Rigid Body Dynamics

$$\frac{ds}{dt} = f(s)$$

- from particles to rigid bodies…

state s: $\vec{p}$ posn, $\vec{v}$ vel

$s \in \mathbb{R}^6$

$\downarrow m\vec{g}$

Newton's equations of motion (EOM)

$$\sum \vec{F} = m\vec{a}$$

$$\begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \sum F \end{bmatrix}$$

initialize $V, P, t, \vec{g} = [0\ -9.8\ 0]^T$

while $(t < T)$ {

① $\vec{F} = m\vec{g} + \dots$ // compute applied forces

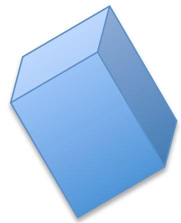
② solve $\vec{F} = M \cdot \vec{a}$ for $\vec{a}$

③ $\vec{p} = \vec{p} + \vec{v}\Delta t$, $\vec{v} = \vec{v} + \vec{a}\Delta t$ } integrate (explicit Euler)

④ $t = t + \Delta t$

}

$s \in \mathbb{R}^{12}$



state s: $\vec{p}$ posn, $\vec{v}$ vel, q, R orientation, $\vec{\omega}, \dot{q}, \dot{R}$

Newton-Euler equations of motion

$$\begin{bmatrix} m & & & \\ & m & & \\ & & m & \\ & & & I \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \\ \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \sum F \\ \sum \tau - \vec{\omega} \times I\vec{\omega} \end{bmatrix}$$

3×3

Coriolis term

Preliminaries

- cross product via a matrix multiply

$$\tilde{a} = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}$$

$$\vec{a} \times \vec{b} = \tilde{a}\,\vec{b}$$

$$[\;] \times [\;] = \left[\boxed{\tilde{a}}\right][b]$$

$$\tilde{b}\,a = \vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$$

Kinematics of Rotation

linear momentum

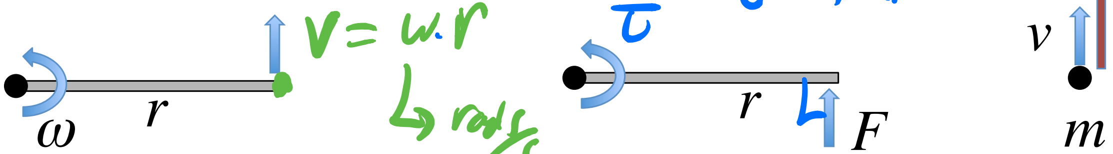
$\vec{p} = m \cdot \vec{v}$

- Intuitively, with scalars:

$v = \omega \cdot r$

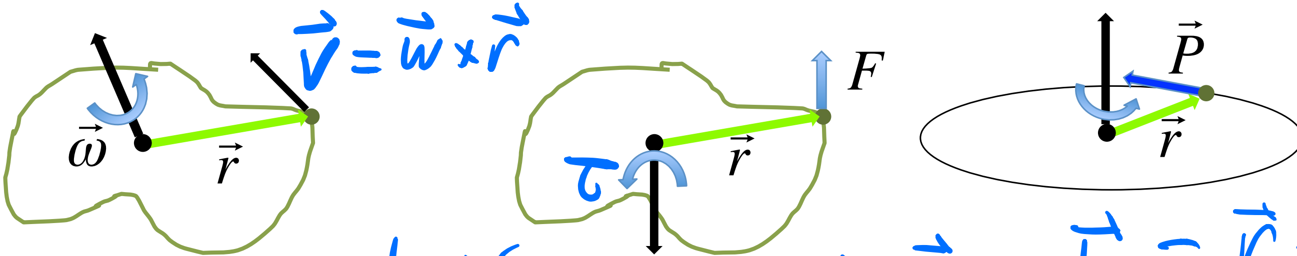
↳ rad/s

$\tau = F \cdot r$



- More generally:

$\vec{v} = \vec{\omega} \times \vec{r}$



$\|\omega\|$ = angular velocity (rad/s)

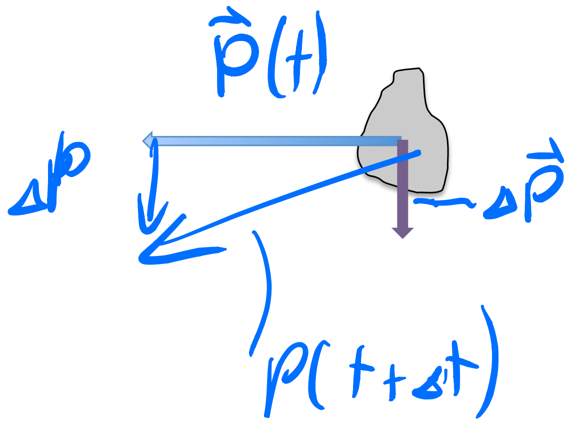
$\vec{\omega}/\|\vec{\omega}\|$ = axis of rotation

$\vec{\tau} = \vec{r} \times \vec{F}$

$\vec{L} = \vec{r} \times \vec{P}$

angular momentum

Newton’s Law



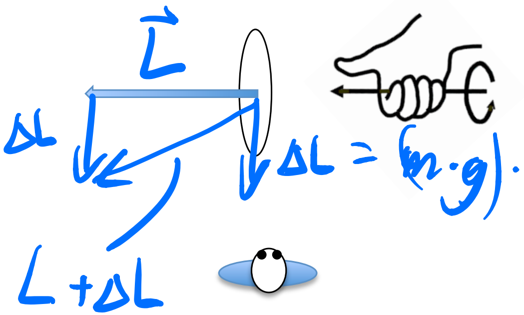
$$\vec{p} = m \cdot \vec{v}$$

$$\frac{d\vec{p}}{dt} = \sum_i F_i$$

$$\Delta\vec{p} = \vec{F}_{tot} \cdot \Delta t$$

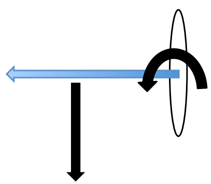
Euler’s Law

top view



$\Delta L = (m \cdot g) \cdot d \, \Delta t$

side view



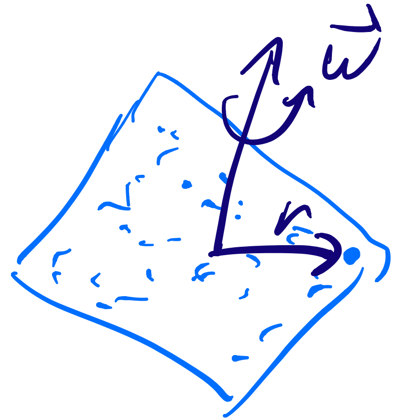
$$\vec{L} = I \vec{\omega}$$

inertia tensor (3×3)

$$\frac{d\vec{L}}{dt} = \sum_i \tau_i$$

torques.

Angular Momentum of a Set of Particles



$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

$$= \sum_i \vec{r}_i \times m_i \vec{V}_i$$

$$= \sum_i \vec{r}_i \times m_i (\vec{w} \times \vec{r}_i)$$

$$= \boxed{-\sum_i m_i \tilde{r}_i \tilde{r}_i}\, \vec{w}$$

I : 3×3 inertia tensor.

$$L = I \vec{w}$$

In~~t~~ertia Tensor

$$\vec{L} = -\sum_i m_i \tilde{r}_i \tilde{r}_i \vec{w}$$

$$= -\sum_i m_i \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \vec{w}$$

$$= \sum_i m_i \begin{bmatrix} z_i^2 + y_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & z_i^2 + x_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{bmatrix} \vec{w}$$

For an object with x, y, z symmetries then

$$I = \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix}$$

Newton-Euler Equations of Motion

Newton

$$\sum F = \frac{dp}{dt} = \frac{d(m\vec{v})}{dt}$$

$$= \cancel{\dot{m}v} + m\dot{v}$$

$$\sum F = m\dot{v}$$

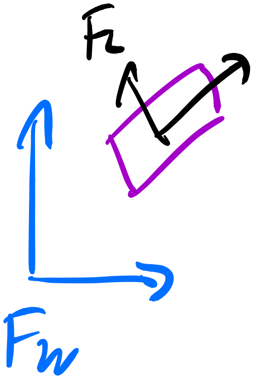
Only valid in an inertial frame

Euler

$$\sum \tau = \frac{dL}{dt} = \frac{d(I\vec{\omega})}{dt}$$

$$= \dot{I}\vec{\omega} + I\dot{\vec{\omega}}$$

$$= \underbrace{\vec{\omega} \times I\vec{\omega}}_{\text{coriolis term}} + \underbrace{I\dot{\vec{\omega}}}_{\text{angular accel}}$$



Updating the Inertia Tensor

Assuming we know I_L, what is I_W?

4 key equations

① $\vec{L}_W = I_W \vec{\omega}_W$

② $\vec{L}_L = I_L \vec{\omega}_L$

③ $\vec{L}_W = R \vec{L}_L$

④ $\vec{\omega}_W = R \vec{\omega}_L \Rightarrow \vec{\omega}_L = R^{-1} \vec{\omega}_W$

② into ③ $L_W = R I_L \vec{\omega}_L$

then into ④ $L_W = R I_L R^{-1} \omega_W$

$I_W = R I_L R^{-1}$

Simulation Loop

$w = I_w^{-1} L$

linear position linear velocity angular orientation angular velocity	x v R w	x v q quaternion L angular momentum
for each timestep		
setup	– compute forces and torques $- I_w = R I_L R^{-1}$	– compute forces & torques $- R = f(q)$ $- I_w = R I_L R^{-1}$ $w = I^{-1} L$ $\dot{q} = 0.5\, \hat{w} \otimes q$
solve eqns of motion	$\Sigma F = m \dot{v}$ $\Sigma \tau = I \dot{w} + w \times I w$	$\Sigma F = m \dot{v}$ $\Sigma \tau = \dot{L}$
integrate	$x = x + v \Delta t$ $v = v + \dot{v} \Delta t$ $R = R + \dot{R} \Delta t$ where $\dot{R} = \tilde{w} R$ $w = w + \dot{w} \Delta t$	$x = x + v \Delta t$ $v = v + \dot{v} \Delta t$ $q = q + \dot{q} \Delta t$ $L = L + \dot{L} \Delta t$