} = R_A I_{bodyA}^{-1} R_A^T \quad \text{and} \quad \boxed{I_B^{-1}} = R_B I_{bodyB}^{-1} R_B^T$$

$$\boxed{r_A} = p_A - x_A \quad \text{and} \quad \boxed{r_B} = p_B - x_B$$

$$\boxed{v_{rel}^{-}} = \hat{n}(\dot{p}_A^{-} - \dot{p}_B^{-})$$

$$\boxed{\dot{p}_A} = v_A + \omega_A \times (p_A - x_A) \quad \text{and} \quad \boxed{\dot{p}_B} = v_B + \omega_B \times (p_B - x_B)$$

Constants

Intermediate Variables

Rigid Body State

Contact Model

j: impulse magnitude

ϵ: coefficient of restitution

m_A, m_B: mass of object A and B

$\mathbf{R}_A, \mathbf{R}_B$: orientation of A and B

$\boldsymbol{\omega}_A, \boldsymbol{\omega}_B$: angular velocity of A and B

$\mathbf{x}_A, \mathbf{x}_B$: centre of mass position of A and B

$\mathbf{v}_A, \mathbf{v}_B$: velocity of A and B

v_{rel}^{-}: pre-collision relative velocity of contact points

$\mathbf{I}_A, \mathbf{I}_B$: world space inertial tensor

$\mathbf{I}_{bodyA}, \mathbf{I}_{bodyB}$: object space inertial tensor

$\hat{\mathbf{n}}$: contact plane normal

$\mathbf{p}_A, \mathbf{p}_B$: contact point on A and B

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APPLYING IMPULSE

Impulse magnitude is given by:

$$j = \frac{-(1+\epsilon)\ v_{rel}^{-}}{m_A^{-1} + m_B^{-1}\ + \hat{\boldsymbol{n}}\left(\boldsymbol{I}_A^{-1}(\boldsymbol{r}_A \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_A + \hat{\boldsymbol{n}}\left(\boldsymbol{I}_B^{-1}(\boldsymbol{r}_B \times \hat{\boldsymbol{n}})\right) \times \boldsymbol{r}_B}$$

The actual impulse vector is simply $\boldsymbol{J} = j\hat{\boldsymbol{n}}$

This is applied to the objects as follows:

- Change in Linear momentum is directly equal to the impulse:

$$\Delta\boldsymbol{P} = \boldsymbol{J} \quad \Leftrightarrow \quad \Delta\boldsymbol{v} = \boldsymbol{J}m^{-1}$$

- Change in Angular momentum is equal to the imulsive torque ($\boldsymbol{\tau}_{\mathrm{IMPULSE}}$):

$$\Delta\boldsymbol{L} = (\boldsymbol{r} \times \boldsymbol{J}) \quad \Leftrightarrow \quad \Delta\boldsymbol{\omega} = \boldsymbol{I}^{-1}(\boldsymbol{r} \times \boldsymbol{J})$$