ABOUT THIS DOCUMENT

For ease of use, this document is made up of three parts, each of which tries to give an account of the Geometry Center at a certain level of detail.

The one-page Introduction (page 1) states the vision and goals of the Center, and gives an idea of the methods and people involved in achieving these goals.

The Overview (pages 2–15) enumerates and discusses briefly each of our programs and activities. It starts with our answer to the question "Why a Center?"

The remainder, starting on page 17, documents in more detail all the activities and resources of the Geometry Center. There are references to this part throughout the Overview, for readers interested in a particular topic.

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Introduction

Mission. The mission of the Geometry Center is to develop, support, and promote computational tools for visualizing geometric structures, for facilitating communication among mathematicians and between mathematicians and the public at large, and for stimulating research in geometry.

The rapid development of computer technology, especially computer graphics, has provided powerful new tools for mathematical experimentation and for communication between mathematicians, between teachers and students, and between the mathematical community and the public. The Geometry Center plays a leading role in developing and promoting these tools as part of a new technological infrastructure in support of mathematics research and education. The goal of the Center is to demonstrate how these tools can be used and to put them in the hands of mathematicians. Among the means employed to achieve this transfer are workshops, visitor programs, courses, tours, books, articles, videos, interactive computer graphics, software development and documentation, and the Internet.

The People. The vitality of the Center is due to the many types of people involved with it. Central to all activities is the role of the administrative staff and of the technical staff (including apprentices). The technical staff ensure the quality of the computational environment: they maintain the hardware and software, provide advice and help, and support the Center's research, education and outreach activities. They also carry on a long-term program of software and video development, which includes a research component. Postdocs (generally appointed for two years) carry out their own research programs, and also take part in software development and outreach.

These "permanent" people create a stimulating environment for visitors representing the international mathematics and computer science research and education communities, as well as for University of Minnesota (U of M) faculty. Visitors come individually or in small groups, and engage in specific projects that would be very hard to carry out at their home institutions. In addition, we have a strong workshop program, which offers, in additional to the advantages of traditional workshops, working sessions in the lab with intense staff support. Visitors often leave with new software and a fuller appreciation of the use of computers in research and education, which they will disseminate at home.

Formerly, members of the Center faculty provided, together with the Director, the overall direction for the Center. These researchers, representing twelve institutions in the US and abroad, were in large part the source of the Center's vision. Several of them visited frequently to work and interact with the permanent staff, suggest projects, organize workshops, and bring other visitors with similar research interests to the Center.

In the future, this role will be played by the Center's Board of Governors.

Why a Center?

A Center concentrates people and resources, and can support activities that a university math department cannot. This makes the research, outreach and education done at a Center qualitatively different from that done at a math department. The work supported by the Geometry Center requires a high-performance graphics and video lab, and, even more importantly, a highly qualified technical staff. All aspects of the Center's program are built upon these resources. The cost of comparable resources would be hard to justify in any individual mathematics department; but on a national scale, this infrastructure is highly cost-effective.

The Geometry Center is an effective vehicle to transfer technology into math departments and industrial labs. We accomplish this through our visitor program, through workshops, through the training of apprentices and postdocs, and through the dissemination of free, high-quality software.

Research. One important role of the Center is as a computation and visualization laboratory designed for research mathematicians, and in particular those who are interested in software development. Visitors use computers, software and video equipment, and the assistance of the technical staff, apprentices and student programmers. These resources are completely unavailable in most departments.

The Center is also a meeting place for researchers sharing ideas and resources, both at workshops and through "accidental" contacts. **Jeff Weeks**, who created the research tool Snappea (page 96), wrote:

> The Geometry Center's full-time staff is a rare collection of individuals who are experts in both geometry and software development. . . The Geometry Center. . . has provided most of the stimulation for Snappea's development. Not only is it the best place to seek help with specific mathematical questions; it has also proven the best place for "accidentally" meeting other geometers with new ideas. . .

The impetus for Weeks's recent theoretical advances in algorithms for drilling and filling hyperbolic manifolds came from an unplanned conversation with **Joe Christy** of MSRI; both had come to the Center to get students started on a video.

Jean Taylor and **Fred Almgren** say of their **Minimal Surface Team**, which has made sustained progress in the understanding of "optimal geometries" (page 101), a branch of mathematics that promises to be rich in practical applications:

> This team would not have come into being had the Center not been created, nor could it continue to function in anything like the manner in which it does except for the resources the Center provides. . . Although each individual on the team is more than able to do good research work on his or her own, they have found that by working together they magnify the impact of ideas and efforts.

Ken Brakke, the author of the extraordinarily successful **Surface Evolver** program (page 105), puts it bluntly:

> The Evolver wouldn't exist if it weren't for the Center.

The Minimal Surface Team and the Center itself have become models for a team research environment: A group of well-known group theorists, geometers and computer scientists proposes the establishment of a Groups, Automata, and Semigroups Team (page 93), while researchers at NIST are proposing a **Center for Computational Materials Science** that borrows many features from the Geometry Center (page 41).

Interaction. The importance of places for visitors to interact away from their home institutions is clear. This interaction, of course, is what institutions such as MSRI and IMA have to offer. But the Geometry Center offers interaction and integration among *several classes of people*, each contributing in a different way.

The interplay of research, outreach and education is especially marked. A good example of this interplay is the development of the **QuasiTiler** program, which is why we chose it as the cover image for this book. The program was written by graduate **student programmer Eugenio Durand** for sabbatical visitor **Marjorie Senechal**, who was writing a book on quasicrystals. Postdocs **Paul Burchard** and **Leonidas Palios** advised Durand. Together with apprentice **Daeron Meyer**, Burchard and Durand adapted the program for the **World-Wide Web**, where it became widely known. "I'd almost forgotten how much I enjoyed mathematics," one network fan wrote about it. Computer scientist Marshall Bern tells us he used the program in a talk at Xerox PARC. Here at the Center, it is being used in a research project by a gifted high-school student.

As was the case for QuasiTiler, many projects at the Center are suggested by visitors and involve apprentices—motivated undergraduates and recent graduates in mathematics and computer science (page 120). Under the supervision of the senior technical staff, the apprentices take on major responsibilities and sometimes initiate their own projects. Similarly, Geometry Center postdocs have greater opportunities to lead projects than postdocs at traditional departments; an example is the new research and software development initiative called Pisces, whose team leader is a postdoc (page 140).

Innovative Outreach. The Center has been a leader in the use of new technology to convey the beauty and excitement of mathematics to a general audience.

The Center excels at communicating mathematics to the public through interactive computer graphics. The Center's **Triangle Tiling** exhibit at the Science Museum of Minnesota (page 31) was made possible because the Center had the necessary software, staff, and commitment to outreach. On the initiative of the apprentices and postdocs, the Geometry Center has become a leader in presenting educational mathematics using interactive computer graphics over the Internet, an increasingly popular forum (page 29). Again, the concentration of software and people makes it possible for the Center to do this much more effectively than it is done in math or computer science departments.

The animations *Not Knot* and *Outside In* (page 32) demonstrate the power of visualization to present the ideas behind sophisticated current mathematics. The creation of these animations required a great deal of effort, because of their high technical and artistic standards. This project could not had been realized at a traditional math department, and it would have cost five times as much if done commercially. Thanks in part to the expertise developed by working on these projects, other, less ambitious, animations are now routine at the Center: a visitor will come, do research, and document it with a video whose production can be left entirely in the hands of the staff.

The Center's experiments with new modes of education have been very rewarding, especially our two summer programs: the undergraduate **Summer Institute** (page 19) and the **Geometry and the Imagination** courses for high-school teachers (page 22). By relying heavily on the former **Center faculty**, we have been able to assemble excellent teams of instructors for the *Geometry and the Imagination* courses. Interacting with these world-class mathematicians is an important part of the students' experience in both programs.

Summary. A Center is a wise investment. It provides the mathematical community with a combination of hardware, software, and technical support that a university math department cannot. A Center provides opportunities, resources, and infrastructure that otherwise would not exist.

Overview: Outreach and Education

Education plays a major role in the Center's outreach program. The Center's **education director** is **Harvey Keynes**, and **Arnie Cutler** is supported half-time as an education consultant.

Undergraduate Education

The Center's **Summer Institute**, an innovative and successful program featured in the **AMS brochure** *What's Happening in the Mathematical Sciences*, brings together twenty undergraduates each summer, for a period of ten weeks, for research and training. Students come from a variety of backgrounds and institutions, but are united in being strongly motivated and interested. The program is directed by the "Head Coach", who for the past two years has been **Tony Phillips** of SUNY (Stony Brook). In consultation with him and other researchers, students develop appropriate summer-long projects involving computation and visualization, often leading to video animations of their results. Upon completion, students write individual reports that are gathered together and distributed as a Center Research Report. The last two summers' animations have been compiled into one-hour tapes that are distributed inexpensively by Media Magic. See page 19 for more details.

K–12 Education

Summer Course for High-School Teachers (page 22). The core of the the K–12 Education program at the Geometry Center is the summer course for teachers. It is offered yearly under the umbrella title "Geometry and the Imagination", but the topic changes from year to year. Each summer, world-class mathematicians introduce contemporary topics to approximately thirty teachers from seventh through twelfth grades. The ability of the Geometry Center to attract these presenters is a major factor in the success of the program, the courses are models of modern teaching practice: hands-on, interactive and challenging. The participating teachers are leaders in their schools and are expected to bring their new skills back to share with their colleagues. An expanded followup program to help track their progress will be in place starting with the 1994 course.

In past years, four to six persons have written materials related to the summer course that are accessible to their own students; this work was supported by a combination of funds from the Center and the Research Explorations for Teachers program operated out of the Extension Division of the University of Minnesota. These materials currently are being reviewed for publication by Janson Publications. Drafts are available at the Education display table.

University of Minnesota Talented Youth Mathematics Project (UMTYMP). Many of the mathematically talented youth of the Twin Cities area graduate from this nationally-recognized high-school program, headed by **Harvey Keynes**. In some ways, the Center acts as a "postgraduate" program for those who are interested in computing, as a large fraction of those accepted into the summer institute have also participated in the UMTYMP program. Beyond its core program of high-school courses, UMTYMP is steadily enlarging its enrichment activities and its outreach activities. The Center has combined efforts with UMTYMP in a variety of ways, of which we highlight two:

Algebraic Geometry in High School (page 25). During the 1993–94 academic year, the Geometry Center and UMTYMP offered an algebraic geometry course to talented Minnesota high school students, with weekly meetings in the Geometry Center classroom. **Victor Reiner**, of the U of M School of Mathematics, led the course, which makes use of the Center's hardware and Center-developed software to incorporate computational and visualization components into the course.

Summer Enrichment Course (page 24). In a joint venture with **UMTYMP**, the Geometry Center is developing curriculum for a four-week summer enrichment course for **female, minority, and economically disadvantaged students** in sixth through eighth grades. Five mornings of this course, which will take place in the summer of 1994, will be spent at the Center using Center-produced software. The curriculum will feature units on planar isometries, symmetry groups, and the Platonic and Archimedean solids. Students will engage in activities and projects for each unit that will involve using interactive visualization software as a discovery tool.

Mentoring (page 28). The Geometry Center's **mentoring** program gives selected local high-school students the opportunity to work closely with research mathematicians to help foster their own interests in mathematics, and broaden their mathematical experiences.

Other Initiatives. The Center interacts with a number of other education and outreach organizations, in Minnesota and nationally. These activities, directed specifically toward K–12 students and teachers, are detailed on page 26.

Global Outreach

The World-Wide Web (page 29). The **World-Wide Web** (WWW) is a global information system that uses a user-friendly hypertext approach to bring together a wide range of services available on the Internet. The Web has grown enormously popular in a short period of time, largely due to this unified approach. While still in its infancy, the Web already represents a powerful tool for information dissemination to an international audience and shows a great deal of promise for the future. The Geometry Center is a leader in using this technology for the communication of mathematics.

Museum Exhibit (page 31). We estimate that roughly 2500 people each week use the Geometry Center's exhibit at the Science Museum of Minnesota. This Triangle

Tiling exhibit allows a visitor to explore the connections between tilings, the Platonic and Archimedean solids. The exhibit is designed to appeal to everyone, from very young children to adults; it is accessible to the casual browser, yet rewards further investigation with deep mathematical content.

Videos. Computer animations play an increasingly important role at all levels of education and research. Consequently, the Geometry Center has involved itself in making videos of different kinds, from quick-and-dirty recordings of live sessions on workstations to the two professional-quality, feature-length animations *Not Knot* (which introduces ideas of knot theory and hyperbolic geometry) and *Outside In* (which introduces differential topology and culminates with a demonstration, due to Thurston, of Smale's result that the sphere can be turned inside out). Because the creation of these videos required a large commitment of resources, we offer, on page 32, a detailed discussion of the importance and effect of such high-quality animations.

The Geometry Forum (Swarthmore College) (page 34). This NSF-funded project establishes an electronic bulletin-board for geometric information and discussion, especially for high school students and teachers. The Center has a three-year subcontract to fund a half-time graduate student in math or math education to post Center-related material to the Forum based on interviews with Center participants and reports on Center activities. **Evelyn Sander** has been fulfilling this function with distinction.

Building a New World (page 34). The Geometry Center provided the mathematical expertise for "Building a New World", a collaborative project that brought together **Skyline Displays** (a local company), hundreds of U of M students and alumni, and some ten thousand grade-school children from schools across the state, to design, manufacture, assemble, and decorate a 42-foot, one-millionth-scale model of the Earth. This model has since been displayed at several sites.

AMS–MAA Mathfest. The joint summer meetings of the American Mathematical Society and the Mathematical Association of America will take place in Minneapolis on August 15–17, and have been nicknamed the **Mathfest**. The Center is sponsoring several exciting activities in connection with the Mathfest. The Center will present a two-day workshop on "Basic Issues in Computer-Aided Math Visualization" on the previous weekend; it is for mathematicians who have little knowledge of computer graphics (see page 35 for a copy of the program). A poster and video session displaying aspects of the Center's work will take place at the main meeting site. The Center will participate in the AMS special session *Computer graphics as a research tool in geometry and topology*, and in the panel *Exploring Mathematics on the Internet*. Finally, we will host an open house all afternoon and a reception in the evening of August 19, with demonstrations of software and videos produced at the Center, plus discussions with students, researchers and staff.

Tours (page 37). The Geometry Center hosts an active **tour** program that brings a wide range of people into contact with the activities of the Center. The many pictures, videotapes and demonstrations, as well as the attractive physical setting, communicate the idea that mathematics is an exciting and modern area of research.

Knowledge Transfer to Industry and Business

The Center transfers knowledge to industry and business through several channels. Our software, particularly the Evolver and Geomview, finds application to industrial problems. Programmers trained at the Geometry Center go on to jobs in industry. People from industry visit and attend workshops at the Center; staff members from the Geometry Center visit industrial labs, and both groups meet at conferences. These contacts sometimes lead to research projects for the postdocs or summer students.

A partial list of examples of knowledge transfer to industry and business can be found on page 40.

Overview: Research and Research Software

The Center research and software development components are intimately connected. Software development is both motivated by the needs of researchers, and conducive to new research. This interconnection enables the Center as a whole to achieve much more than would be possible for programmers working independently. In particular, each project benefits not only from the involvement of its primary architects and developers, but also from the role played by the Center staff—as part of the main development team or as technical consultants and graphics implementors.

This interconnectedness means that any presentation of this material involves arbitrary choices. We have chosen an outline that appears to minimize the drawbacks of a necessarily linear exposition, without implying a ranking between the various subjects. Accordingly, this section has two parts: one for the various research programs and the people that participate in them; one devoted to research and research software by area; and one that covers the "infrastructure", including non-research-specific software.

People and Programs

Postdoctoral Research Program. The **postdoc** program provides the Center with in-house mathematical and computational expertise, and an energetic group of talented researchers who can work with visitors, develop software, mentor students, and plan events. At the same time, the Center gives these young mathematicians valuable experience and a unique vision of mathematics. They take this experience and outlook with them when they leave the Center for industry or academia. A more detailed discussion of the postdoc program, including a summary of the accomplishments of the eight postdocs the Center has had so far, starts on page 45.

University of Minnesota Faculty. Three members of the School of Mathematics were also members of the Center Faculty: **Al Marden**, our Director until February 1994; **Richard McGehee**, our Interim Director; and **Harvey Keynes**, our Education Director. A few other members of the School of Mathematics have been closely involved with the Center, using the facilities to carry out their research, organizing programs and events, and participating in our weekly seminar series. They are the subject of a section starting on page 58.

Workshops. Most of the Center **workshops** are oriented towards computation, and have exploited the unique resources of the Geometry Center. Formal presentations are not the main focus; instead, participants interact informally in the laboratory, in front of the computers, with intensive help from the staff. This format is conducive to learning and collaboration, and participants report very productive experiences. We list the research workshops held in the Center's three years of existence, and those planned through the end of the 1994–95 academic year; details can be found starting on page 62.

1. Discrete Conformal Geometry Symposium, May 3–4, 1991
2. Computational Crystal Growers Workshop, February 24–28, 1992
3. Visualization of Invariant Sets for Symplectic Maps in Dimension Four, October 11–16, 1992
4. Knot Workshop, March 22–26, 1993
5. Advances in Nonlinear Astrodynamics, November 6–8, 1993
6. Geometric Group Theory, January 3–14, 1994
7. Elliptic and Parabolic Methods in Geometry, May 23–27, 1994
8. Topology and Geometry of DNA and RNA, July 29–30, 1994
9. Computational Geometry Software, January 18–20, 1995
10. Wavelet Analysis, Spring 1995
11. Noninvertible Dynamical Systems, Spring 1995
12. Computer Vision, Spring 1995

Visitor Program (page 66). The Geometry Center has a strong visitor program, which has brought over 200 **visitors** (apart from workshop participants) to the Center in the last three years. Visitors come to take advantage of the unique mix of hardware and staff support. Their projects include mathematical research and interaction with other researchers, software development, curriculum development, learning new software packages, creating animations and still images, and writing.

Major Research Areas

The main areas of research at the Geometry Center largely reflect the interests of the former **Center Faculty**, whose leadership role is summarized on page 90. In the future, new areas may be explored and new directions may be set by the **Board of Governors**.

Optimal Geometries (page 101). The **Minimal Surface Team** was established by **Jean Taylor** and **Fred Almgren** to study geometric optimization problems. The team includes **Ken Brakke**, whose **Surface Evolver** (page 105) is one of the most successful and widely used software packages from the Center. The Evolver has been used for practical problems outside mathematics, but also plays an essential part of much of the research at the Center. **John Sullivan** and others have used it to investigate Willmore surfaces, other quadratic curvature energies, knot energies, energies for embedded submanifolds, and other problems.

In recent years, the Minimal Surface Team has investigated problems of crystal growth, modeled with a combination of surface tension and heat-flow effects. The computer programs here have given insight essential for the development of new mathematical theories for such evolutions. Crystal growth was also the subject of a workshop (see page 62).

Combinatorial Group Theory. The Geometry Center, and its predecessor the Geometry Supercomputer Project, were to a large extent the cradle of the young and thriving branch of combinatorial group theory dealing with so-called **automatic groups** (see page 91). Work in this field proceeds apace. **David Epstein** and others have been able to prove significant new results in the last two years (page 91), in large part inspired by the use of the **automata** program developed in part with Geometry Supercomputer Center support.

The successes of the Minimal Surfaces Team have led David Epstein and other colleagues to propose the creation of an **Automata and Groups Team** along similar lines (page 93).

The Center hosted a workshop in January 1994 on Combinatorial Group Theory, which brought together twenty-five leading researchers in the field. This was yoked with a more theory-oriented workshop at DIMACS in March.

Three-Manifolds, Hyperbolic Geometry and Knot Theory. An apprentice, **Nathaniel Thurston**, has written a program to try to solve an important problem posed by **David Gabai**: briefly, to search in the six-dimensional parameter space of two-generator Kleinian groups for those that satisfy certain geometric conditions, such as the existence of a thick tube (in the appropriate sense) in the associated hyperbolic manifold. An affirmative solution to this problem would be a major step in a possible proof of the Thurston Geometrization Conjecture, and would lead to a strengthening of the Mostow Rigidity Theorem (page 98).

Jim Cannon's work on the combinatorial Riemann mapping theorem (page 95) was carried out substantially during visits to the Center. **Oliver Goodman** (page 50) has written several **Mathematica** packages for studying hyperbolic geometry. The program **Snappea** by **Jeff Weeks** (page 96) has led to many new results on hyperbolic manifolds and knots. Optimal configurations for knots have been studied by John Sullivan using the Evolver (page 101).

Computational Geometry (page 108). The Geometry Center is having a growing impact in **computational geometry**. Drawing on its special strengths, the Center is promoting the production of efficient computational geometry software and the communication of algorithms through computer animation, two rapidly growing areas. In addition, the Center is becoming known in the community through the research of Center postdocs **Brad Barber** (page 48), **Leonidas Palios** (page 51) and **Nina Amenta** (page 52). Barber developed **QuickHull**, an algorithm to compute convex hulls in arbitrary dimension taking into account imprecise data; he and a student implemented this algorithm and the second release of the program is in beta test. Palios studies the decomposition of three-dimensional objects into polyhedra, and has also considered problems in graph algorithms and VLSI design. Amenta works on geometric optimization problems related to linear programming.

A workshop on computational geometry software is planned for January 1995, with a list of distinguished participants. Two computer animations produced at the Center have appeared in the Annual Video Review of the ACM Symposium on Computational Geometry. See page 108 for details on these projects and their impact.

Dynamical Systems (page 111). The main focus of research in dynamical systems at the Geometry Center has been in understanding symplectic mappings, bifurcation surfaces, and KAM tori. Richard McGehee, his graduate students, and visitors, have heavily used existing Center tools and have also stimulated new software development. The workshop on the visualization of invariant sets of 4D symplectic mappings (page 63) showed that few tools were available for computing or visualizing structures associated with these mappings. Since then, the Geometry Center has begun to address these needs. Next year's conference on noninvertible maps is likely to produce similar results.

Computer Graphics. The mathematical visualization challenges faced by the technical staff lead to original research in computer graphics, notably techniques for the visualization of curved spaces in correct perspective, as if the viewer were "living" inside the space under consideration. See page 114.

Publications

Books. The belief that good communication is as important as good research has let us to exploit communication possibilities beyond the printed word, such as animations and the **World-Wide Web**. But also within the realm of traditional printed documents directed at an audience of peers, the Center has had an impact. We put an emphasis on clarity in writing and on the use of illustrations. Software was developed at the Center to facilitate the creation of mathematical documents in TEX, and especially the inclusion of figures (page 146).

Two important books owe their existence (at least in their current form) to the Geometry Center. The first, *Word Processing in Groups*, by David Epstein and others, was published by Jones and Bartlett in 1992; its history is detailed on page 91.

The impact of the Center on the second, **Bill Thurston**'s *Three-Dimensional Geometry and Topology* (to be published by Princeton University Press), is equally important. This book was extensively edited and illustrated by **Silvio Levy**, and significant contributions have been made by **Richard Canary**, David Epstein, **Bill Floyd**, **Steve Kerckhoff**, **Yair Minsky**, and **Lee Mosher** in the course of **book-writing workshops**. The involvement of other people means that standards of exposition are indeed high, and that many errors have been corrected. We believe that this work will break new ground for advanced texts in mathematics. Currently the book awaits finishing touches by Thurston, but the draft has been widely distributed because of high demand.

Other books that benefited greatly from their authors' stay at the Center are **Marjorie Senechal**'s *Quasicrystals and Geometry* (to appear in 1994, Cambridge Univ. Press) and **Alfred Gray**'s *Modern Differential Geometry of Curves and Surfaces* (CRC Press, 1993).

Experimental Mathematics (page 89). The Center supports the publication of *Experimental Mathematics*, a quarterly devoted to research results and conjectures arising from experimentation, as well as survey articles on areas of math where experimentation plays an important role. The journal is primarily edited by David Epstein and Silvio Levy, with Fred Almgren, **Henri Cohen**, **Robert Devaney**, **David Hoffman**, **Rafael de la Llave**, **David Mumford**, **Ulrich Pinkall**, and **Peter Sarnak** as Associate Editors.

Other Publications. A list of publications reported by their authors to have arisen from work done at the Center is given on page 86.

Overview: Infrastructure

Staff. We have a highly efficient **administrative staff**, consisting of the Administrative Director (Angie Vail), an accountant (Charlene Videen), a visitor program coordinator (Lee Raue), a senior secretary (Stan Glodowski) and a part-time student office assistant (Caryn Coopmans). Their work and dedication are crucial to the operation of the Center.

Equally central to all activities is the work of the **technical staff**, which takes many forms. The **senior staff** carries out a long-term program of software and video development, frequently in collaboration with visiting researchers, apprentices, and students. Many mathematicians are unaware of the amount of labor and expertise that goes into the creation of robust, well-documented and user-friendly software for distribution. The staff are knowledgeable about software engineering, interface design, and major programming paradigms such as object-oriented and functional programming. They work with researchers to identify appropriate data structures, and must devise standards for data exchange when they do not exist. Such issues require research in their own right. Video production, likewise, requires a variety of technical skills. The staff also ensures the quality of the Center's technical environment by maintaining the hardware and software and providing advice and help.

An **apprentice** (page 120) is a highly gifted and motivated student in mathematics or computer science, who works full-time at the Center for at least six months, typically during a leave of absence from college or graduate school. Usually we have three to five apprentices working on outreach activities and assisting the senior staff. They also have the opportunity for independent work, for example in mathematics research, software development or video production.

Student programmers are (generally) local undergraduate or graduate students who work part-time on programming projects assigned by the senior staff or postdocs. Examples of their work are described on page 3 and 25.

Hardware. The Center maintains a first-class laboratory featuring a variety of graphics workstations, personal computers, and video production equipment. We have a wide range of installed software to meet the needs of the staff and of the many visitors who come to make use of the Center's facilities. Most of our visitors do not have access to such equipment at their home institutions. Visitors frequently ask the technical staff for advice on hardware purchases, and use the Center's environment to try out various configurations that they are considering buying. The computational resources of the Center also are available over the Internet to many people who are not physically here.

Software. The Center supports the development and exchange of mathematical software. All software written at the Center is distributed free through the **Internet**. We focus on making programs widely usable and distributable, whether they be long-term staff projects like **Geomview**, ongoing projects such as **Evolver** and **Snappea** written by visiting researchers, or short-term projects by researchers and students.

Overview: Management

There have been major changes recently in the Center management, both internally and externally, both in structure and in personnel. A new supervisory structure for the Center staff is in place, with the goal of increasing the efficiency of the staff. A new governance structure has been established, with the goal of increasing the accountability and outreach to the national mathematical and scientific community.

Albert Marden, the original Center Director, resigned on February 14, 1994, and was replaced by **Richard McGehee** as Interim Director. There will be a national search for a new Director beginning in the summer of 1994.

Internal Structure. In August, 1993, the Center supervisory structure was relatively flat. Most of the Center staff reported directly to the Center Director. The exception was the administrative staff, who reported to the **Administrative Director**. There was a search in progress for a **Director of Technology**, but it was not yet complete. There was a plan to appoint an **Education Director** contingent on the funding for a major new proposal in materials development, but the proposal had not yet been submitted.

The search for a Director of Technology was completed in the fall of 1993, and **Silvio Levy** was appointed to the position. **Harvey Keynes** has agreed to resume the position of Education Director, which he held at the inception of the Center.

The new supervisory structure is reflected in the organizational chart on page 154. Changes to note are that the senior technical staff now report to the Director of Technology instead of to the Director and that the apprentices now report to members of the senior technical staff instead of to the Director.

External Structure. In August, 1993, the Center was governed by the **Center faculty**, a highly distinguished group of mathematicians and computer scientists who provided the scientific leadership of the Center. A subgroup of the Center faculty, the **Executive Committee**, was appointed to provide advice on management issues.

Among the criticisms of the previous management system were the following. First, there was a perception in the community of a "closed shop," that is, there was no mechanism for rotation among the faculty so as to provide a broader range of scientific interest and to insure significant outreach to the community. Second, the automatic funding for the Center faculty was provided without any analysis of how the activity furthered the goals of the Center. Finally, some of the Center postdocs were hired on the basis of one or two letters from Center faculty.

The role of the Center faculty has been replaced by the **Board of Governors**. The membership of this board will rotate as described in more detail in the **Center Governance** section, page 156. There will be no automatic funding for any project. The Board will grant proposed projects based on budgetary considerations and on how the projects further the Center's goals. The Board will review ongoing projects periodically. The **External Advisory Board** will provide further review. The Board of Governors will solicit project proposals from the community at large as well as from among its own membership, although members of the External Advisory Board will be excluded from submitting formal proposals for funding.

A new **Programs Committee** takes on the role of the former Executive Committee. The Programs Committee will be appointed by the Board of Governors and will be composed of persons actively involved in Center projects. This committee in turn will appoint subcommittees for specific functions, such as searching for postdocs and visitors, organizing a particular workshop, and selecting students for the summer institutes. The Programs Committee also will provide scientific and managerial advice to the Director and the Center staff.

With these changes the Geometry Center has addressed the major issues raised by the August site visit and is now in a position to increase dramatically its role as an engine of innovation in the way that mathematicians pursue research and education.

Documentation: Outreach and Education

Summer Institute

by **Olaf Holt**

The **Summer Institute** is a ten-week research and training program primarily directed toward undergraduates. It brings together a diverse group of people, including students, staff, and visitors, in an environment that encourages creativity and learning.

Backgrounds. Students have come from a variety of high schools and colleges, including Caltech, Carleton, Duke, Harvard, and the University of Minnesota. Most of these students are majoring in mathematics, computer science, or physics. The following table summarizes the composition of the classes:

year	high-school	college	minority	women	total
1991	5	11	2	2	16
1992	0	19	5	6	19
1993	2	18	3	9	20
1994	0	20	4	7	20

Program Outline. At the beginning of each summer, students attend "project-generating seminars", where visiting researchers from both academia and industry, Center postdocs and staff, and the "head coach" suggest possible summer projects. Project ideas have come from **Bill Thurston**, **Pat Hanrahan**, **John Conway**, and other distinguished scientists. For example, **Nicholas Coult**'s project on three-body dynamical systems was suggested by John Hubbard (see page 20 for descriptions of these projects), and **Blaise Morton** of Honeywell suggested **Mark Meloon**'s project on control systems. The students are invited to attend the two-week summer course for high school teachers (page 22), which has been a fertile source of project ideas. A few students come up with project ideas on their own, relating to other areas of interest they have already developed, like **Elizabeth Callaghan**'s diving simulation. The students are advised, both in their choice of topics and throughout the summer, by the "head coach". Tony Phillips, a topologist from SUNY–Stony Brook, is returning for his third year as head coach this summer. In 1991, the coaches were **Stan Wagon** of Macalester College, **Anthony Iano-Fletcher** of the University of Warwick, and **Jeff Ondich** of St. Olaf and Carleton. While most students work on individual projects, some form teams. Self-directed work on projects then fills most of the summer, with varying degrees of feedback and input from the project mentors, the head coach, and Center staff.

Students are encouraged to give informal seminars describing their work, and have many opportunities for contact with visitors to the Center throughout the summer. At the end of the program each student writes a project report, and most create an accompanying video. The reports are published in the Center preprint series and the videos are compiled and distributed commercially.

Available Resources. The students have a special appreciation for the people, equipment, and expertise at the Geometry Center. Acknowledgements at the end of student reports often express gratitude to those who suggested their projects, those who helped in their implementation (especially the Center staff), and those with whom they shared ideas. Some students focus on the personal contact: "The one experience from which I gained the most here is discussing mathematics with various people, learning new ideas and teaching what I know," wrote **David Ben-Zvi** in the 1992 summer report. Others focus on the technical resources: "The Center resources are great. . . The computer network was excellent. The only problem I encountered was that at peak times, all the Irises would be occupied." The students also take advantage of the staff expertise: "I am not the type to readily ask for help, but I felt more comfortable among the staff members than I do among most people I need to ask for help. It was rare for someone not to be able to answer my question."

Results. Student projects show an impressive depth of mathematical research and the creative application of tools. The Summer Institute blends inspiration from noted scientists with excellent facilities, in an atmosphere of friendly collaboration. Those who participate in the Summer Institute often go on to pursue graduate studies in mathematics or computer science.

The following short descriptions from an article by Tony Phillips in the Center newsletter summarize some of the students' projects:

Linktool, by **David Broman**, Rice University (1991). David wrote **Linktool**, an application for the NeXT that allows knot theorists to enter descriptions of knots, display them, and convert between different knot descriptions. It incorporates several previous programs written for knot manipulation, and has since been widely used by knot theorists. Linktool is available via anonymous ftp from the Center software archive.

The Three-Ring Circus, by **Jacques Friedman**, Brown University (1992). Jacques analyzed the motion of three concentric hoops, each of the two inner ones rotating about an axis fixed in the next one out, the outermost rotating about a fixed axis. The global dynamics of this system are extremely intricate. Following a suggestion of Bill Thurston, he illustrated them "optically" by examining image distortion when the kinetic energy function is interpreted as the metric tensor in a three-space.

Special Three-Body Systems, by **Nicholas Coult**, Carleton College (1992). Nick's project addressed the question, why does the moon always show us the same face? In this model, suggested by John Hubbard, the moon is represented by an orbiting system of two masses joined by a damped spring, and Nick's interactive simulation illustrates under what conditions the moon will stabilize with one face always toward us.

Penumbral Shadows, by **Adrian Mariano**, University of Washington, and **Linus Upson**, Princeton University (1992). Adrian and Linus investigated the shadows cast by a polygonal occluding object illuminated by a polygonal light source. The light intensity distribution is very complex. Using a direct simulation of the illumination process, they noticed and explained the close relationship between these shadows and projections of four dimensional polytopes. (The importance of this work is discussed on page 109; their video will appear in this year's ACM Symposium on Computational Geometry videotape).

Lindemayer Systems and Space-Filling Music, by **Stephanie Mason**, Virginia Polytechnic Institute and State University (1992). Stephanie interpreted space filling curves as musical scores, following P. Prusinkiewicz. In particular, she generated right-angle canons by playing a curve, rotated ninety degrees, against itself as a second voice. These curves are generated by Lindemayer systems (L-systems), and the consequent self-similarity gives coherence to the score.

The Diving Simulator, by **Elizabeth Callaghan**, University of Minnesota (1993). Elizabeth, a diver herself, devised an interactive program that simulates various dives used in competition. The user chooses the type of dive (forward, backward, inward), sets parameters (height and weight of diver, takeoff angle and speed) and then adjusts the times at which the diver begins and ends the moment-of-inertia changing maneuvers which, together with the law of conservation of angular momentum, shape the dive. The computer solves the differential equations and a **Geomview** mannequin executes the actual dive.

Navigating Teichmüller Space, by **Deva Van Der Werf**, Bowdoin College (1993). Deva worked with several other students in complex analysis on software to explore various models of Riemann surfaces. Her program, the **Teichmüller Navigator**, allows one to move around in the space of all Riemann surfaces of genus two, by deforming an octahedral tiling of the hyperbolic plane. The user initiates the deformation by moving any vertex to a new location with the mouse; her program finds the "hyperbolically closest" tilable octagon, and then creates the tiling. (Center staff have subsequently adapted this program for the **World-Wide Web**; see page 29.)

> "I think this is one of the key ingredients of the Center: math is alive!!! And I thank Al Marden and the Center staff for making it this way."
>
> – **Craig Sutton**, Summer Institute project report, 1992.

Educational Programs for Teachers

by **Arnie Cutler**

The core of the the K–12 Education program at the Geometry Center is the summer course for teachers. It is offered yearly under the umbrella title **Geometry and the Imagination**, but the topic changes each year:

- 1991 *Geometry and the Imagination*. Instructors: **Bill Thurston**, **Peter Doyle**, **John Conway** and **Jane Gilman**
- 1992 *Geometry and the Imagination: Chaos and Fractals*. Instructors: **John Hubbard**, **Bodil Branner**, **Bjørn Felsager** and **Mette Vedelsby**
- 1993 *Geometry and the Imagination: Computation, Visualization and Graphics*. Instructors: **David Dobkin**, **Pat Hanrahan**, **Diane Souvaine** and **Vibeke Sorensen**
- 1994 *Geometry and the Imagination: Chance*. Instructors: **Peter Doyle**, **J. Laurie Snell** and **Joan Garfield**

Each year, world class mathematicians introduce contemporary topics to approximately thirty teachers from seventh through twelfth grades. The ability of the Geometry Center to attract these presenters is a major factor in the success of the program, and the courses are models of modern teaching practice: hands on, interactive and challenging. One of my roles is to facilitate communication between the instructors and the course participants, to assure that the level of challenge is high but attainable. The course is residential and includes evening and weekend activities.

Beyond giving the teachers current mathematical content for their own classes, this course provides them with invaluable contacts with the presenters. For many, this is the first time that they have actually worked with or known a world-class mathematician. It positions them to talk to their students about what mathematicians do, and how that relates to high-school mathematics. What a difference it makes in a classroom to be able to say "John Hubbard last summer said. . . " rather than just quoting a journal article written by John Hubbard. It encourages discussion about careers in mathematics and enhances a teacher's credibility.

Teachers are selected so as to balance rural, suburban, and urban participation. We have enough quality applications to be able to select an equal number of males and females and have had some representation from teachers in groups that are typically underrepresented.

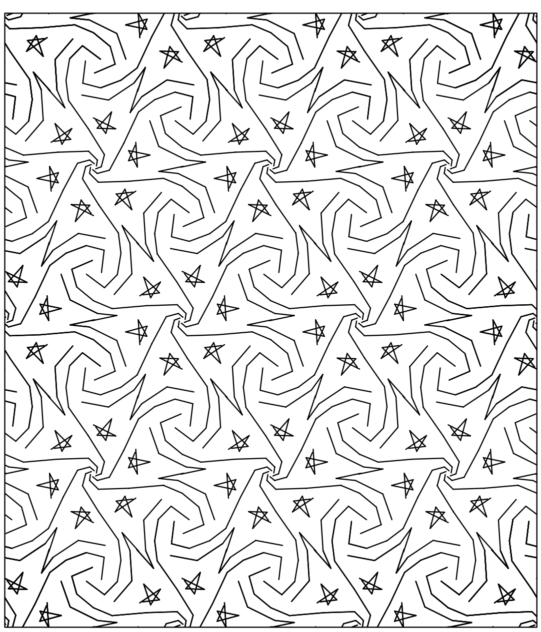
The basic model of the course is to "train the trainer": we invite teachers who are leaders in their schools and expect that they will share their new knowledge with their colleagues. Their school principals certify that they will encourage the teachers to use the new content in at least one course in the following year, and as a result, many teachers have given presentations on the topics of the courses at regional meetings of the **MCTM** and **NCTM** (Minnesota and National Councils of Teachers of Mathematics). Some have sent us local newspaper clippings of work their students have done based on the course materials. All participants get **Internet** accounts through the Center, so that they can communicate with the Center and each other electronically throughout the year.

The teachers filled out evaluations after each summer course. To date, they have shown overwhelming support for the philosophy of these courses. In most cases the content was unfamiliar, was liked by the participants, was usable in classrooms, stretched participants minds and was well learned in an intense yet relaxed environment. Summaries of the annual surveys appear at the Education display table.

The major advertising for the summer course has been in the Upper Midwest and primarily in Minnesota using MCTM, **M**^{3} (another teacher network), and the State Department of Education. This year we have announced on the **Geometry Forum**, and a number of out-of-state participants have enrolled. We would prefer to recruit participants nationally, and for 1995 we plan to recruit through the magazines *Mathematics Teacher* and *Teaching Mathematics in the Middle School*.

Followup. Teachers who enroll for the full six-credit program are expected to complete a related project after the course is over. Moreover, in cooperation with the Extension Division of the University of Minnesota, the Center supports selected summer course participants to continue on with their research and learning, and to write **curriculum** related to the course. This program, jointly funded by the Center and the Extension Division, is an expansion of the Division's Research Explorations for Teachers program. Six teachers were funded in 1992 and four in 1993; the resulting curriculum materials are being reviewed by **Janson Publications** for possible publication, and are available for viewing at the Education display table.

In response to comments at the 1993 site visit on the need for a more active followup on the summer course teachers, we arranged a "reunion" for the 1993 summer course teachers. David Dobkin, one of the course instructors, came to the Center and participants were invited to return and discuss their work with David and each other. Not many were able to attend because of scheduling problems, but many others asked for summaries of the day's activities. This year we plan to establish a number of reunion dates while the course is in session, so that more participants can attend.



This figure, created with **kali**, an interactive symmetry pattern editor written by **Nina Amenta**, is taken from the Symmetry and Orbifolds section of the 1991 Geometry and the Imagination course notes, which are available on-line through the **World-Wide Web** (page 29). The caption in the course notes reads:

This pattern has rotational symmetry about various points, but no reflections. The rotations are of order 6, 3 and 2. The quotient orbifold is a triangular pillow, with three cone points.

Summer Enrichment Program for Females, Underrepresented Minorities, and Economically Disadvantaged Students in Grades 6–8

by **Rick Wicklin**

In a joint venture with the University of Minnesota Talented Youth in Mathematics Program (**UMTYMP**), the Geometry Center is developing curriculum for a four-week summer enrichment course to take place in the summer of 1994. Five mornings of this course will be spent at the Center using Center-produced software. The curriculum will feature units on planar isometries, symmetry groups, and the Platonic and Archimedean solids. Students will engage in activities and projects for each unit that will involve using interactive visualization software as a discovery and research tool. For example, students will be able to use software written at the Center to produce patterns that are invariant under certain groups, and to continuously deform Platonic solids into related Archimedean solids.

The goal of this program is to motivate young women, minorities, and economically disadvantaged students to pursue interests in the mathematical sciences, and to make mathematics more accessible by using technology to help visualize mathematical objects. Eighty students will participate in the first two weeks of activities; twenty-one of those students will continue on for the second two weeks.

During the first two weeks, participants will engage in activities where the mathematical focus is directly linked to invited speakers (including professional mathematicians and scientists), exploration of career opportunities, tours of local business and industry that use mathematics in daily operations, demonstrations by faculty from related disciplines, and visits to research facilities such as the virtual reality lab on the campus of the University of Minnesota. Prior to each event, mathematical activities will be structured to give participants a foundation from which they can formulate questions, express interest and engage in discussions during the event.

The last two weeks of the summer institute will be focused on geometry and will make extensive use of the Center. A possible "final project" of the solids unit may be the construction of a large model icosahedron (the symbol of the Mathematics Association of America) for display at the **Mathfest** (page 35).

The current draft of the curriculum includes handouts and a teachers guide, and is available at the Education display table.

This joint venture is an outgrowth of the tour program: for the past two summers, one morning of the course was devoted to a group tour at the Center. The curriculum development fully integrates the unique resources of the Center into the course, and will furthermore be a resource for all teachers who bring school groups on tours of the Center in the future.

Advanced UMTYMP Courses at the Geometry Center

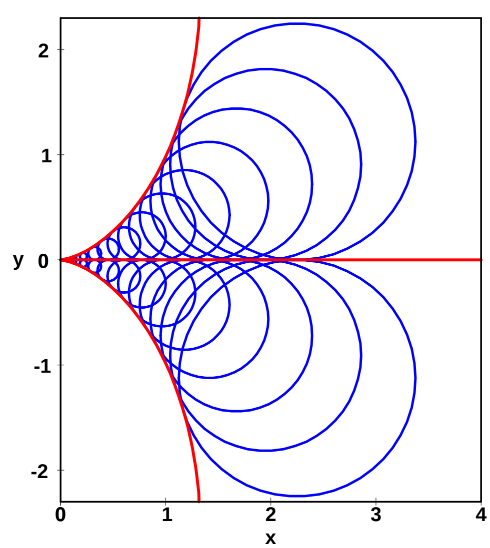
During the 1993–94 academic year, the Geometry Center teamed up with the University of Minnesota Talented Youth in Mathematics Program **(UMTYMP)** to present an algebraic geometry course to talented Minnesota high-school students. The course is held weekly in the Geometry Center classroom. The presence of graphic workstations allows the instructor, **Victor Reiner**, School of Mathematics, to easily integrate computational and visualization components into the course.

Besides using commercial symbolic algebra software to help with difficult or unwieldy algebraic computations, students in this course use software written by Center student workers **Scott Wisdom** and **Dan Krech**. The software computes and displays affine and projective plane algebraic curves—one of the primary objects of study in algebraic geometry. Laboratory exercises help to build students' intuition and increase their understanding of abstract formulations; computer-aided visualization of examples reinforces fundamental theorems. Center postdoc **Frederick Wicklin** helps with the course and serves as a lab assistant during class meetings.

This experimental course has been a great success. The course will almost certainly be taught in subsequent years as a junior-level undergraduate course. Difficult concepts such as Gröbner bases and singular points on curves are discussed with real understanding because the students have actually computed and seen these objects. The students have a firm grasp on the importance of algorithms in modern algebraic geometry. Proofs tend to be constructive and algorithmic. Examples are often presented graphically as well as algebraically. The students are absorbing material that was once thought to be far beyond their abilities.

Next year, Center postdocs Wicklin and **Davide Cervone** will be teaching two sections of UMTYMP multivariable calculus and will use computation and visualization tools to help present the main ideas and concepts of calculus and differential equations.

A family of circles and its envelope



Other K–12 Outreach Initiatives

by **Arnie Cutler**

Gallery of Mathematicians. Building on an idea of Nancy Guldberg, a summer course participant, we have started a **Gallery of Mathematicians**, to counteract the common impression that all mathematics was created prior to the twentieth century. The Center visitor roster appears to be a good starting point for a project designed to show students contemporary mathematicians, what they are like, and what they do.

Each visitor to the Center is asked to be a participant in the Gallery. This means that they permit their picture to be taken and used in posters, calendars, etc., and that they write a short paragraph directed to school students. This paragraph might indicate what they do, it might include some of the things that they enjoy outside of mathematics, it might talk about their education or family, with the goal of showing that mathematicians are alive, human, employed in a variety of occupations, and enjoy life!

Currently, we have over fifty photos and statements ready for use, pending the signing of release forms. We will initially publish a poster with six persons featured and a removable calendar attached. Details of a contract with **Janson Publications** are being finalized and the poster should be on the market in time for the opening of school in the fall.

Geometry Software Conference. During the summer of 1993, under a planning grant from the NSF and the leadership of **Gene Klotz** (Swarthmore College), we hosted a **Geometry Software Conference**. The task of the invited participants was to study the current state of visualization as it pertains to K–12 education, and make recommendations for future developments. Representatives were present from all of the major K–12 curriculum projects funded by NSF, software developers, mathematics education professionals, testing experts, mathematicians and publishers.

The overwhelming outcome was the need for more user-friendly and adaptable software for 3D viewing. Students live in a three-dimensional world and they often appear more interested in spatial than planar geometry. Intuition and judgements about three-dimensional objects, and in particular how they are represented on a two-dimensional screen, is required in industry, in science , and in many aspects of everyday life. There is a pressing need for a technologically literate work force. Unfortunately, three-dimensional geometry has all but disappeared from both the school and college curricula. A recent analysis of existing educational software carried out by a publisher found three-dimensional software almost totally lacking.

The conference presented an opportunity for interaction between portions of the mathematics and mathematics education community that rarely communicate.

K–12 Networking and Collaborations. One of the goals of the K–12 Education Outreach has been to encourage networking among the various groups that have an interest in improving mathematics education. For example:

- The Center is a cooperating organization with the Educational Cooperative Service Unit of the Metropolitan Twin City Area in the Teacher Enhancement Proposal to NSF entitled Enhancing Mathematics and Science Teaching by Using Computational Mathematics. The Center will offer tours to participants and make software developed at the Center available to training sites.

- The Center is a cooperating organization with Technology and Information Educational Services (TIES) and the Anoka Hennepin School District in an Annenberg-funded project entitled Teacher On-Line Projects (TOPS). The project links rural schools in the Upper Midwest electronically to exchange information concerning community projects which each community develops to solve a local problem. The Center provides a person to serve on their Advisory Board and offers tours and assistance with e-mail access.
- The Center collaborates with Sci/Math$^{\mathrm{MN}}$ to effect improved math and science education. Sci/Math$^{\mathrm{MN}}$ grew out of a State Systemic Initiative (SSI) proposal that was not funded. With funding from the state and Cray, Honeywell, Medtronic, and Alliant Techsystems, they are sponsoring a teacher academy and working to improve the graduation requirements in math and science. The Center hosted planning meetings in the formative stage and continues to host meetings. One of my functions is to be an advocate for appropriate content and performance standards as the graduation rule is revised.
- The Center interacts with the **NCTM** (National Council of Teachers of Mathematics) and the **MCTM** (the Minnesota council, of which I am currently president). The Center hosts the Delegate Assembly of the MCTM annually and Board of Directors meetings as necessary. Speakers from the summer course and the Center have been on the programs of the Fall and Spring Conferences of the MCTM and the Minneapolis Regional Conference of the NCTM.
- The Center maintains an ongoing relationship with the Geometry Group of the National Center for Research in the Mathematical Sciences Education at Madison, Wisconsin. This group held a convening conference in fall 1992 and continues to meet at ASCD (Association for Supervision and Curriculum Development) meetings and last summer at the Geometry Software Conference. Ideas about the status of research in pedagogy surrounding geometry is the main topic of interest to this group.
- The Center works with the High Tech Council of Minnesota in collaborative ways to promote technology education. Member companies were cooperating organizations in the Materials Development Proposal submitted by the Center and Gene Klotz to NSF.
- The Center collaborates with the Minnesota Mathematics Mobilization (**M**^{3}**)**. The Center provides regular information for the newsletter and hosts workshops sponsored by the Mobilization. The Center's Education Director, **Harvey Keynes**, is the Codirector of the Mobilization, and I serve on the Board of Directors of the Mobilization.
- The Center supports the NSF St. Olaf Geometry project by providing e-mail access to participants in the St. Olaf project who live in the metro area and thereby can access their St. Olaf colleagues without a long distance phone charge.

Mentoring

In the 1991–92 and 1992–93 academic years, the Center experimented with an informal **mentoring** program for **high-school students** who learned math and computer graphics through individual projects guided by a staff member. For the pilot year, staff member **Tamara Munzner** mentored four gifted female high-school students taking upper-level **UMTYMP** calculus classes. The goal of the program was to encourage women to stay involved in computing and mathematics, and the students chose the Fibonacci Sequence, photorealistic rendering, mathematical surfaces, and 3D computer graphics as project topics. The next year, four non-gifted high-school students who might not have otherwise been encouraged to continue on in math and computing were recruited. The **apprentices** and staff member **Mark Phillips** mentored the two male and two female students, who used computer graphics software developed at the Center to create animations. While the eight students had valuable experiences, we now believe that the unique computational and human resources of the Center are more effectively harnessed through interactive **tours** that reach a larger audience of students.

The **Mentor Connection** has placed one student each year at the Center for the past two years. This formal mentoring program provides highly motivated students with extensive in-class preparation before the biweekly meetings with the mentor begin, and weekly meetings with the other students for the duration of the mentorship. Last year a student mentored by Mark Phillips wrote a ray tracer for a computer graphics programming project, and this year postdoc **Nina Amenta** is working with a student on problems in combinatorial and computational geometry.

Snapshot from "School", an animation by Maria Nagan, a high-school student mentored by Tamara Munzner. The modeling was done in Mathematica using parametrized quadric surfaces, and the rendering in Geomview. The animation can be seen at the Video display table.

The World-Wide Web

by **Paul Burchard**

The **World-Wide Web** (WWW) is a global information system that uses hypertext to retrieve information from Internet-connected sites around the world. Over the past year, the Web has gained enormous popularity as a result of **Mosaic**, a multimedia navigation tool developed at the National Center for Supercomputing Applications (NCSA), with funding from NSF. One of the strongest advantages to the Web is that it brings together diverse services under a unified and essentially platform-independent interface.

The Geometry Center has been a part of the Web for nearly half a year. During that time, the Web has become an important part of the way that the Center accomplishes its mission. Moreover, the Center has become a recognized leader in applying this technology for the communication of mathematics: our Web pages feature prominently in the Best of the Web '94 competition, now being conducted in association with the First International Conference on the World-Wide Web (the conference is hosted by CERN and the Centre Universitaire d'Informatique of the University of Geneva). The Center has been nominated for awards in three of the fourteen categories: best educational service, best use of interaction, and most technical merit. Winners will be announced at the conference on May 26.

Reaching more people with graphical mathematics. Many graphically stunning interactive illustrations of mathematical concepts have been developed at the Center, based largely on the advanced graphical capabilities of SGI and NeXT computers. Now we have begun to make these programs available to a much wider audience by using Mosaic as a "multimedia terminal" that allows users to run these applications remotely. (See page 145 for a description of **W3Kit**, the underlying Center-developed technology that makes this possible.)

Currently we have six programs that make up the Geometry Center's "Interactive Gallery" on the Web, as listed in the table on page 30. Each of these applications includes an explanation of the mathematics involved and interactive controls that allow users to adjust the parameters for the objects they are viewing. What makes this unusual is that users do not have to download or set up any special software (other than Mosaic itself) to use these programs. By installing a single program (Mosaic) remote users gain access to all the available WWW applications. This offers an appealing platform for the distribution of mathematical software applications because it greatly simplifies the task of distribution for both the end user and the developer.

Although the system is limited by the speed of current networks, many mathematical applications can benefit from distribution on the Web. For the user, the complexity of downloading, compiling, installing, and periodically upgrading software is replaced by a few mouse clicks. At the other end, the software author can provide access to users on a wide variety of computer platforms by setting up a server program on a single machine. The program runs on the server and the results are transferred to and displayed on the user's machine. This is quite tempting in view of the time required to port, compile, package, and distribute conventional software for multiple platforms.

QuasiTiler	Generates Penrose and other quasiperiodic tilings.
Lafite	Calculates the fundamental region and generators for any discrete symmetry group of the hyperbolic plane, and draws the corresponding tiling.
Kali	Draws symmetry patterns using any of the seventeen planar symmetry groups.
Cyberview	An semi-interactive 3D viewer closely related to Geomview.
Unifweb	Displays a Riemann surface with a given group of symmetries.
Teichmüller Navigator	Explores the space of all angle geometries on a surface of genus two.

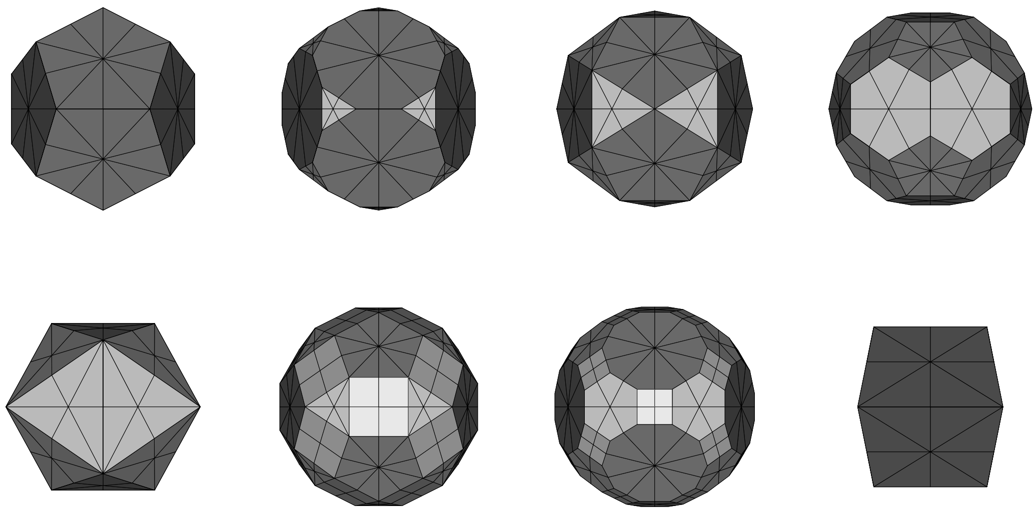
Taking advantage of multimedia. The Web is an ideal medium for presenting educational materials. At a basic level, it offers the benefits of hypertext so that explanatory links can be placed at any point in the text without breaking the flow of ideas. In addition, Mosaic makes it easy to add images, movies, and sounds when words do not suffice. The Center is offering a rapidly growing body of hypertext documents to the network community, including the **Geometry and the Imagination** course notes (see page 22), the illustrated **Geometry Forum** articles by **Evelyn Sander** (page 34), apprentice **Olaf Holt**'s Tesseract demo, which uses animation to explain the concept of higher dimensions, and graduate student **Eugenio Durand**'s documentation about quasiperiodic tilings based on **Marjorie Senechal**'s notes. Thousands of people have looked at the latter and gone on to use the QuasiTiler application remotely.

Finding new growth paths for existing center software. Geomview makes a good external viewer for 3D objects in Mosaic, because it is free, widely portable, and easily extensible. The idea of using Geomview as a viewer for 3D objects on the Web came from the Center for Innovative Computer Applications (CICA), who explained how to do this in their Web page on 3D file formats. Extensibility is critical here, because it allows Geomview to be adapted to serve as a viewer for new types of 3D data. For example, in collaboration with the Keck Center for Genome Informatics, we are investigating the use of **Geomview** as a viewer for Protein Data Bank files.

The Web enhances our distribution of existing software. Apprentice Daeron Meyer used tools he developed at the Center to put together a Motif interface for MPEG, a popular movie-playing program written at the Computer Science Division at Berkeley. Thousands have come to his Web page to download it, and out of those, hundreds who probably had never heard of the Center before have gone on to read more about the Geometry Center, and even downloaded Geomview.

Future plans. We hope to make more tools available as W3Kit applications (page 145); summer students can publish their work as WWW pages or W3Kit applications; we can summarize the important results from conferences or even place their proceedings on the Web; and we can have on-line application forms for our various programs.

The Geometry Center will continue to be a leader in the use of the Web to further the understanding and appreciation of mathematics.



Museum Exhibit

by **Tamara Munzner**

The Triangle Tiling **museum exhibit** is a collaboration between the Geometry Center and the Science Museum of Minnesota. Every week roughly 10,000 people see the Geometry Center's exhibit at the museum in St. Paul; about 25% of that number stop to investigate it. The exhibit allows museum visitors to explore the connections between symmetry groups, tiling, the Platonic and Archimedean solids, and non-Euclidean geometry through interactive 3D graphics. The exhibit was designed for the science museum environment with visitors ranging from very young children to adults, so it is accessible to the casual browser, yet also informs and entertains mathematically literate viewers.

Software by **Charlie Gunn** was adapted for museum use by **Stuart Levy** and **Tamara Munzner**, with assistance from **Olaf Holt**. The software is built on top of **Geomview**, and runs on an SGI workstation. The exhibit consists of explanatory text and graphics in addition to a workstation running the software. Both the customization of the software and the development of explanatory materials was done in collaboration with museum exhibit developers, who have also created a "Geometry Lab" with manipulatives like Polydrons and joined mirrors. Center staff have devoted roughly three person-months of work to the project. The exhibit has been under development for the past year, and an evolving version has been installed at the museum's hands-on Experiment Gallery for the past six months.

The Triangle Tiling program has been used extensively at the Center itself during interactive tours. The exhibit has been accepted for display at SIGGRAPH, the computer graphics conference, in July 1994. As part of "The Edge", a gallery of interactive graphics applications, it will be seen by many of the 30,000 attendees.

The museum exhibit is duplicated at the Museum display table.

Not Knot and *Outside In*

by **David Epstein**

The Geometry Center has produced two professional-quality computer-generated mathematical animations: *Not Knot* and *Outside In*. We started *Not Knot* and *Outside In* with the objective of explaining some particular mathematical ideas. We then tried to make the movies comprehensible to as large an audience as possible. For many mathematicians, the definition-theorem-proof approach is the only acceptable means of communication; this is one of several reasons why so many research seminars and lecture courses are incomprehensible to an overwhelming proportion of their audiences. While definition-theorem-proof will always remain an essential aspect of the mathematician's work, this is not necessarily the most efficient method of communicating ideas. A great deal of effort goes into the process of coding a picture in one mathematician's head into symbols and then decoding the symbols into a picture in another mathematician's head.

The goals. Here are some reasons for the creation of *Not Knot* and *Outside In*.

- The importance of outreach has been more widely acknowledged in the scientific community in recent years, and our movies form part of a movement to inform and explain. All of us depend either directly or indirectly on the good opinion of Congress and the general public. We believe that our movies make an unusual and effective contribution in this direction, as an antidote to the belief of 90% of the world's educated population that has been no new mathematics for centuries.
- We need to attract the best minds of the next generation into science instead of into areas that may offer society and the individual more in the short term, but in the long term lead to stagnation. Those who were at the world première of *Not Knot* will have observed how it intrigued and puzzled Fields Medalists. A postcard at the Movies display table shows a similar effect on a nine-year-old girl. We doubt there has ever been anywhere an exposition of mathematics that has appealed to so many people of such diverse backgrounds.
- It is important for staff at the Geometry Center to be familiar with the techniques necessary to make videos of the highest standard, so they can be in a position to advise visiting mathematicians and computer scientists when they need to make videos about their own work.
- Many people have learned a lot of mathematics by watching our videos. Of course, in the training of professional mathematicians, videos are likely to be only a supplement to a more formal education.

The success of *Not Knot*. *Not Knot* has won several awards: it was shown at the Electronic Theater, SIGGRAPH 1992 (this shows the best in computer graphics for the year); it received the AVA Multimedia 1991 Award in the Computer Graphics Category at Nicograph and the Nicograph Tenth Anniversary Special Prize (Nicograph is an international computer graphics meeting sponsored by the Nippon Computer Graphics Association); and it received a "Distinction of Prix Ars Electronica 1992" (this international competition is sponsored by the Austrian Broadcasting Corporation).

Not Knot has sold over 4,000 copies and continues to sell. It is one of the best-selling mathematical videos ever made. To gauge its effects, one also needs to bear in mind that it is often shown to substantial audiences. **Klaus Peters**, who has thirty years experience in publishing mathematics, says: "[This sales level] is a remarkable testimony to the appeal of this product. It has reached viewers from the elementary school level through high schools and colleges to research mathematicians and computer scientists. Teachers and students [say] that the video conveys the excitement of contemporary work in mathematics. . . It has demonstrated the potential of sophisticated computer graphics and animation in the visualization of mathematical research. The awards it has won show the appreciation of the professional community in this regard."

The influential magazine *The Mathematics Teacher* concluded a review of *Not Knot* as follows:

> The best thing about this product is the direct visual access it offers to some of the most important and beautiful contemporary mathematical ideas
>
> Suggestion: Any large or even moderate-sized city in the United States has universities with mathematicians acquainted with the work of Thurston [and] Mostow. . . A high school class might view the tape a few times, read the supplement, record questions and problems, and then invite a geometer from a local university for an afternoon of discussion and questioning. Professional mathematicians should let high schools know that they are willing and eager to get involved in such activities to increase public understanding and awareness of their work.

We realize that the NSF's goal is not to make aesthetic contributions to our culture. Nevertheless, it is significant that *Not Knot* has entered the popular culture to the extent that the rock group Grateful Dead has licensed *Not Knot* for use "in the entire universe", and Odyssey Productions has included an excerpt from *Not Knot* in the popular computer graphics compilation *Beyond the Mind's Eye*. Although this does not make a direct contribution to mathematical understanding, it indicates to the general public the power of visualization as a medium and the power of mathematics to create such pictures.

Not Knot has been used as the main source for a graduate course on geometry at Warwick University, as the basis of many popular talks on mathematics given to non-specialist audiences, and as a bedtime story for a six-year old, who learned the script word for word. Clearly, different ideas are communicated to people at different levels; but at all levels it communicates the wonder and beauty of mathematics.

According to **Delle Maxwell**, *Not Knot*'s art director and a longtime consultant for firms that produce computer animations, the cost of making *Not Knot* commercially would have been between one and two million dollars.

The Potential of *Outside In*. *Outside In* demonstrates a proof due to **Thurston** that the sphere can be turned inside out. The proof is by "corrugations", which is a widely applicable method of integrating structures only given at the infinitesimal level. This technique was pioneered by **Gromov**, Thurston and **Eliashberg** and is explained in Gromov's book *Partial Differential Relations*. It is possible that *Outside In* will help to spread knowledge of the power of this method.

Showings of *Outside In* drafts to audiences of undergraduates, graduate students and mathematicians indicate that it will be as successful as *Not Knot*. The movie will be exhibited at SIGGRAPH 1994 and the 1994 SIAM annual meeting. A supplement is in preparation.

Writing for the Geometry Forum

by **Evelyn Sander**

I am a mathematics graduate student employed as a writer for a set of NSF-funded Internet newsgroups called the Geometry Forum. I interview visitors and staff at the Geometry Center and write articles about their mathematical and educational ideas. The articles are short but informative accounts of the Center's activity, on topics such as available math software, current education ideas, and active areas of research. Despite the sophistication of the math, the articles have few formulas and most are accessible to those with a background in calculus.

My Forum articles have met with a very positive response. From researchers to high-school teachers, readers often continue discussing ideas from articles on the Forum. They also indicate that they download software, request Geometry Center preprints, make contact with researchers who have been featured, and distribute articles to colleagues and students.

Over twenty articles have been written, on all aspects of the Center's activities: education (for instance, Vic Reiner's algebraic geometry course, *Outside In*, Building a New World); software (qhull, Hyperbolic.m, QuasiTiler); and research (among others topics: automatic groups, ballistic lunar capture, dendritic growth, visualization, Hamiltonian systems, quantum filed theory, quasicrystals, Seifert conjecture). The articles are available at the Outreach display table.

Building a New World

Staff member **Stuart Levy** was the technical engine behind **"Building a New World"**, a 1993 U of M Institute of Technology project that drew the involvement and captured the imagination of dozens of companies, hundreds of from schools across the state.

The project's tangible result was a 42-foot, one-millionth-scale model of the Earth in the form of a geodesic sphere. Painting the world's geography onto each of its 1620 facets became a class project for the thousands of fourth- through ninth-grade students who assembled it in a ceremony on the University campus in May of 1993.

The Center's role was to design the geodesic sphere, provide assembly maps, and produce template maps of geographic features for each of the globe's facets.

Several consequences have followed from the project. The globe itself has been publicly displayed in other places, such as the Concordia language camps, the Mall of America in Minneapolis, and the Minnesota State Fair. Publicity from the project has led to several inquiries to the Geometry Center, varying from geographers interested in mapping large databases onto the globe to carpenters curious about building geodesic structures. The 1993 **summer course** for high school teachers (page 22) included a spinoff from the project: a class exercise was the assembly of a miniature globe, simple enough to be built in a single classroom in a few hours. Summer course attendees took back to their schools copies of the templates, which we also make available to other teachers on request. Finally, **Skyline Displays**, the Minneapolis firm that did the structural design, is providing the Center with a grant to write software that will translate from geographic coordinates to their grid system; this software will be incorporated into several projects of the company.

Tours

The Center conducts tours for a wide range of interested people—from research mathematicians, scientists and corporate officers to politicians, science writers, and students at all levels. People requesting tours have heard of the Center through a wide variety of channels. Members of the general public are often introduced to the Center by magazine articles such as the **Discover** or **Scientific American** cover stories (page 39). Many teachers who attended the **Geometry and the Imagination** course later bring classes of students for group tours.

Tours usually consist of an overview of the Center and its mission, demonstrations of Center software, and showings of our computer-generated mathematical animations. Some groups have gone beyond watching demos and have had the opportunity to experiment individually with selected software by prior arrangement. Such tours demand far more staff and hardware resources than the usual ones. There have been twelve such interactive tour groups since December 1991.

Since July 1991, over 1100 people toured the Center. The 45 group tours included forty mathematically gifted female and minority students in sixth through ninth grades from the **Summer Enrichment Institute** run by the University's Special Projects Office, sixty tenth- and eleventh-grade low-income students showing promise in math and science in the Lakewood Community College Upward Bound Math and Science Program, fifty teachers attending the ISACS conference, thirty people attending the FISEA (Fourth International Society of Electronic Artists) conference, and twenty minority high school students from the Chicago Astrophysics STC. See the next page for a complete list.

List of Geometry Center Tours, 7/25/91–4/30/94

Total tour attendees: 1107 people, including 45 groups

Individuals:		176

Interactive Demos: 14 groups		attendees: 313
12/5/91	UW-LaCrosse undergraduates	12
6/29/92	UYMTYMP Summer Enrichment (female and minority junior high)	40
8/5/92	Lakewood Upward Bound Math and Science (low-income high school)	20
11/13/92	ISACS high-school teachers	53
2/15/93	Woodbury Girl Scouts (grades 4-6)	9
3/3/93	Prairie Creek school (2nd and 3rd grade)	17
6/30/93	Geometry Center summer course high-school teachers	30
7/22/93	Lakewood Upward Bound Math and Science (low-income high school)	40
7/26/93	Science Museum of Minnesota summer class	6
7/27/93	Girl Scout Council/Alternative Learning Center	8
10/23/93	Lake Country School 4th graders	14
2/1/94	Cutler's MathEd 5322 class on Assists to Teaching Mathematics	16
4/5/94	Orono High School geometry students	18
4/15/94	More Orono High School geometry students	30

Groups: 31 groups		attendees: 618
7/25/91	AHPCRC supercomputing summer program (minority undergraduates)	20
4/3/92	Loras college undergraduates	13
6/15/92	Geometry Center summer undergraduate students	20
7/8/92	Geometry Center summer high-school teacher course	30
7/10/92	TCITY summer class "Math and the Mind's Eye"	12
10/31/92	STC Administrators	15
11/24/92	St. Thomas Academy high school students	5
2/27/93	Chicago STC minority students	20
2/27/93	UMTYMP junior high students	38
3/9/93	St. Louis Park junior high students	5
3/31/93	St. Paul Open School (5th and 6th graders)	16
4/23/93	Math Counts state winners junior high students	5
5/5/93	Fairbault High School students	7
5/11/93	MNUG (Minnesota NeXT Users's Group) meeting	20
5/20/93	SIGGRAPH meeting	15
6/14/93	Geometry Center summer undergraduate students	20
6/18/93	UMTYMP Summer Enrichment (junior high)	42
7/20/93	TCITY Shape of Space class (gifted students)	17
9/21/93	Vic Reiner's Algebraic Geometry UMTYMP class	12
10/9/93	Dick McGehee's graduate dynamical systems course (DsTool)	12
10/19/93	Chicago Astrophysics STC (minority high-school students)	20
10/25/93	Minnesota HighTech Council	7
11/6/93	FISEA (Fourth International Society of Electronic Artists)	30
11/8/93	Low Energy Space Travel conference (Ed Belbruno)	35
11/20/93	UMTYMP enrichment (junior high female, minority, low-income)	80
1/19/94	Macalaster College undergraduates (including Calc 2 students)	20
1/28/94	Webb's graduate Group Theory class	8
3/24/94	IT alumni in alumni/student mentoring program	6
3/28/94	Professors who train elementary math teachers, Taipei Teachers College	6
3/29/94	8th graders, Science Museum Lab Partner program	12
4/22/94	8th graders taking high-school geometry, Ramsey Junior High	50

New Technology. Although most of the Center's research is abstract, some may have direct practical applications to industrial problems. For example:

- Postdoc **Leonidas Palios** worked on a computational geometry problem that arises in the design of VLSI array processors, and created an improved algorithm for utilizing "spare" processors on the chip to compensate for manufacturing flaws.
- **Edward Belbruno**, an independent consultant supported by the Center, organized a workshop on Nonlinear Astrodynamics that attracted participants from **Ford**, **McDonnell–Douglas**, **Los Alamos**, and the Jet Propulsion Lab (**JPL**). He and Center summer student **Nicholas Coult** developed a program, SPAM, that provides the most precise and visually accurate simulation of the solar system that exists for a moving spacecraft. SPAM and videos made with it have been used as a visualization tool at JPL and WISCAR (University of Wisconsin Space and Robotics Laboratory).
- **Skyline Displays**, a Minneapolis design firm, did the structural design for the globe in the **"Building a New World"** project (page 34). The collaboration continues since staff member **Stuart Levy**, who worked on the project, is now providing them with software to translate from geographic coordinates to their grid system; this software will be incorporated into several projects of the company.

Other Activities

- In the summer of 1993 the Center hosted a **Geometry Software Conference** (page 26 to examine the state of visualization software in K–12 education. Representatives of publishers, software companies, and producers of educational manipulatives attended.
- The Materials Science and Industry Laboratory of the National Institute of Standards and Technology (NIST) is working on forming a **Center for Computational Materials Science**, largely modeled on the Geometry Center. The major goal of CCMS will be to identify and solve industrially relevant problems in materials science that have computationally feasible solutions. The know-how we are transfering to NIST in this case is organizational: they think our setup with a large workroom and a series of workshops at which computational work takes place is ideal for their situation. Andy Roosen, one of the planners, also says: "We intend to have a [World-Wide Web] server as the 'nerve center' of the CCMS, and have software available for 'evaluation' much in the same way that the Geometry Center has. In fact, when we gave the big pitch to the Lab director just this morning, we used the Geometry Center's server as a demonstration of what we'd like to be able to do. He was extremely impressed." A planning meeting will be held here from May 31 to June 3.
- Silvio Levy, the Center's Director of Technology, was for three years the editor of the quarterly *The Mathematica Journal*, devoted to the symbolic-manipulation system most popular in the industry and business. He remains on the editorial board, and is also a member of the Mathematics Advisory Board for WRI, the company that produces **Mathematica**.
- People from industry sometimes are referred to the Center, usually through the University, for answers to mathematical questions ranging from the trivial to the unsolvable. Recent examples are a pool manufacturer who wanted to design an octagonal tub as big as possible for a given square space (we solved this one), and a mail-order firm who needed to solve a problem in time-series analysis (we recommended texts and software).

Documentation: Research and Research Software

The Postdoctoral Research Program

The **postdoc program** gives young mathematicians and computer scientists an opportunity to pursue their own research. At the same time, the presence of the postdocs gives the Center a more permanent research focus and an in-house source of mathematical expertise. In the course of their research, the postdocs maintain ties with leading researchers in their fields, go to conferences, and visit other institutions, connecting the Geometry Center to the larger mathematical community. They participate in Center activities: developing software, talking to and working with visitors, mentoring students and apprentices, teaching and developing courses, and organizing workshops. While at the Center, the postdocs are exposed to a wide range of mathematicians and their work, and a variety of visual and computational approaches to problem solving. When they leave the center for more traditional mathematics or computer science departments, they take these experiences with them, and, we hope, incorporate the vision of the Geometry Center into their own research and classes.

Past and Present Postdocs

Name	Adviser	University	Dates
John Sullivan	F. Almgren	Princeton University	1991–1993
(Currently at University of Minnesota)			
C. Bradford Barber	D. Dobkin	Princeton University	1992–1994
(Currently a postdoc at Harvard University)			
Paul Burchard	R. Narasimhan	University of Chicago	1992–
Oliver Goodman	D. Epstein	University of Warwick	1992–
Leonidas Palios	B. Chazelle	Princeton University	1992–
Nina Amenta	R. Seidel	University of California	1993–
Davide Cervone	T. Banchoff	Brown University	1993–
Frederick Wicklin	J. Guckenheimer	Cornell University	1993–

New Postdocs (Tentative)

Ioannis Emiris, University of California, Berkeley
Chaim Goodman-Strauss, University of Texas, Austin
Guillermo Sapiro, Massachussetts Institute of Technology
Ayellet Tal, Princeton University

Postdoc Publications. The following papers have been published or written by postdocs since the Center's inception. For talks given by postdocs, see page 55.

- Almgren, F. and Sullivan, J. (1992). Visualization of soap bubble geometries. *Leonardo*, 24(3/4):267–271.
- Amenta, N. Helly theorems and generalized linear programming. *Discrete and Computational Geometry*. To appear.
- Amenta, N. (1994a). Bounded boxes, Hausdorff distance and a new proof of an interesting Helly theorem. *Proc. 10th ACM Symposium on Computational Geometry*. To appear.
- Amenta, N. (1994b). Helly theorems and generalized linear programming. Geometry Center Preprint GCG62.

- Barber, C. B., Dobkin, D. P., and Huhdanpaa, H. The quickhull algorithm for convex hull. ACM Transactions on Mathematical Software.
- Barber, C. B. and Hirsch, M. D. A robust algorithm for point in polyhedron. Submitted to Computational Geometry: Theory and Applications.
- Chazelle, B. and Palios, L. Decomposing the boundary of a non-convex polyhedron. Submitted for publication to *Algorithmica*.
- Hsu, L., Kusner, R., and Sullivan, J. (1992). Minimizing the squared mean curvature integral for surfaces in space forms. *Experimental Mathematics*, 1(3):191–207.
- Kusner, R. and Sullivan, J. Möbius energy of Hopf links, and electrons on the sphere. In preparation.
- Kusner, R. and Sullivan, J. A nonlocal, conformally invariant energy for embedded submanifolds. In preparation.
- Kusner, R. and Sullivan, J. (1994). Möbius energies for knots and links, surfaces and submanifolds. In *Georgia International Topology Conference*. To appear.
- Morgan, F. and Sullivan, J. (1993). Monotonicity of the location of absolute minima.
- Morgan, F., Sullivan, J., and Larché, F. (1993). Monotonicity theorems for two-phase solids. *Arch. Rat. Mech. Anal.*, 124(4):329–353.
- Palios, L. Connecting the maximum number of nodes in the grid to the boundary with non-intersecting line segments. Submitted for publication to the *Journal of Algorithms*.
- Palios, L. On the number of extrema of a polyhedron. Submitted to the *6th Canadian Conference on Computational Geometry* (1994).
- Palios, L. Optimal tetrahedralization of the 3d-region 'between' a convex polyhedron and a convex polygon. Submitted to the *6th Canadian Conf. on Computational Geometry* (1994).
- Palios, L. *Tetrahedralizing the 3d-Region 'between' a Convex Polyhedron and a Convex Polygon: The Movie*. The Geometry Center, University of Minnesota. Video.
- Palios, L. Upper and lower bounds for optimal tree partitions. Submitted to the *2nd Annual European Workshop on Algorithms* (1994).
- Palios, L. (1994). Connecting the maximum number of grid nodes to the boundary with non-intersecting line segments. In *Proc. 3rd Scandinavian Workshop on Algorithm Theory*.
- Palios, L. and Phillips, M. (1992). Tetrahedral breakup. In *Animation of Algorithms: A Video Review*, publicaiton 87a. DEC Systems Research Center, Palo Alto, CA. Video.
- Sullivan, J. (1991). Generating and rendering four-dimensional polytopes. *The Mathematica Journal*, 1(3):76–85.
- Sullivan, J. (1992a). The existence of lattice coverings with small multiplicity.
- Sullivan, J. (1992b). Using max-flow/min-cut to find area-minimizing surfaces. In *Computational Crystal Growers Workshop*, AMS Sel. Lect. Math., pages 107–110 plus video.
- Sullivan, J. (1993). *Knots Minimizing a Möbius-Invariant Energy*. Six-minute video.
- Sullivan, J. (1994a). Computing hypersurfaces which minimize surface energy plus bulk energy. In *Motion by Mean Curvature and Related Topics*, pages 186–197. de Gruyter.
- Sullivan, J. (1994b). Sphere packings give an explicit bound for the Besicovitch covering theorem. *J. Geometric Analysis*, 4(2). To appear.
- Sullivan, J. and Altschuler, S. (1992). Self-similar solutions to the curve-shortening flow in space.
- Wicklin, F. Maple. In *Handbook of Software for Engineers and Scientists*. CRC Press. To appear.

John Sullivan

John Sullivan was a postdoctoral fellow from September 1990 until September 1993. In September 1991 he was also appointed Assistant Professor in the School of Mathematics at the University of Minnesota. He received his Ph.D. from Princeton University under **Fred Almgren**.

Sullivan's research, on topics including area-minimizing surfaces, Willmore surfaces, knot energies (and their extensions to surfaces), sphere packings, and self-similar solutions to curvature flows, is described in the research section starting on page 101. This page describes some of Sullivan's other activities at the Center.

Education

- In the summer of 1991, Sullivan assisted in teaching **Geometry and the Imagination**. This experience led him to propose a similar course for the University of Minnesota's College of Liberal Arts Honors Program. He is now teaching this course, Spring Quarter 1994, and will offer it again for high school students in the Summer Honors College this July.
- In the 1992–93 academic year, Sullivan taught the undergraduate Differential Geometry course (5375/6/7) at the U of M. For the course, he developed on the Center's NeXTs some software for students to interact with curves and surfaces.
- In the summer of 1993, Sullivan was a consultant to the REU program at Smith College. A major part of this program consisted of teaching the students to use Brakke's **Evolver** to investigate problems in knot energies and minimal foams.

Outreach

- Sullivan developed a RenderMan shader for realistic soap films. The pictures created with this have been the focus of expository papers published in *Leonardo* and the *Mathematica Journal*, and have also been printed in *Science News*, the *American Mathematical Monthly* and the *Mathematical Intelligencer*. Sullivan's research has been mentioned elsewhere in the popular press, for example in *Popular Science*.
- Sullivan has presented talks around the world, showing *Not Knot* and his own videos at math departments and industry research labs. He has given special talks for undergraduates at Williams College and Smith College, as well as at the Geometry Center.

Software Development

- Sullivan implemented his algorithm to find area-minimizing surfaces; he has continued to update his program vcs for computing Voronoi diagrams in three dimensions. These are described in more detail on pages 109 and 108.
- Building on work of Oliver Goodman and Nathaniel Thurston, Sullivan added to **Geomview** a capability for viewing in the conformal model of spherical (or Euclidean) space. This means that the full ten-dimensional Möbius group is now accessible to the user of Geomview through the mouse.
- Sullivan has written several modules for Geomview, including one to compute self-similar solutions to curve flows (see page 104), and one demonstrating stereographic projection for educational purposes.

Bradford Barber

Bradford Barber was a Postdoctoral Fellow at the Geometry Center from July 1992 to January 1994. He received his Ph.D. in Computer Science from Princeton University under the direction of **David Dobkin** (1992). While at the Center, Barber's research revolved around algorithms that take into account imprecise data and finite-precision arithmetic, two fundamental aspects of computer programming that are frequently overlooked by algorithm developers and implementors.

Research and Software Development

- **Point-in-polyhedron.** Considered a topological algorithm for determining point inclusion under imprecise arithmetic and data. A polyhedron consists of a set of facets and geometric information about each facet, where this geometric information may be exact or imprecise, say a bounding box that contains the facet. The algorithm determines those facets that a test ray emanating from the point might intersect and computes their topological boundary. This reduces the problem by one dimension. The solution becomes trivial in one dimension (an odd number of intersections means the point is inside the polyhedron).
- With **Michael Hirsch**, wrote a paper describing the point-in-polyhedron algorithm for publication in *Computational Geometry: Theory and Practice*. The paper was presented at the 1993 Canadian Conference on Computational Geometry.
- **QuickHull.** Developed an efficient process for computing the convex hull of a collection of points in any dimension. With David Dobkin, wrote a paper describing this algorithm, and presented it at the 1993 Canadian Conference on Computational Geometry. The paper is under revision for the ACM Transactions on Mathematical Software.
- **qhull.** With **Hannu Huhdanpaa**, an undergraduate programmer at the Center, wrote qhull, a program that implements the QuickHull algorithm. The first version was released in June 1993. This program has been retrieved by ftp over 700 times, and is one of the most discussed programs for convex hull and Delaunay triangulation in the Internet newsgroup comp.graphics.algorithms.

  Barber has recently released an enchanced version of qhull. This version handles imprecise data and arithmetic and selective processing. It also provides a library interface to the qhull algorithms so that they can be included into user-written programs. This is the first convex-hull code to handle roundoff errors in three dimensions and higher.

Other activities

- While at the Center, Barber ran the computing labs for the summer course on computer graphics, and supervised student programmers. He also continued development of a programming and information retrieval system. Barber is now a postdoctoral fellow at Harvard University.

Paul Burchard

Paul Burchard has been a Postdoctoral Fellow at the Geometry Center since the fall of 1992. He received his Ph.D. from the University of Chicago under the direction of R. Narasimhan in 1989. During 1989 and 1990, he was a member of the Institute for Advanced Study at Princeton, and from 1990 to 1992 he was an instructor at the University of Utah. He is currently on leave from Utah during his appointment at the Geometry Center.

Research and Software Development

- **CADiff.** Burchard's main goal in coming to the Center was to begin the design and implementation of a flexible software system for differential geometry. After a process of careful research into design issues, he is embarking on implementing the core functionality of this system. The project is an extension of his research interests, which center on the formal properties of differential equations.
- **W3Kit.** This software toolkit, designed to make it easier to build interactive **World-Wide Web** applications, is described on page 145.
- **CRSolver.** The Cauchy–Riemann Solver is an interactive application for experimenting with conformal mapping, complex analytic functions, and meromorphic sections of line bundles on Riemann surfaces of genus one. It uses an innovative interface and a novel relaxation algorithm to make it both intuitive and responsive to the user.
- **Dynamic Kit.** This software toolkit consists of an interface builder and Objective-C class library for making highly interactive mathematical applications under the NeXTSTEP operating system. It has been used as the foundation for several packages developed at the Center, including CRSolver, QuasiTiler (see cover illustration and page 2), and AlgCurve, a program for finding implicitly defined curves in the plane, which is being used by an **UMTYMP** course (see page 25).

Education and Outreach

- **Summer Student Institute, 1993.** Burchard, together with Center apprentice David Ben-Zvi, directed several students in projects related to complex analysis and other topics. One student's project has become an interactive Web application (the Teichmüller Navigator, see page 29).
- Burchard presented a talk "Distributing Graphical Applications on the World-Wide Web" as part of the Keck Seminar Series in Informatics at the W.M. Keck Center for Genome Informatics, Institute of Biosciences and Technology, Texas A&M University. This may lead to W3Kit and/or Geomview being used as part of an interactive Web application for viewing Protein Data Bank files.

Oliver Goodman

Oliver Goodman has been a Postdoctoral Fellow at the Geometry Center since March of 1992. He received his Ph.D. in Mathematics from the University of Warwick under the direction of **David Epstein** (1992). His work at the Center has included:

Software Development

- **CirclePack.m.** A Mathematica program to solve the circle packing problem described by Thurston in his book on three-manifolds in the context of Andre'ev's Theorem.
- **Hyperbolic.m.** Goodman took charge of this hyperbolic geometry package written by Silvio Levy, adding several new features including improved 3D graphics, improved support for transformations, and support for totally geodesic subspaces.
- **Poincaré model graphics for Geomview.** This is a Poincaré ball model viewer for Geomview that shows 3-dimensional hyperbolic polyhedra and other objects in the Poincaré's ball model for hyperbolic space. Postdoc **John Sullivan** later built on this work to write a conformal model viewer for spherical geometry.
- **Mathpad.** An interactive input and output device for Mathematica that runs on Iris workstations, and provides the user with faster and more direct control over graphics generated by Mathematica. This is one possible solution to a fundamental shortcoming of the Mathematica interface.

Mathematical Research

- **Andre'ev's theorem.** Goodman is working on a paper to describe a constructive proof of Andre'ev's theorem as given by **Thurston**. This was inspired by efforts to design an algorithm to find circle packings on a sphere.
- **Three-manifolds.** Goodman has on several occasions suggested corrections or possible improvements to Thurston's book on three-manifolds.

Outreach and Education

- Helped **Peter Waterman**, Northern Illinois University, with using the Hyperbolic.m package. In response to his needs, Goodman worked on the TriangleGroup sub-package of this package.
- Worked with **Andre Rocha**, **Greg McShane** and **Ian Redfern** on Kleinian groups by using Hyperbolic.m and CirclePack.m to compute hyperbolic polyhedra satisfying certain angle constraints to serve as fundamental domains for these groups.
- Goodman is currently working on a Maple program to implement **Jane Gilman's** algorithm testing for discreteness of subgroups of PSL$(2, \mathbf{R})$.
- Worked with **Deva Van der Werf** (summer student, 1993) on her Teichmüller Navigator project. This was to write a program enabling the user to interact with a fundamental domain for a genus-two hyperbolic surface. Goodman helped her to understand the necessary hyperbolic geometry, to use Mathematica, and to design a numerical algorithm to solve the problem.
- Goodman presented several talks, including one on Andre'ev's theorem at North Illinois University, and one on Hyperbolic.m at the Mathematical Sciences Research Institute.

Leonidas Palios

Leonidas Palios has been a Postdoctoral Fellow at the Geometry Center since April 1992. He received his Ph.D. in Computer Science from Princeton University under the supervision of Bernard Chazelle (1992).

Palios' research interests involve the design and analysis of algorithms, and in particular, computational geometry and graph theory. While at the Geometry Center, he has been engaged in a number of projects, described in more detail in the section starting on page 108:

Research

- Computed a tight upper bound on the number of the extrema of a polyhedron;
- Described an algorithm, which, for a given set of points enclosed in a rectangle R, computes the maximum number of these points that can be connected to the boundary of R by means of non-intersecting (horizontal or vertical) line segments;
- Established upper and lower bounds on the cut size of optimal partitions of trees;
- Described an algorithm to tetrahedralize the three-dimensional region "between" a convex polyhedron and a convex polygon.

He is currently working on two new projects:

- Unfolding of a convex polyhedron. It is an interesting open problem whether, for a given convex polyhedron, there always exists a way to cut along some of its edges (thus disconnecting the incident facets) such that the boundary of the polyhedron can be unfolded on the plane without overlaps. Olaf Holt (an apprentice at the Center) and Palios are trying to gain intuition on the problem by using Geomview to animate possible algorithms implemented by Holt in Perl.
- Convex hull of a set of points in three dimensions. Paul Burchard and Palios are trying to use a new approach to obtain a new divide-and-conquer algorithm for the convex hull of a set of points in three dimensions.

Algorithm Implementations and Video Animations

- Augmented an existing C implementation of the polyhedron tetrahedralization algorithm of Chazelle and Palios, so that graphical output readable by Geomview is produced. This led to the production of the video "Tetrahedral Break-up" with staff member Mark Phillips.
- Implemented an early version of his algorithm to tetrahedralize the 3d-region "between" a convex polyhedron and a convex polygon. The program was further augmented to produce graphical output, which can be piped to Geomview, yielding an interactive animation. Moreover, the program was used to create the video "Tetrahedralizing the 3d-region between a convex polyhedron and a convex polygon: The Movie" animating the algorithm.

Teaching

In the winter quarter of 1993, Palios worked as an adjunct professor at the Computer Science Department, University of Minnesota teaching the course CS5400 (Introduction to Automata Theory) for advanced undergraduate and beginning graduate students.

Nina Amenta

Nina Amenta began as a postdoc in November 1993, after completing her Ph.D. in computer science at U.C. Berkeley under the direction of **Raimund Seidel**. Amenta is working on the following projects:

- **Computational Geometry Software Workshop.** Amenta is organizing a Geometry Center Workshop on Computational Geometry Software, for January 1995. Most of the senior researchers in computational geometry who are involved in software development, as well as other prominent computer scientists and mathematicians who use computational geometry algorithms, have agreed to attend. Some additional funding is being provided by the Office of Naval Research (ONR). Selected papers will be published in a special issue of the *International Journal of Computational Geometry and Its Applications*, with Amenta as Guest Editor.
- **Software Library.** In conjunction with the workshop, Amenta has begun to locate publicly available computational geometry software, which will be made available on the **World-Wide Web** from the Geometry Center. This service has been suggested by many computer scientists, and the Geometry Center is a logical place for it to be located.
- **Research.** Amenta's doctoral research concerned a class of geometric optimization problems related to the Helly-type theorems from combinatorial geometry. She showed that these problems could been solved by known combinatorial algorithms designed for linear programming, such as the simplex algorithm. Amenta's approach gives an algorithmic paradigm that applies to many geometric optimization problems, some of which had previously be solved by ad-hoc methods. For instance, she gives expected linear-time algorithms for finding the translation and scaling that minimizes the Hausdorff distance between two convex polygons in the plane (a problem with applications in computer vision) and for finding the largest axis-aligned box in the intersection of a family of intersecting spheres (or other easily described convex sets).

  Since arriving at the Center she has completed an invited paper on this topic for *Discrete and Computational Geometry*, and a follow-up paper was accepted at this year's Association of Computing Machinery Symposium on Computational Geometry. She plans to continue this line of research during a visit to the Freie Universität, Berlin, this coming September through December, funded by a Deutscher Akademischer Austauschdienst (DAAD) Postdoctoral Scholarship.
- **Computer Vision.** With David Jacobs of NEC Research Labs in Princeton, New Jersey, Amenta is exploring the application of her optimization algorithms to object-recognition problems in computer vision. Also in computer vision, she is currently reviewing an article on line detection for the *IEEE Transactions on Pattern Analysis and Machine Intelligence*.
- **Mentoring.** Amenta is mentoring a Minneapolis high-school student, **Derek Farmer**, who is exploring the use of the vertex set of a Penrose tiling as a sampling pattern for applications such as numerical integration.

Davide P. Cervone

Davide P. Cervone has been a Postdoctoral Fellow at the Geometry Center since August 1993. He received his Ph.D. in Mathematics from Brown University under the direction of **Thomas Banchoff** (1993). At the Center, he has been involved in the following projects:

Research and Software Development

- Continued study of vertex-minimal immersions and tight immersions of simplicial surfaces in space, the subject of his Ph.D. thesis.
- Began investigation into vertex-minimal immersions of surfaces with constraints on the shapes of the faces. Obtained several interesting examples and preliminary results.
- Discovered a significant error in a published result of **Ulrich Pinkall**, and generated a new solution to the problem (to be submitted to *Topology* for publication).
- **Pisces** (page 140): developed a new algorithm for determining level-sets of function graphs using an adaptive subdivision process to obtain greater resolution near singularities. The algorithm can locate very small components and even places where the graph reaches the level tangentially rather than crossing it transversely (such points are missed by most other algorithms). The algorithm generalizes to any dimension and codimension. Cervone implemented the new algorithm for the specific case of curves in the plane, using an interactive "workbench" design that allows the user to modify parameters that control the operation of the algorithm. This makes it possible to tailor the algorithm interactively to produce even better results, and to see the effects of different aspects of the algorithm. Implementations of the algorithm for higher dimension and codimension are planned.
- **Geomview** modules: designed the specifications for an interactive polyhedral-modeling Geomview application. Cervone's research requires the ability to create and modify polyhedral objects in a very interactive way (by adding and removing faces, moving vertices, looking only at the faces around one vertex, etc.), with special attention paid to self-intersection and other singular behaviors. Currently there is no easy way to accomplish this in Geomview. Ideally, the planned module will allow mathematical relationships to exist among the values that specify the positions of vertices, and will allow symmetry mappings to be applied to portions of the object to make specifying other portions easier.
- Implemented a multiline input object for the FORMS library of input devices available on SGI machines. Geomview, as well as most of its external modules, uses FORMS to maintain its input panels.

Education

- Gave two lectures as part of the Geometrical Analysis seminar run by the Department of Mathematics (January 1994).
- (with Harvey Keynes) Will teach Calculus III to UMTYMP students (1994–95).

Frederick J. Wicklin

Frederick J. Wicklin has been a Postdoctoral Fellow at the Geometry Center since August 1993. He received his Ph.D. in Applied Mathematics from Cornell University under the direction of **John Guckenheimer** (1993). While at the Center, Wicklin has been involved in the following projects:

Education

- (with **Tracey Bibelnieks**, **Harvey Keynes**) Designed and assembled curriculum for the summer enrichment course described on page 24.
- (with Harvey Keynes) Serves on the University of Minnesota NSF Curriculum Initiative, which is proposing a reformation of the University's sequence of calculus courses.
- (with Harvey Keynes) Provides technical support for the **UMTYMP** course in Algebraic Geometry held weekly at the Geometry Center (page 25).
- Spoke to the Graduate Seminar on Teaching Mathematics: "Teaching ODEs using Technology."
- (With Harvey Keynes) Will teach Calculus III to UMTYMP students (1994–95).

Research and Software Development

- Proposed, planned, and served as team leader for **Pisces** (page 140). This included researching existing algorithms, assessing needs within the mathematical community, and learning about the numerical techniques used in continuation algorithms. Adapted the central "data manager" from DsTool (see below) and wrote code that enables users to add algorithms in a modular fashion. Wrote code that implements algorithms of A. Geisow and R. Morris for finding implicitly defined algebraic curves and surfaces.
- (with **Arlie Petters**, Princeton) Used Pisces to compute caustics for equations that model gravitational lensing. The numerical experiments suggest the existence of previously unsuspected caustics when there are multiple lenses between the source and the observer.
- Gave two talks to the Dynamics Seminar at the U of M School of Mathematics: "The Geometry of Resonance" and "Resonance Regions in Torus Maps and Twist Maps."
- Wrote an article on Maple to appear in the *Handbook of Software for Engineers and Scientists*, CRC Press.
- Installed and continued development on **DsTool**, a dynamical systems toolkit written at Cornell University. Included in this is the modification of DsTool code so that it runs properly on SGI workstations.
- (with Alfred Gray) Wrote numerous Maple functions for Alfred Gray that compute objects in differential geometry (e.g., tangents, curvature, torsion). Gray has written a textbook on learning differential geometry with Mathematica.
- Wrote Maple functions that translate 3D Maple output into **Geomview** format. This enables users to compute surfaces in Maple and then display them using Geomview.

Talks Given by Geometry Center Postdocs

This list does not include talks given at the Geometry Center as part of our weekly seminar series.

John Sullivan May 1991
Crystalline approximation: computing minimum surfaces via maximum flows
Lehigh University, Geometry and Topology Conference

John Sullivan July 1991
Spherical bubble clusters
Williams College Summer Colloquium

John Sullivan July 1991
Crystalline approximation: computing minimum surfaces via maximum flows
NSF Regional Geometry Institute (Five Colleges): Optimization in Geometry

John Sullivan September 1991
Crystalline approximation: computing minimum surfaces via maximum flows
GADGET Workshop (Granada): Minimal Surfaces

John Sullivan February 1992
Crystalline approximation: computing minimum surfaces via maximum flows
Differential Geometry Seminar, University of Minnesota

John Sullivan March 1992
Using max-flow/min-cut to find area-minimizing surfaces
Computational Crystal Growers Workshop, Geometry Center

John Sullivan March 1992
Computing Willmore surfaces at the Geometry Center
CMA Colloquium, Australian National University

John Sullivan March 1992
Finding area-minimizing surfaces
Advanced Computational Seminar, Australian National University

Leonidas Palios May 1992
Tetrahedralizing a nonconvex polyhedron
Computer Science Department, University of Minnesota

Mark Phillips, Leonidas Palios June 1992
Film: "Tetrahedral Break-up"
ACM Symposium on Computational Geometry (Berlin)

John Sullivan July 1992
Computing Willmore surfaces
NSF Regional Geometry Institute (Utah): Nonlinear PDEs in Differential Geometry

Leonidas Palios July 1992
Decomposing the boundary of a nonconvex polyhedron
Third Scandinavian Workshop on Algorithm Theory

John Sullivan October 1992
Self-similar solutions for the curve-shortening flow
Second MSRI Conference on Geometric Visualization

Oliver Goodman October 1992
Hyperbolic.m
Second MSRI Conference on Geometric Visualization

Bradford Barber November 1992
The Flashlight algorithm for point-in-polyhedron
Computer Science Department, University of Minnesota

John Sullivan May 1993
A nonlocal, conformally invariant energy for embedded submanifolds
Midwest Geometry Conference, University of Missouri

John Sullivan May 1993
A nonlocal, conformally invariant energy for embedded submanifolds
Geometric Analysis Seminar, University of Minnesota

John Sullivan June 1993
A nonlocal, conformally invariant energy for embedded submanifolds
Celebration in honor of Almgren's 60th Birthday, Princeton University

John Sullivan July 1993
Conformal knot energy and its extension to surfaces
Williams College Summer Colloquium

John Sullivan July 1993
Regular polytopes as bubble clusters
NSF Regional Geometry Institute (Five Colleges): Discrete Geometry

Bradford Barber August 1993
The robust Flashlight algorithm for point-in-polyhedron
Canadian Conference on Computational Geometry, Waterloo

Bradford Barber August 1993
General dimension Quickhull
Canadian Conference on Computational Geometry, Waterloo

John Sullivan August 1993
Using duality to find area-minimizing hypersurfaces
Fields Institute Microprogram: Riemannian Geometry

Frederick Wicklin September 1993
The geometry of resonance
Dynamics Seminar, University of Minnesota, Department of Mathematics

John Sullivan September 1993
Möbius energy of knots and links, surfaces and submanifolds
ITCS Workshop, University of Kansas: Discretization and Geometric Visualization

Oliver Goodman September 1993
Andre'ev's theorem and circle packings
North Illinois University

Frederick Wicklin October 1993
Resonance regions in torus maps and twist maps
Dynamics Seminar, University of Minnesota, Department of Mathematics

Leonidas Palios November 1993
Tetrahedralizing the 3D region between a convex polyhedron and a convex polygon
Computer Science Department, Princeton University

Davide P. Cervone January 1994
Vertex-minimal immersions of simplicial surfaces into three-space
Geometric Analysis seminar, University of Minnesota, Math Department

Davide P. Cervone January 1994
Tight immersions of simplicial surfaces into three-space
Geometric Analysis seminar, University of Minnesota, Math Department

Leonidas Palios February 1994
Optimal tetrahedralization of the 3D region between a convex polyhedron and a convex polygon
Computer Science Department, University of Illinois at Urbana–Champaign

John Sullivan March 1994
Symmetry and curvature: computer graphics in curved spaces
Research Group Seminar, Microsoft, Inc.

Frederick Wicklin March 1994
Using technology in upper-division ODE courses
University of Minnesota, Math department graduate teaching seminar

Nina Amenta March 1994
Convex programming, Hausdorff distance, and Helly theorems
Theoretical Computer Science Seminar, University of Illinois at Urbana–Champaign

Nina Amenta March 1994
Generalized linear programming and a new proof of an interesting Helly theorem
Combinatorics Seminar, University of Minnesota, Math Department and IMA

Frederick Wicklin March 1994
The dynamics of circle maps: an undergraduate research topic
Conference on Undergraduate Research in the Mathematical Sciences, Greensboro, NC

Nina Amenta April 1994
Generalized linear programming and a new proof of an interesting Helly theorem
Special Seminar, Carnegie Mellon University

Paul Burchard April 1994
Distributing graphical applications on the World-Wide Web
Keck Seminar Series in Informatics, W. M. Keck Center for Genome Informatics, Institute of Biosciences and Technology, Texas A&M University

University of Minnesota Faculty

We quote here excerpts from statements of U of M School of Mathematics faculty who have been closely involved with the Center. For **John Sullivan**, a former postdoc who keeps strong ties with the Center, please refer to pages 47, 64 (EPMG Workshop), and 101.

Victor Reiner

[Regarding the **UMTYMP** course described on page 25]

Our topic for the year was Introduction to Algebraic Geometry. We used the new book by Cox, Little, and O'Shea, *Ideals, Varieties, and Algorithms*, which worked extremely well. The book and the course emphasized understanding how varieties relate to the polynomials that define them, and how to compute with these polynomials explicitly using the technique of Gröbner bases. To this end, we used Maple on the Geometry Center's NeXT machines to do Gröbner basis computations.

The Geometry Center also developed for me a graphics package that runs on the NeXTs, called **AlgCurve**. This program draws algebraic curves in the real affine plane and in the real projective plane, and is an extremely useful tool for teaching about projective closures of affine curves. I spent a good deal of time with two of the Center's undergraduate programmers (Scott Wisdom and Dan Krech) developing AlgCurve, and eventually Krech produced a very nice version that we have made good use of in class.

Robert Gulliver

I received a course reduction during 1992–93 to participate in the program of the Geometry Center. I started with the idea of a project related to a research paper I was then beginning to work on. . . The research paper did not get far enough in time for me to work on computer implementation that year; nonetheless, I was able to develop large parts of a future implementation, with substantial help from several of the apprentices and postdocs, especially Paul Burchard. I also benefited greatly from being able to learn about the Evolver, directly from Ken Brakke. This is a powerful tool for geometric experimentation.

A number of Center postdocs have been active in the Geometric Analysis Seminar, which has opened up some interesting mathematical topics. The Center is an excellent institution and a great tool for the furthering of mathematics.

Peter Olver

My activities at the Geometry Center have been centered on a new project on computer **vision** and image processing. . . in collaboration with Allen Tannenbaum, of the Electrical Engineering Department, and **Guillermo Sapiro**, who will be a Geometry Center postdoc next year. The project is concerned with the use of certain geometrically based diffusion equations to process and reconstruct two and three dimensional images. . . We have, using the theory of differential invariants, successfully characterized all such evolution equations, and the goal now is to understand how they process images. To this end, we have been using the **Surface Evolver** program of **K. Brakke**, although several difficulties are already apparent when we use the simplest evolution by mean curvature, including numerical singularities. The long-term goals of the project

> Furthermore, a workshop such as this could only have been held at the Geometry Center. We used its reliable large network of workstations and video equipment plus the open layout of the Center with its many whiteboards to promote one-on-one and small group interactions. ... The staff ... was fantastic, always available and very helpful; their activities contributed enormously to the success of the workshop, especially since the participants came from a variety of computational environments.

Workshop on Visualization of Invariant Sets for Symplectic Maps in Dimension Four, October 11–16, 1992. Organized by **Richard McGehee**. From the original proposal:

> Most questions about the dynamics of symplectic maps in dimension four remain unanswered. Even the basic problem of determining the stability of an elliptic fixed point is unresolved, a problem which is rooted in the classic question of the stability of the solar system. This workshop will bring together researchers interested in exploring these questions through computer simulation and visualization. Talks will be limited to two hours per day; most of the time will be devoted to informal discussions centered around computer simulations of symplectic maps.
>
> Topics will include: (1) the breakdown of invariant two-dimensional tori; (2) the computation of Arnold's "whiskered tori", which, in this case, are invariant circles; (3) the computation of stable and unstable manifolds for hyperbolic periodic orbits and for hyperbolic invariant circles.

Knot Workshop, March 22–26, 1993. Organized by **Louis Kauffman** (University of Illinois, Chicago), **DeWitt Sumners** (Florida State University), **Jeff Weeks** (Canton, NY) and **James White** (UCLA). From the proposal:

> The workshop will focus on research and applications involving knots, including knot polynomials, the geometry of knot complements, physics and biology. The workshop will emphasize those aspects that involve the use of computing and visualization both as investigative tools and for purposes of displaying and communicating results.

And from Weeks' article in the Center newsletter:

> The Geometry Center Knot Workshop was tremendously successful in its goal of facilitating communication between workers in diverse aspects of knot theory. As well as the aforementioned applications to molecular biology, we saw the role of random knots in polymer chemistry; applications of differential geometry to the twisting, writhing and linking of DNA; the mathematical similarities between the new knot polynomials and quantum mechanics; and many other aspects of knot theory too numerous to list here.
>
> Some sessions, such as a vigorous four-dimensional group which met daily, were a complete surprise to the organizers. These surprises were made possible by the workshop's unusual, yet highly effective format. On the first day each participant gave a five-minute introduction to his work, so that we would all have at least a rough idea of each other's interests. An introductory talk and discussion began each subsequent day (to introduce basic concepts and terminology to non-specialists in the topic), but then the rest of the day was "self-organizing". That is, at the conclusion of the introductory talk, participants would propose talks, discussion groups, and software demonstrations for the remainder of that day, according to their interests, and the schedule was written on the blackboard. This self-organizing approach extended to individual sessions, so that speakers could take just as much time as they needed,

without having to either squeeze their ideas into too short a time slot, or stretch them to fill a longer a time slot.

Advances in Nonlinear Astrodynamics, November 6–8, 1993. Organized by **Ed Belbruno** (Geometry Center). This workshop focused on the solution of three- or four- or n-body problems arising in the computation of energy minimizing trajectories for spacecraft. Belbruno is an independant consultant based at the Geometry Center. The workshop was essentially self-supported and was attended by a large number of representatives from the aerospace industry.

Geometric Group Theory, January 3–14, 1994. Organized by **Gilbert Baumslag** (CCNY), **David Epstein** (University of Warwick), **Robert Gilman** (Stevens Institute of Technology), **Charles Sims** (Rutgers) and **Hamish Short** (CCNY). From the writeup in the Center newsletter:

> About twenty people specializing in a variety of different areas of mathematics and computer science participated. All of them had a particular interest in group theoretic programming. The workshop included short talks and computer demonstrations, but most of the time was left free for informal interactions.
>
> According to the organizers, the special layout of the computer workstations and the excellent technical staff of the Geometry Center were important in making the workshop a success. In addition, they said that the Center had been very helpful with the organization of the workshop and that all the participants felt very much at home here.
>
> This workshop was the first of two related workshops. The second one was held March 17–20 at DIMACS, the Center for Discrete Mathematics and Computer Science, a consortium of Rutgers, Princeton, AT&T Bell Laboratories and Bellcore located at Rutgers. The second workshop covered areas of more theoretical interest than the first one.

The following workshop is about to take place this month:

Elliptic and Parabolic Methods in Geometry, May 23–27, 1994. Organized by **Ben Chow** (University of Minnesota), **Robert Gulliver** (University of Minnesota) and **John Sullivan** (University of Minnesota); partly funded by the **IMA**. From the announcement:

> A number of challenging geometrical problems have appeared in recent years whose statements or likely methods of solution involve methods of elliptic or parabolic partial differential equations or geometric measure theory. Interaction between these areas has been remarkably productive.
>
> The availability of powerful hardware and software makes it possible to address geometrical questions in low dimensions by numerical simulation and to represent results in a meaningful and flexible visual format. These questions were previously accessible only to the imagination. The majority of mathematicians pursuing these questions in traditional pencil-and-paper ways, however, have no ready access to the intelligent use of computers to attack geometrical problems. At the same time, the best ideas and most fundamental contributions to the understanding of geometrical phenomena have not always been available to the people who are writing software. One goal of this workshop is to bring these two sides together in fruitful ways.

Five further workshops are being planned:

Topology and Geometry of DNA and RNA, July 29–30, 1994. Organized by **DeWitt Sumners** (Florida State University), and partly funded by the IMA. From the program:

> During the last decade, experimental scientists have become increasingly aware that the powerful analytical techniques of geometry and topology can be used in the interpretation and design of experiments [on polymers]. Chemists have long been interested in developing techniques. . . to synthesize molecules with interesting three-dimensional structure. . . Models for molecular structure must be built and understood; reactions which produce specific three-dimensional shapes must be designed; proof of three-dimensional structure must be produced, and these proofs often involve the use of geometry and topology.

Computational Geometry Software, January 18–20, 1995. Organized by Bernard Chazelle (Princeton University), David Dobkin (Princeton University) and Nina Amenta (Geometry Center); additional support from ONR. This workshop is intended as a meeting place for computational geometers interested in software development and people in other areas such as such as graphics, robotics, biochemistry, vision, math and engineering who use computational geometry algorithms. The format will include software demonstrations, interdisciplinary survey talks, technical presentations, and explicit goal-setting sessions.

Wavelet Analysis as a tool for Geometric Synthesis and Analysis, Spring 1995. Organized by **Robert Coifman** (Yale University) From the synopsis provided by the organizer:

> Wavelets, and more generally adapted waveform analysis, provide a rich collection of (orthogonal) templates that can be used for the efficient representation and construction of geometric structures, from fractals to surfaces and images. These methods provide a simple unifying mathematical framework for such tasks as singularity analysis, feature and parameter extraction, and data compression. This efficient representation permits faster manipulation of geometric data, providing new fast algorithms for CAD as well as image matching and deformation.

Non-Invertible Dynamical Systems, Spring 1995. Organized by **Ray Adomaitis** and **Ioannis Kevrekidis** (both Princeton University) and **Richard McGehee** (University of Minnesota); partly funded by DARPA.

Computer Vision, Spring 1995. Organized by **David Mumford** (Harvard University) and **Allen Tannenbaum** (University of Minnesota).

Geometry Center Visitor Program

Visitors to the Geometry Center find themselves in an excellent computing environment designed by and for mathematicians. This is a unique experience in the mathematical community, and many visitors have made good use of it.

There have been over 380 Geometry Center visitors since February, 1991. (This does not include tour groups, workshop or meeting participants.) Visits last from one day to several weeks or months. Some visitors use the Center facilities and the computer expertise of the staff to develop programs, videos or images. Others share programs or research ideas, or collect teaching materials.

Facilities are arranged so that visitors can get to work immediately, with none of the usual difficulties associated with a new computing environment. Much of the work at the Center is driven by visitor research, so staff members, apprentices and student programmers are eager to get involved, helping with technical problems, offering suggestions, and sometimes writing programs. For example, the ability to view 4D data was originally added to Geomview in response to requests by Dennis Roseman, and the facility continued to evolve through his cooperation with the programmers.

Some visitors use the local facilities and expertise to make videos for seminars and conferences. For instance, **Charles Peskin** and **David McQueen** made a video that helped them win a Smithsonian/Cray research award. This is one concrete way in which the the Center affects the conduct of mathematics. Another is the support given to extended visitors for the production of books with extensive computer-generated illustrations (page 12).

Restructuring of the Visitor Program. In response to suggestions of the **External Advisory Board** and the previous site visit team, a more formal application process for short-term visitors has been adopted, which will be advertised in the standard journals such as the *Notices of the AMS*, and electronically through newsgroups and the World-Wide Web. The application process is simple: the form reproduced on page 68 can be submitted either by electronic mail, by physical mail, or on the **World-Wide Web**. Currently, applications are evaluated by the Director and Director of Technology, but when the **Board of Governors** is in place, they may review the candidates.

Quotes from Visitor Reports

Additional quotes are interspersed with the table of all visitors to the Geometry Center, which starts on page 70.

Marjorie Senechal, March 15–19, 1993:

I would like to thank you again for inviting me to the Geometry Center; my visit exceeded my most optimistic expectations.

I came to the Center with two goals: one, to develop the little **Mathematica** program I had written for simulating simple optical diffraction patterns into one powerful enough to compute and display the patterns produced by large sets of points, and two, to gather materials and ideas for this summer's Regional Geometry Institute at Smith.

As you know, by the time I left three and a half days later the program had been improved so much—thanks to the enthusiastic assistance of **Silvio Levy** and **Stuart**

Jun 26 – Jul 5, 1991 WRT–AUT–HYP *David Epstein,* Warwick	Had discussions with Thurston and others. Worked on *Three-Dimensional Geometry and Topology* and on *Word Processing in Groups*.
Jun 27–30 HYP–ED *Helaman Ferguson,* artist	Made plans for Center sculpture. Learned more hyperbolic geometry from Thurston, Epstein, etc.
Jun 30 – Jul 7 WRT–HYP *Bill Floyd,* Virginia Polytechnic	Had discussions with Thurston and others. Worked on *Three-Dimensional Geometry and Topology*.
Jun 30 – Jul 4 WRT–HYP *Steve Kerchkhoff,* Stanford	Had discussions with Thurston and others. Worked on *Three-Dimensional Geometry and Topology*.
Jul 27 – Aug 17 RES–OPT–PGM–RES–PR–ADV *Rob Almgren,* Courant Institute	Minimal Surface Team research; see quote below.
Jul 28 – Aug 18 RES–OPT *Fred Almgren,* Princeton	Minimal Surface Team research.
Jul 28 – Aug 18 RES–OPT *Jean Taylor,* Rutgers	Minimal Surface Team research.
Jul 28 – Aug 18 PGM–VID–RES–OPT–PR *Andy Roosen,* Rutgers	Vastly expanded and modified program for simulating crystal growth using the method of crystalline curvature and taking into account the diffusion of latent heat.
Jul 29 – Aug 21 OPT–PGM–RES *Ken Brakke,* Susquehanna	Worked on Surface Evolver.
Aug 1 – Sep 4 PGM–VID *David Ben-Zvi,* Princeton	Familiarized himself with software. Started work on *Outside In* with Silvio Levy.
Aug 2–5 HYP–PGM–RES *Jeff Weeks,* Middlebury College	With Mark Phillips, made plans for a rewrite of Snappea to allow easy porting.
Aug 4–30 PGM–VID *Matt Headrick,* Princeton	Familiarized himself with software. Started work on *Outside In*.
Aug 6–31 WRT–AUT *David Epstein,* Warwick	Worked on *Word Processing in Groups*.

This visit marked a real breakthrough for the Minimal Surface team. Our two-year project to simulate the growth of real crystals is beginning to be successful; this success is in the form of two working computer programs, one by myself and one by Andy Roosen, which reproduce solidification behavior of physical materials.

I like the NeXT computers very much, and I am glad that the Center is moving toward them. . . I was impressed by quality of the undergraduate students.

From report by Rob Almgren (Jul 27 – Aug 17)

The center's facilities were of enormous help to me. . . The working environment was also of great help, allowing me to produce more lines of code in two weeks than in the previous four months.

Also thanks to Stuart Levy for helping make the videotape of the resultant crystal possible. . . I cannot emphasize enough the value of the distraction-free work environment.

From report by Andy Roosen (Jul 28 – Aug 18)

Aug 7–21, 1991 RES–PGM–PR *Mike Casey,* UC San Diego	Worked on programs that compute Betti numbers for discrete spaces and that numerically study a cubic map as a dynamical system.
Aug 11–14 BUS *Bjørn Dahlberg,* Volvo	Discussed and gave talk on mathematical models for computer-aided car design.
Aug 14–18 RES–CG *Konrad Polthier,* Bonn	Discussions with John Sullivan on minimal surfaces and with staff on the GRAPE graphics environment.
Sep 4–9 ED–PGM–ADV *Bill Goldman*, U. Maryland, and *John Harer,* Washington and Lee U.	Wrote a proposal to establish an undergraduate RGI course. Discussed various technical aspects of the course with Mark Phillips.
Sep 4–8 RES *Paul Scofield,* Washington and Lee U.	Worked on solving natural equations for space curves. "My visit to the Geometry Center helped me make rapid progress on my problem."
Sep 28 – Oct 7 OPT–RES–GV *Lucas Hsu,* MSRI	Used the Evolver to test conjectures related to the Willmore Problem. "I benefited greatly from interactions with John Sullivan and Ken Brakke. . . I find MinneView to be indispensable."
Oct 7–9 BUS–ED–RES–VID–GV *Jim Cobb*, IBM, and *Herb Clemens,* Utah	Their goal was "to make use of insights gained at the Geometry Center from the making of *Not Knot*" for the production of a visualization video on the minimal triangulation of $\mathbf{CP}^2$.
Oct 10–14 ED–VID *George Francis,* UIUC	Discussed graphics software and videos with staff
Oct 26 – Nov 3 PGM–VID *Matt Headrick,* Princeton	Worked on *Outside In*.
Oct 31 – Nov 6 HYP–PGM–VID *John Parker,* Maryland	Produced pictures of the boundary of certain Dirichlet domains of groups of complex hyperbolic isometries. Continued joint research with Bill Goldman.
Nov 4–10 OPT–PGM–GV–ED–PR *Rob Kusner,* U Mass Amherst	Developed Willmore algorithms for Surface Evolver. Discussed "vertically integrated programs", as the research coordinator of the NSF Five Colleges Geometry Institute.
Nov 6–13 HYP–RES–WRT *Clifford Earle,* Cornell	Studied the extension of holomorphic motions invariant with respect to some group of projective transformations of the sphere. With Marden, studied the geometry of the boundary of Teichmüller spaces.
Nov 13–20 PGM–WRT–VID–PR *Alfred Gray,* U. Maryland	Worked on his book *Curves and Surfaces*. Made *Strange Surfaces* video with Mark Phillips and Tamara Munzner.
Dec 12–16 RES *Don Mitchell* and *Pat Hanrahan,* Princeton	Worked on paper about illumination from curved reflectors.

Jan 12–18, 1992 RES–DYN–PR–GV–PGM *Alessandra Celletti,* Univ. of L'Aquila	Computed and viewed KAM tori for a class of simplectic maps in $\mathbf{R}^4$ with Dick McGehee, Eduardo Tabacman, and Wenxiong Liu.
Feb 1–28 VID–ED–PGM–PR *Audun Holme,* UCLA	Collaborated with Joel Roberts of the School of Math. Involved in math education (geometry text in progress). Inspired construction of algebraic curves program. Video on algebraic curves with Mark Phillips.
Feb 10–25 DYN–ADV–WRT–RES *Taka Shiota,* Kyoto	Wrote a paper on numerical simulation of a nonlinear differential equation with D. Mumford. Discussions with School of Math members on soliton equations.
Feb 12–18 CG–RES–ED–PGM–GV *George Francis,* UIUC	Made many contacts: Audun Holme, Dennis Roseman, Arnie Cutler, Taka Shiota. Discussed graphics and mathematics teaching. Generated graphics for publication.
Feb 13–17 PR–GV–VID-RES *Dennis Roseman,* Iowa	Worked on visualization of knotted spheres in four-space.
Feb 24 – Mar 4 OPT–PGM–VID–RES *Ryo Kobayashi,* U. Ryukoku	Conducted crystal growth simulations, attended workshop.
Mar 4–31 ED–RES–DYN *Jürger Moser,* ETH, Zürich	Gave a talk on mathematics of billiards for high school teachers. Advised researchers attending symplectic workshop.
Mar 8–23 PR–VID–ADV–RES *John Maddocks,* U. Maryland	Visualization of Symplectic Maps workshop. Ported his MC^2 program for interactive graphics and multi-parameter path following from DEC to IRIS. Made video. Used Minneview and Explorer.
Mar 14–22 PGM–VID *David Ben-Zvi,* Princeton	Worked on *Outside In* with Silvio Levy.
Mar 20 – Apr 1 DYN–RES *Alfredo Poirier,* SUNY Stony Brook	Worked on a problem, proposed by Milnor, in dynamics of rational maps,
Mar 21–24 OPT–VID *Jean Taylor,* Rutgers	Worked on video compilation from crystal growth workshop, for publication by the AMS.
Mar 21–25 RES–PGM *Igor Rivin,* NEC	Discussions with Silvio Levy, Al Marden and others on symbolic software development.

I found the visit to be extremely useful. The Geometry Center is clearly a tremendous facility and resource for the uses of computer graphics in mathematics. . . I think the Center can play a valuable scientific role in acting as a clearing house of information and expertise concerning purely technical issues in computer graphics as it applies to mathematics. It certainly helped me in that regard.

I wanted to officially let you know how helpful, responsive and pleasant all the technical staff at the Geometry Center were both before, during, and after my visit to Minneapolis. . . Tamara was cheerful and cooperative. . . she is a gem of an employee.

From report by John Maddocks (Mar 8–23)

Mar 26–29, 1992 ED *Gene Klotz,* Swarthmore College	Conversations with Al Marden and others on possible plans for educational software production setup.
Mar 30 – Apr 29 HYP *Seppo Rickman,* Helsinki	Studied the Center as a model for a possible similar in Helsinki. Discussed hyperbolic geometry software.
Apr 4–7 BUS–ED–RES *Jim Cobb*, IBM, and *Herb Clemens,* Utah	Continued discussions with staff and work on $\mathbf{CP}^2$ animation (see earlier entry in October 1991).
Apr 8–12 *Brad Barber*, Princeton, and *Paul Burchard,* Utah	Prospective postdocs.
Apr 18–21 *Louis Kaufmann,* U. Ill. Chicago	Make plans for Knot Theory workshop.
Apr 22–26 WRT–CG–RES *Todd Drumm,* MSRI	Wrote paper on "crooked planes". Got help from staff on Mathematica and Geomview graphics.
Apr 22–26 CG–PGM–PR *Mika Seppala,* Helsinki	Installed programs to define groups of Moebius transformations interactively.
Apr 30 – May 10 RES–WRT–ED *Audun Holme,* Bergen, Norway	Research in algebraic geometry, visualization using Mathematica, book writing; see earlier visit in February.
May 14–17 PRG *Gilbert Baumslag, Hamish Short, William Sit*, CUNY, *and Robert Johnson,* St. Cloud State	Consulted about building their group-theory software Magnus (see quote below; all returned for the 1994 Group Theory conference, presenting finished software.)
May 16 – Aug 15 PGM–VID–CG *Nathaniel Thurston,* UC Berkeley	Integrate Salem motion paradigm into Geomview. Modeled sphere eversion for *Outside In*.
May 16–30 RES–WRT *David Epstein,* Warwick	Interacted with staff, Al Marden, Magnus developers (see previous entry). Worked on *Three-Dimensional Geometry and Topology*.
May 21–29 RES–WRT *James Cannon,* Brigham Young	Contacts with Thurston, Epstein and others. Worked on *Three-Dimensional Geometry and Topology*.
May 21–26 PGM *Igor Rivin*, NEC, and *Henry Cejtin,* consultant	Discussions and presentation leading to proposal for symbolic software development exploratory grant.

> There are many problems in a wide variety of disciplines which can partially be distilled into problems about groups given by generators and defining relations. There are a number of primitive techniques that can, in principle, be used to obtain some information about the groups involved. In the past one has made some, usually ineffectual, efforts to apply these techniques by hand. Our project is designed to implement these techniques on machines. Theoretically there is no chance of success. In reality, however, these experiments are precisely what mathematicians do when they try to get some idea as to whether or not something is true.
>
> On a technical level our time spent here has been very beneficial. [Stuart] Levy and Phillips have. . . taught us many things and have shared many of their ideas with us. This will be of great value when we start on our next round of programming. . .

From report by Gilbert Baumslag et al. (May 14–17)

May 24–31, 1992 RES *Walter Parry,* Michigan	Work with Cannon and others on Combinatorial Riemann Mapping Theorem.
May 27–29 ED *Gene Klotz,* Swarthmore College	Continued plans for educational software production grant proposal.
May 31 – Jun 5 PGM–VID–GV–CG–RES *Dennis Roseman,* Iowa	Constructing knotted spheres, examined with Geomview/4Dview.
Jun 12 – Aug 15 PGM–ED *Ted Stanford,* Columbia	Worked on knot-theory programs. Helped summer students.
Jun 12 – Aug 15 ED *Tony Phillips,* SUNY Stony Brook	Head coach for summer institute.
Jun 13–20 PGM–ED *Jeff Weeks*, Middlebury College, and *Joe Christy,* MSRI	Discussed Snappea directions with Mark Phillips. Interacted with summer students, three of whom began work on *Shape of Space* video.
Jun 13–19 PGM–RES–OPT *Andy Roosen,* Rutgers	Crystal growth research.
Jun 15–19 ED *Stan Wagon,* Macalester College	Taught short Mathematica course to summer students. Helped suggest summer student projects.
Jul 6–17 ED *John Hubbard*, Cornell, and *Bodil Branner, Bjørn Felsager, Mette Vedelsby,* Denmark	Taught summer course on Chaos and Fractals. Hubbard also came June 15–19 and 22–25 to prepare and give summer student project generating seminar.
Jun 15–19 ED–BUS–RES *Blaise Morton,* Honeywell	Guided summer student in research topic.
Jun 15–19 ED–RES *Jean Taylor,* Rutgers	Crystal growth research. Gave project generating seminar for summer students.
June 21–23 ED–BUS *Nick Jackiw,* Key Curriculum Press	Planning for educational software production grant proposal with Klotz.
Jul 1992 – Aug 1993 OPT–PGM–WRT–VID–CG–RES *Ken Brakke,* Susquehanna	Added new features to Evolver, including new types of energies and constraints, and command language extensions. Versions under development for several parallel machines. Created a Surface Evolver Newsletter, distributed by e-mail. Worked on developing a new mathematical model of soap films, and created related program polycut. Worked with Knud Andersen on software to find soap films without prior knowledge of their topology. Assisted summer students and directed a project.
Jul 20–21 ED–GV–VID–PR *Richard Millman,* Cal State at San Marcos	Learned software. Got ideas from Harvey Keynes and Tony Phillips for his non-Euclidean geometry course; decided to use *Not Knot* in the course.
Jul 27 – Aug 2 RES *Miller Maley,* UC Berkeley	

Aug 1–6, 1992 ED *Liisa Kinnunen,* Helsinki	Studied the Center's initiatives in education.
Aug 2–22 WRT–OPT–PGM–CG–RES *Fred Almgren,* Princeton	Wrote two papers with Wang and Taylor. Worked with Ken Brakke on multi-sheeted covers of knot complements.
Aug 2–22 PGM–WRT–OPT–RES *Jean Taylor,* Rutgers	Research on three-dimensional crystal growth by crystalline curvature; see also previous entry.
Aug 2–9 ED–VID–PGM–GV–CG *François Apéry*, Haute-Alsace, and *George Francis,* UIUC	Helped summer students; created video of polyhedral sphere eversion. Discussed sphere eversion with David Ben-Zvi and Tony Phillips. Worked on plans for Heidelberg conference.
Aug 2–22 PGM–OPT–CG–PR *Rob Almgren,* Courant	Wrote three-dimensional crystal growth code.
Aug 3–6 CG–RES *Pat Hanrahan*, Princeton, and *Seth Teller, UC Berkeley*	Discussed Voronoi visibility walk-through and other graphics with staff.
Aug 6 – Sep 6 RES *David Epstein,* Warwick	Wrote *Not Knot* supplement with Charlie Gunn.
Aug 8–14 VID *Delle Maxwell,* consultant	Worked on plans for *Outside In*.
Aug 14–28 PGM–OPT–RES *Andy Roosen,* Rutgers	Implemented C++ data structure for 3D crystal program, with Taylor.
Aug 14–28 GV–PGM–VID *Richard Morris,* Liverpool	Worked on algebraic surface program and module for Geomview. Made video *Parade of Surfaces*.
Aug 16–22 PGM–CG–OPT–RES *Nelson Max,* UC Davis	Volume visualization of Rob Almgren's temperature fields. Showed Topology Films Project films.
Aug 18–25 OPT–RES *Lucas Hsu,* MSRI	Worked on Willmore surfaces with John Sullivan.
Aug 19 – Sep 10 HYP–PGM–CG–AUT–DYN *Greg McShane,* Warwick	Began a thorough computer investigation of a certain dynamical system arising from work of Brian Bowditch on arithmetic torus bundles.
Sep 1 – Nov 30 RES *Dan Freed,* U. Chicago	Studied models of topological quantum field theory.
Sep 1 – Nov 30 ED–RES *Matt Richey,* St. Olaf College	Developed curriculum material for upper-level undergraduate courses.
Sep 1 – Nov 30 RES–DYN–VID *Bruce Peckham,* U. of MN, Duluth	Worked with McGehee on dynamical systems problems. Created video on resonant surfaces.

> In two dimensions. . . , for example, numerical instabilities in the interior of the domain show up as waves on the surface. In 3D one is working blind; one may visualize only a very few aspects of the data. . . The rate at which one can advance the state of the art in computation is closely linked to the current state of the art in visualization techniques.
>
> From report by Rob Almgren (Aug 2–22)

Sep 1 – Oct 30, 1992 RES–DYN *Antoni Susin,* Barcelona	Worked on planar three-body problem with triple collisions.
Sep 8–13 PGM *Nina Amenta,* UC Berkeley	With Tamara Munzner, worked on new version of her program for interactively creating patterns with planar symmetries.
Sep 17–21 VID–GV–RES *Dennis Roseman,* Iowa	Generated animations of surfaces in $\mathbf{R}^4$. Nice presentations of the torus and Klein bottle, "an example I thought I knew very well and yet I discovered something 'new' about it using Geomview."
Sep 17–23, Oct 20–28 VID *Delle Maxwell,* consultant	Worked on *Outside In*.
Oct 8–11 WRT–OPT–RES *Lucas Hsu,* MSRI	Worked with Sullivan on Willmore surfaces, finishing paper.
Oct 18–21 PR–HYP *Yasushi Yamashita,* Nara College	Looked into hardware to use for computing in mathematics. Discussed issues of software standardization.
Oct 28 – Nov 4 WRT–RES *Michael Hirsch,* UC Berkeley	Worked with Brad Barber on paper on point-in-polyhedron algorithm.
Oct 28 – Nov 1 VID–BUS–ED *Jim Cobb*, IBM, and *Herb Clemens,* Utah	Continued discussions with staff and work on $\mathbf{CP}^2$ animation.
Oct 29–31 ED *Gene Klotz,* Swarthmore College	Educational software production grant proposal.
Nov 4–9 RES *John Lott,* U. Mich. Ann Arbor	
Nov 7–14 PGM–AUT *Steve Rumsby,* Warwick	Installed current versions of Warwick automatic groups software. Learned Geomview.
Nov 12–17 PGM *Igor Rivin*, NEC, and *Henry Cejtin,* consultant	Work on symbolic software development exploratory grant.
Nov 15–20 PGM–CG–GV–HYP *Robert Riley,* SUNY Binghamton	Interfaced with Geomview and Mathematica his PNCRE program that deals with discrete subgroups of $\mathrm{SL}(2,\mathbf{C})$. Held discussions with Charlie Gunn, Oliver Goodman, and other technical staff.
Nov 19–22 RES–HYP *Troels Jorgensen,* Columbia	Worked on discrete groups and hyperbolic geometry with Al Marden.
Dec 14–21 VID *Delle Maxwell,* consultant	Worked on *Outside In*.

Many visitors seem to come to the Center to produce something, [but] my primary aim was to see how it is organized and to find what kind of research people are doing.

Simply speaking, what I have done during my stay in the Geometry Center was just to watch and talk... I would like to express my thanks... especially [to] Prof. A. Marden, Ms. A. Vail, Dr. O. Goodman, Dr. M. Phillips.

From the report by Yasushi Yamashita (Oct 18–21)

Jan 3–17, 1993 WRT–RES *David Epstein,* Warwick	Worked on paper on the Poincaré polyhedron theorem.
Jan 4–5 VID *Delle Maxwell,* consultant	Worked on *Outside In*.
Jan 6–29 VID–GV–RES *Dennis Roseman,* Iowa	Continued research on knotted spheres. Made video.
Jan 18–24 VID–GV *George Francis,* UIUC	Planned Heidelberg math visualization conference with Mark Phillips. Discussed collaboration with UIC's EVL lab to incorporate hyperbolic geometry, and possibly others, into the Cave. Discussed porting Evolver to the CM-5 with Ken Brakke. See full report on page 66.
Jan 23–27 PGM–ADV–GV–CG *Andy Hanson,* Indiana	Learned Geomview and Evolver, and used them in 3 and 4 dimensions. Ken Brakke added new "electrostatic" energy for him. Held discussion group on philosophy/plans/designs for visualization tools at the Center and Indiana.
Jan 27 – Feb 2 RES *Henry Wente,* Toledo	
Feb 1–7 PGM–CG *Adam Deaton, David Dobkin,* Princeton	Completed release of yaep (which later evolved into lafite), and gathered knowledge (from Charlie Gunn and Oliver Goodman) towards senior thesis project, advised by Dobkin and Conway.
Feb 2–7 PGM–RES *Jeff Weeks,* Clarkson	Snappea development.
Feb 3–8 PGM–PR–GV–RES *Rob Kusner,* IAS	With Ken Brakke, Charlie Gunn, John Sullivan, implemented Freedman–He knot energy evolution in Evolver. Experiments gave rise to later paper.
Feb 8–19, Mar 1–12, 21–27 VID–GV–RES *Dennis Roseman,* Iowa	Continued research on knotted spheres. Made video.
Feb 14–17 RES *Ulrich Pinkall, Matthias Heil,* Technische Univ. Berlin	With Charlie Gunn, discussed closer collaboration between Berlin group and Center.
Feb 15–19 ED–VID *Audun Holme,* Bergen	Continued work on video with Mark Phillips.
Feb 16–21 *Karin Johnsgard,* UIUC	Prospective postdoc.
Feb 26 – Mar 1 RES *Dan Freed,* U. Chicago	Studied models of topological quantum field theory.
Mar 5–15 GV–RES *Amir Assadi,* U. Wisconsin, Madison	Learned about Geomview and SGIs. Arranged for Brakke, N. Thurston and Ben-Zvi to give seminars at Madison.
Mar 14–21 WRT *Alfred Gray,* U. Maryland	Worked on differential geometry textbook. Made video with David Ben-Zvi.

Mar 14–21, Apr 7–15, 1993 VID *Delle Maxwell,* consultant	Worked on *Outside In*.
Mar 15–19 PGM–PR–ED–CG–RES *Marjorie Senechal,* Smith College	Worked on computing diffraction patterns for quasicrystalline tilings. Interested in materials available at the Center and the the teachers' summer program. Arranged for Tamara Munzner and David Ben-Zvi to speak at upcoming RGI. See full report on page 66.
Mar 18 – Apr 15 PGM–ADV–PR–CG *Mitsuyuki Ochiai,* Nara College, Japan	Took part in Knot Theory workshop. Installed his knot theory program at Center, demonstrated it. Wrote program to compute matrices of two-variable Jones polynomials.
Mar 21–26 RES *Philip Holmes, Robert Ghrist,* Cornell	Attended Knot Theory Workshop. Worked on knots as periodic orbits in flows on three-manifolds given by differential equations.
Mar 27 – Apr 3 *Frederick Wicklin,* Cornell	Prospective postdoc.
Apr 11–23, May 1–8 VID–GV–RES *Dennis Roseman,* Iowa	Worked on energy-minimizing knotted surfaces. Made video.
Apr 16–19 RES *Dan Freed,* U. Chicago	Studied models of topological quantum field theory.
Apr 17–23 RES–PGM *Igor Rivin,* NEC	Attended knot theory workshop. Continued work on symbolic software development.
Apr 18–25 PGM–OPT–RES *Harold Parks,* Oregon State	Used his least-gradient method with Evolver to construct minimal surfaces with given boundary, applied to some sample problems.
Apr 26–27 *Maria Klawe,* U. British Columbia	Made an early tour of the Center since she could not come for External Advisory Board meeting in May.
Apr 26–30 CG–GV–PGM *Ayellet Tal,* Princeton	Received feedback on her work in algorithm animation. Was introduced to Evolver. Discussed software system design.
Apr 27–28 RES–PGM *Ruth Gornet,* Washington U.	Gave seminar talk at the School of Math on her results in spectral geometry of nil-manifolds. Discussed adding Heisenberg geometry to Geomview's capabilities.
May 5–12 PGM–WRT *Andy Hanson,* Indiana	Demonstrated his and Hui Ma's work on 4-D computer graphics software. Wrote summary of features important for 4-D viewing. Participated in discussion group on 4-D extensions for Geomview.
May 8–22 PGM *Hui Ma,* Indiana	(See also previous entry.) Demonstrated 4-D viewer with Scheme interface. Wrote program to find self-intersections of surfaces in 4-space projected into 3-space. Had discussions with staff and visitors Chris Hartmann and Glenn Chappell. Learned from watching people work.

May 10–23, 1993 WRT–RES *Jim Cannon*, Brigham Young, *Bill Floyd*, Virginia Polytechnic, and *Walter Parry,* Eastern Michigan U.	Finished paper on the combinatorial Riemann mapping theorem. Also proved related results (fifteen of which are mentioned in their report).
May 10 – June 3 PGM–HYP–WRT–AUT–RES *Greg McShane* and *Ian Redfern,* Warwick	Worked on improving speed of Knuth–Bendix code. Tried examples, some from Cannon and Floyd. Worked on estimating the Hausdorff dimension of a circle-packing limit set. McShane also worked on a preprint about homeomorphisms of surfaces preserving simple geodesics.
May 12–16 PGM *Barry Merriman,* UCLA	Worked on CADiff with Paul Burchard.
May 17 – Jun 18 RES *Dan Freed,* U. Chicago	Studied models of topological quantum field theory.
May 17–22 PGM–ED *Chris Hartman* and *Glenn Chappell,* UIUC	Worked on program for reconstructing surfaces from level curves (toward animating Tony Phillips' sphere eversion).
May 19–30 VID *Delle Maxwell,* consultant	Worked on *Outside In* and on designing the *Not Knot* poster.
May 20–22 PGM *Frank Quinn,* Virginia Polytechnic	Discussed software issues with staff.
May 20 – June 2 PR–PGM–HYP–AUT–CG–RES *Michihiko Fujii,* Tokyo Inst. Tech.	Discussed circle packings and Kleinian group limit-sets with Ian Redfern and Greg McShane. Learned to use their software. Tried Snappea.
May 20 – Jun 2 WRT–PR–RES *Katsuhiko Matsuzaki,* Tokyo Inst. Tech.	Improved proof of result that stability implies geometric finiteness for Kleinian groups, and results on ergodic properties of discrete groups.
May 22–31 ED *Gene Klotz,* Swarthmore College	Educational software production grant proposal.
May 24–27 RES–GV *Dan Asimov,* NASA Ames	Possible candidate for Director of Technology. Discussed with staff visualization of minimal surface of genus 73 in three-sphere.
Jun 7 – Jul 31 PGM–ED *Ted Stanford,* Columbia	Rewrote his Vassiliev knot-invariant program, allowing for extension to invariants of links and three-manifolds as well. Spoke to undergrads about their projects, and about Vassiliev invariants. Michael Sullivan, a summer student, integrated Vassiliev-invariant computation into the LinkTool knot editing program.

> I have been surprised that attitudes of people here for mathematics are slightly different from the one I knew. They are in atmosphere of freedom, in pleasure of mathematics. They would stimulate inspirations to solve problems. Indeed, I have found answers for long unsettled problems mentioned [above] so easily.
>
> From Katsuhiko Matsuzaki's report (May 20 – Jun 2)

Jun 13 – Aug 20, 1993 ED *Tony Phillips,* SUNY Stony Brook	Head coach for summer course.
Jun 14–18 ED–VID–CG–PR Charles Peskin and David McQueen, Courant	Inspired student projects on applications of mathematics and computing to biology. Gave two talks. Created a video of their simulation of the beating heart, which was later nominated for a Smithsonian/Cray Research award.
Jun 19 – Jul 3 ED–CG *Pat Hanrahan* and *David Dobkin*, Princeton, *Diane Souvaine*, Rutgers, and *Vibeke Sorensen,* Cal. Inst. for the Arts	Taught the Computer Graphics summer course
Jun 23–27 VID *Delle Maxwell,* consultant	Worked on *Outside In* and on *Not Knot* poster.
Jul 7–9 GV–HYP–ED–PR *Robert Miner,* Oklahoma	Learned about Geomview and Mathematica.m. Discussed educational software project he'll be undertaking. Got initial tutorial on programming on the NeXT.
Jul 14–18 ED *Gene Klotz,* Swarthmore College	Led the Geometry Software Conference.
Jul 14–15 ED–RES *Bill Thurston,* MSRI	Participated in Geometry Software Conference. Discussed *Outside In*.
Jul 16–18 OPT–GV–RES *Rob Kusner,* U. Mass. Amherst	Carried on discussions and experiments with Ken Brakke on self-similar mean curvature flow, Willmore surfaces, sphere eversion. Planned future projects with Lucas Hsu, John Sullivan, George Francis.
Jul 17–19 ED–VID *Herb Clemens,* Utah	Continued discussions with staff and work on $\mathbf{CP}^2$ animation.
Jul 17–21 PGM–HYP–ED–GV–RES *Joe Christy,* MSRI	Worked with Jeff Weeks (see next entry). Participated in Geometry Software conference discussions. Discussed Geomview and computer hardware with staff.
Jul 18–21 PGM–HYP–ED–GV–RES *Jeff Weeks,* consultant	Worked with Joe Christy on Snappea programming and design. Discussed and experimented on a conjecture: all orientable hyperbolic 3-manifolds may be obtained by surgery on a chain of unknots with minimal twist.
Jul 22–29, Aug 4–7 VID–GV–RES *Dennis Roseman,* Iowa	Continued research on energies of knotted spheres.
Jul 28–31 RES *Dan Freed,* U. Texas, Austin	Studied models of topological quantum field theory.
Aug 1–8 RES–HYP *Troels Jorgensen,* Columbia	Worked on discrete groups and hyperbolic geometry with Al Marden.

Aug 5–27, 1993 PGM–OPT–RES *Dave Anderson,* Rutgers	Worked on implementing a wavelets approach that will ultimately do heat flow in dendritic crystal growth.
Aug 7–30 *Fred Almgren,* Princeton	
Aug 7–30 PGM–OPT–RES *Jean Taylor,* Rutgers	Rewrote in C++ major pieces of her program for moving crystalline surfaces by their crystalline weighted mean curvature.
Aug 7–28 *David Caraballo,* Princeton	
Aug 7–28 *Nung Kwan Yip,* Princeton	
Aug 9–21 *Rob Almgren,* U. Chicago	
Aug 15 – Sep 5 *David Epstein,* Warwick	After the NSF site visit, worked on Center restructuring.
Aug 16–27 PGM–OPT–RES *Andy Roosen,* NIST	Worked on his program for dendritic crystal growth in the plane, achieving an important step in the conversion of the program from using a uniform grid to a more efficient system.
Aug 20–24 *Steve Kerchkhoff,* Stanford	
Aug 21–24 *John Harer,* Duke	
Aug 21–27 PGM–OPT–RES–VID *Craig Carter,* NIST	Improved Roosen's code for motion of crystalline curves in the plane. Made a video of his use of Evolver to compute forces involved in sintering.
Aug 27–28 ED *Richard Lehrer,* U. Wisconsin, Madison	Discuss with Cutler future interaction of NCRMSE (National Center for Research in the Mathematical Sciences Education) with Center under the materials development proposal.
Aug 27–29 VID–GV–RES *Dennis Roseman,* Iowa	Finished work on video *Twisting and Turning in Four Dimensions*.
Aug 31 – Sep 5 PR–RES *Peter Doyle,* UC Sand Diego	Discussed Möbius energy of knots with Sullivan and Kusner. Discussed CRSolver with Burchard.
Sep 1–5 OPT–RES *Rob Kusner,* U. Mass. Amherst	See previous entry.
Sep 7 – Jan 15, 1994 WRT *Alfred Gray,* U. Maryland	Completed his book on differential geometry.

> This exchange will help coordinate the work of the numerous people who are currently looking at these [knot] invariants. . .
>
> [Paul Burchard's CRSolver] will be an excellent tool for understanding line bundles over Riemann surfaces, and more.
>
> From Peter Doyle's report (Aug 31 – Sep 5)

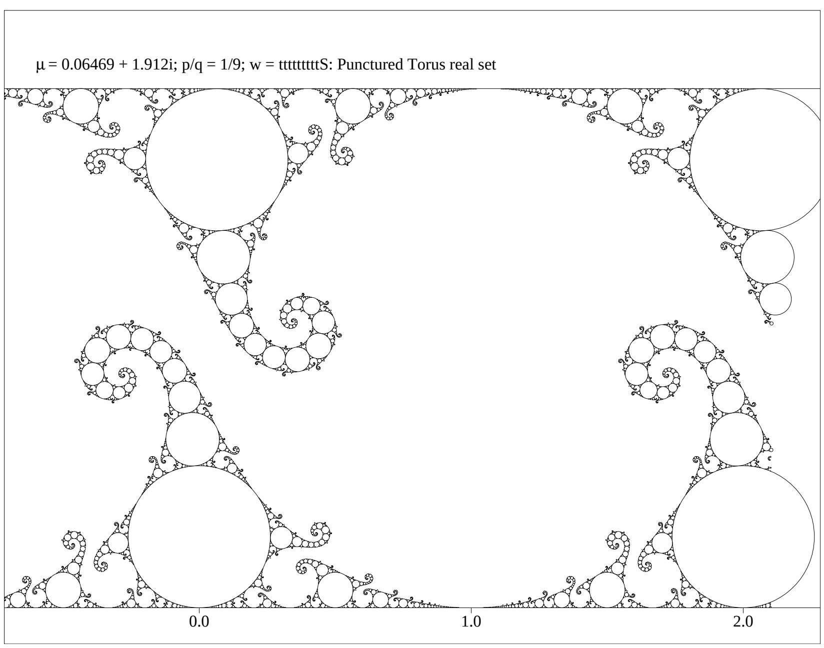
Sep 11–14, 1993 HYP–PGM–RES *Robert Meyerhoff,* Boston College	Collaborated with David Gabai on problems in three-manifold topology, and interested Nathaniel Thurston in attacking one such problem by computer (see page 98).
Sep 13–14 HYP–PGM *Bill Goldman,* U. Maryland	Discussed hyperbolic geometry software with Mark Phillips.
Sep 20 – Dec 17 RES *Dan Freed,* U. Texas, Austin	Continued work on topological quantum field theory.
Oct 1–14 VID *Delle Maxwell,* consultant	Worked on *Outside In*.
Oct 17 – Nov 19 PGM–ED–CG-RES *Marjorie Senechal,* Smith College	Worked on her book *Quasicrystals and Geometry* (to appear in 1994, Cambridge Univ. Press), with programming assistance from Stuart Levy and Eugenio Durand.
Oct 21–23 RES *Alex Selby,* U. Texas, Austin	Worked with Dan Freed on topological quantum field theory.
Oct 21–24 HYP–PGM–CG–PR–ADV *Ken Stephenson,* U. Tennessee	Discussed computer graphics. Got feedback and technical help with his circle packing programs. Made suggestions for Geomview improvements.
October 23–25 ED–BUS *Gene Klotz,* Swarthmore College, and *Nick Jackiw,* Key Curriculum Press	Educational software production grant proposal.
Nov 1–5 VID *Delle Maxwell,* consultant	Worked on *Outside In*.
Nov 3–6 *Steen Markvorsen,* Tech. Univ. Denmark	
Nov 4–11 *Helaman Ferguson,* artist	In town for FISEA conference. Discussed *Outside In*.
Nov 7–20 RES–DYN *Carles Simó,* Barcelona	Worked on computation of invariant manifolds in problems of celestial mechanics.
Nov 11–16 VID *John Sullivan,* MSRI	Made a video about experiments with knot energies.

I enjoyed many other features of the Center—indeed, there were none that I did not enjoy. It was a pleasure to get to know the postdocs as well as the staff, to meet other visitors, and to learn something about what other people are doing. The Center is a wonderfully informal place where learning takes place in many ways—including by osmosis.

Such an atmosphere does not come about by accident. The longer I was at the Center, the more I realized that its design is the key to its functioning. The combination of private offices and open spaces, and the grouping of workstations within the open spaces, is optimal for encouraging interchange without in any way discouraging intensive individual work.

From Marjorie Senechal's report (Oct 17 – Nov 19)

Nov 14–17, 1993 *Leo Guibas,* Stanford	Gave talk at CS department. Worked with Leonidas Palios and Nina Amenta.
Nov 18–19 DYN–RES *Mark Levi,* Rensselaer Polytechnic	Discussed problems in dynamical systems with McGehee, Wicklin; gave talk at School of Math.
Nov 18–21 RES *Veit Elser,* Cornell	Worked with Marjorie Senechal on zonohedra.
Nov 19 BUS–RES *Peter Shor,* AT&T Bell Labs	Gave seminar on biological applications of computational geometry. Talked with Nina Amenta.
Nov 27 – Dec 17 WRT *Renzo Caddeo,* Univ. di Cagliari	Worked with Alfred Gray, translating his book into Italian.
Dec 4–8 PGM *Igor Rivin*, IAS, and *Henry Cejtin,* NEC	Continued work on symbolic software development.
Dec 12–19 VID *Delle Maxwell,* consultant	Worked on *Outside In*.
Dec 14–16 *David Dobkin,* Princeton	Worked on Center restructuring.



The limit set of the group of Möbius transformations generated by $z \mapsto z + 2$ and $z \mapsto \mu + 1/z$, where $\mu = 0.06469 + 1.912i$. Taken from the paper by Greg McShane, John R. Parker and Ian Redfern, "Drawing limit sets of Kleinian groups using finite state automata", submitted to *Experimental Mathematics*. Produced with algorithms partly developed during the McShane and Redfern's visit to the Center (May 10 – June 3, 1993); see pages 80 and 95.

Jan 7–9, 1994 *John Guckenheimer*, Cornell University, and *Al Thaler,* NSF	Took part in Center restructuring meeting with Executive Committee.
Jan 17–28, Feb 12 – Mar 5 VID *Delle Maxwell,* consultant	Worked on *Outside In*.
Mar 3–5 *David Dobkin,* Princeton University	Worked on Center restructuring.
Mar 8–16 HYP–RES *David Epstein,* University of Warwick	Worked on Center restructuring. Talked to N. Thurston and R. Meyerhoff.
Mar 9–14 PGM–HYP–RES *Robert Meyerhoff,* Boston College	Worked with Nathaniel Thurston on hyperbolic groups parameter space program.
Mar 14 – Apr 21 VID *Delle Maxwell,* consultant	Finished *Outside In* !!!
Mar 21–22 *Chaim Goodman-Strauss,* U. Texas, Austin	Prospective postdoc.
Mar 29 RES–WRT *Bruce Peckham,* U. of MN, Duluth	Worked with McGehee on dynamical systems problems. Finished a joint paper.
Mar 30 – Apr 14 ED *James King,* U. Washington	Discussed undergraduate and high-school education and teacher enrichment. Gathered materials for undergraduate program in which students would produce mathematical videos. Saw World-Wide Web applications and arranged for software to be available at Park City/IAS Math Institute this summer.
Apr 14–17 ED *Ioannis Emiris,* UC Berkeley	Prospective postdoc.
Apr 16–30 VID–RES–CG–PGM *Yolanda Furuya* and *Noritsuna Furuya,* U. Fed. São Carlos, Brazil	Collected ideas and information for their home institution. Noritsuna attended MSI conference, talked with EE researchers on tomography, and learned about the GL graphics. Yolanda talked about Pisces with Silvio Levy. Made a short video.
Apr 17–30 ED–HYP *Judy Moran,* Trinity College	Familiarized herself with resources of the Center, including videos and software. Discussed quasiperiodic tilings with Al Marden and Paul Burchard, graphics hardware choices with staff.
Apr 19 *Guillermo Sapiro,* LIDS-MIT	Prospective postdoc.
Apr 29 – May 1 ED *Gene Klotz,* Swarthmore College	Discussed possible future of materials development proposal.
May 6–14 ED *Audun Holme,* University of Bergen, Norway	

Publications that Arose Substantially from Work Done at the Center

The following publications were reported by their authors as having arisen from work done primarily or substantially at the Center. For publications by Center staff and postdocs, see pages 119 and pages 45, respectively.

A number of these publications have appeared in the Geometry Center preprint series, of which a list can be found at the Publications display table. (A few preprints did not arise from work at the Center, and were issued in the series as a service to the authors. They are not included in this list.)

- Almgren, F. and Taylor, J. Flat flow is motion by crystalline curvature for curves with crystalline energies. *Journal of Differential Geometry*. To appear.
- Almgren, F., Taylor, J., and Wang, L. (1992). A variational approach to motion by weighted mean curvature. In *Selected Lectures in Mathematics*. Amer. Math. Soc.
- Almgren, F., Taylor, J., and Wang, L. (1993a). Curvature driven flows: A variational approach. *SIAM Journal of Control and Optimization*, (31):387–437.
- Almgren, F. and Wang, L. Mathematical existence of crystal growth with Gibbs-Thomson curvature effects. Submitted for publication.
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Experimental Mathematics

In 1992, encouraged by **David Mumford** and **Klaus Peters** (a publisher), **David Epstein** founded the journal ***Experimental Mathematics***, to encourage researchers to explain how their rigorous theoretical ideas arise from experiments. This high-quality research journal is primarily edited by Epstein and **Silvio Levy**, with **Fred Almgren**, **Henri Cohen**, **Robert Devaney**, **David Hoffman**, **Rafael de la Llave**, **David Mumford**, **Ulrich Pinkall** and **Peter Sarnak** as Associate Editors.

It publishes research results and conjectures arising from experimentation, as well as survey articles on areas of math where experimentation plays an important role. Articles by **J. Milnor**, **B. Mazur**, **J. Sethian**, **M. Pohst** (a leading authority on computational number theory), **J. Buchmann** (the father of Gröbner bases), and many other top mathematicians have been published or submitted.

In accordance with the philosophy of good exposition for good mathematics, papers are edited with great care, which is quite unusual for mathematics journal. In the representative words of one author: "I am grateful that you do this [extra editing]. . . this is truly an extraordinary service. I have published articles in many journals, including the *Bulletin of the AMS*, the *Journal of Differential Geometry* and *Inventiones*, and wish their editors would do one-tenth of what you do. . . to [ensure the clarity of the prose] and make constructive criticism to the authors. In this age of easy electronic distribution of preprints and TEX 'published-look' formatting, the main role I can see for the editing and publishing process is *exactly* this."

A copy of Experimental Mathematics' statement of philosophy and copies of the journal itself can be found at the Publications display table.

The Geom Style Package

This collection of LATEX programs by **Silvio Levy** helps writers of mathematics in many ways: it provides automatic indexing and cross-referencing, and a versatile mechanism for the definition of theorem-like environments; it largely lifts LATEX's restrictions on the use of "fragile" commands; and it simplifies the inclusion of PostScript figures, especially those generated by Mathematica and Illustrator (**Nathaniel Thurston** wrote the Illustrator interface). The package was publicly released in 1992 and has been increasingly used in the mathematical world for the production of both books and papers—it is one of the most popular items in our ftp directory. The Geom style package arose from the production of ***Three-Dimensional Geometry and Topology***, and was refined through its use in ***Word Processing in Groups*** (page 12).

The Former Center Faculty

by **Nina Amenta and Silvio Levy**

Albert Marden founded the Center and imparted it his vision. His perseverance brought together the core group of researchers that became the Geometry Supercomputer Project and later the Geometry Center, which he led until February 1994. Marden devised many of the Center's successful innovations, such as the apprentice program.

William Thurston was a major influence in the establishment of the Center, persuasively arguing for the importance of visual communication in mathematics. He served as codirector until 1993. His work inspired the videos *Not Knot* and *Outside In*.

James Cannon and **David Epstein**, together with Thurston, set the direction for the extensive research on hyperbolic geometry, topology, knot theory and automatic groups. Their work benefited greatly from the Center, and at the same time infused it with vitality, especially as it brought in numerous other researchers. The same can be said of **Jean Taylor** and **Fred Almgren**, who led the research on optimal geometries.

Richard McGehee is the Interim Director of the Center. He has led the Center's involvement in dynamical systems.

Harvey Keynes has been tireless in furthering the Center's educational mission.

Patrick Hanrahan began the **Geomview** project by writing its predecessor, MinneView. He has been a continuing source of support and inspiration for the technical staff as computer graphics researchers. Hanrahan and **David Dobkin** taught the summer course on Visualization and Graphics. Dobkin has been central to the reorganization of the Center; he and **Bernard Chazelle** are helping organize the 1995 workshop on computational geometry.

John Conway and Thurston were two of three people who taught and designed the Geometry and the Imagination summer course. **John Hubbard** was one of the teachers of the Chaos and Fractals summer course, and directed the projects of two summer institute students.

Charles Peskin has visited and gave seminars for the summer institute students, one of which inspired a student project.

Almgren, Dobkin, Epstein, Hanrahan, Marden, **David Mumford**, Taylor, Thurston, and **Allan Wilks** served on the Executive Committee. Subsets of this group formed the search committees for postdocs and for the Director of Technology.

Adrien Douady, **Mike Freedman**, **Benoît Mandelbrot** and **John Milnor** visited on different occasions.

Many other contributions by the Center faculty could be cited, some tangible, some remote. In the name of the postdocs and staff, we thank the Center faculty for their inspiration and guidance.

The Center faculty members each received a subcontract from the Geometry Center grant, which they spent on hardware or on support for graduate students, postdocs or research faculty. This practice has been discontinued.

Combinatorial Group Theory

by **David Epstein**

David Epstein has continued to work on combinatorial group theory, and particularly automatic groups. He and his associates at Warwick have been studying **growth functions** of groups, which have been of considerable interest since **J.-P. Serre** first showed that they were equal to rational functions of z with integer coefficients in the case of Coxeter groups.

The same rationality result is true for automatic groups of particular kinds, for example Gromov's **word-hyperbolic groups**. Computing these growth functions for a given group can be difficult; the most efficient known method for an automatic group is the Massey–Berlekamp algorithm, working modulo primes. This is necessary because the integers involved in the computation grow exponentially. A paper with Anthony Fletcher and Uri Zwick is in preparation about this work.

Using these highly efficient algorithms, in the form of usable software, the authors were able to find the growth functions for many examples in a few days, make relevant conjectures and prove the following fact, which substantially generalizes a conjecture of **K. Saito**: the growth function associated with any finite configuration in the Cayley graph of a word-hyperbolic group is a rational function, for any choice of generators. Surprisingly, the denominators for the rational functions depend only on the choice of generators, and not on the configuration.

Epstein has recently produced a new procedure for carrying out the Knuth–Bendix process, using finite state automata to store the rules. This promises to provide a very much improved method for computing the automatic structure of an automatic group.

Preface to *Word Processing in Groups*

These are excerpts from the preface of *Word Processing in Groups*, a groundbreaking book by **David Epstein**, **Jim Cannon**, **Derek Holt**, **Silvio Levy**, **Mike Paterson** and **Bill Thurston** on the theory of **automatic groups**. This theory, briefly, applies to the study of certain groups techniques from the theory of finite-state automata, objects of interest in areas as diverse as linguistics, computer science, psychology and mathematical logic. *Word Processing in Groups* was published by Jones and Bartlett in 1992 and, as can be seen from this preface, its existence—and that of the theory itself—owe a lot to the Geometry Center and its predecessor, the Geometry Supercomputer Project (see especially the last paragraph).

by **David Epstein**

Connections between the theory of hyperbolic manifolds and the theory of automata are deeply interwoven in the history of mathematics of this century... Ergodic theorists have been motivated by the consideration of geodesic flows on hyperbolic manifolds, amongst other things, to consider shifts of finite type... Max Dehn ... was the first person to point out the importance of the word problem in group theory. His solution in the case of fundamental group of a surface is very much in the spirit of what we are doing, namely a geometric approach to group theory. He was aware that geodesics in the Cayley graph of such a group follow certain rules (formalized in this book by the use of a finite state automaton), and that rules also characterize pairs of geodesics ending at the same point. His work was confined to the use of generators with geometric significance.

. . . In 1985, Epstein was developing a computer program for making group invariant drawings in the hyperbolic plane, and battling with the complexities of floating point inaccuracy and the gross inefficiencies of a naive approach to enumerating group elements up to a certain length. Around the same time, Al Marden was beginning to formulate his plans for a research project involving geometry and computing, centered in Minneapolis. During the discussion of these plans and the preparation of an associated grant proposal, Epstein learned from Thurston and Cannon an outline of what was to become known as the theory of automatic groups.

On returning to Warwick, Epstein explained these ideas to Holt and Paterson, and the three of them started to work out in detail many basic aspects of the theory. In 1986 Holt started to produce computer programs to find the automata associated to an automatic group. The original programming was based on the theory underlying Cannon's proofs. However, this method seems to lead to an exponential explosion in complexity and very few examples could be handled successfully in this way.

The experimental work of **Bob Gilman** was specially important to us. . . it led Holt to make the Knuth–Bendix procedure the basis for our computer programs. The Knuth–Bendix approach turned out to be quite effective, and it is a central element of our current suite of programs. These programs were written by Epstein, Holt and Sarah Rees.

In 1987 Epstein started to write a paper describing the results so far achieved. . . The paper expanded steadily as additional results were discovered, and was widely distributed in preprint form, in many different versions. . . Eventually the paper expanded to such an extent that the only reasonable way to publish the work was as a book. . .

We hope that [the abundant use of illustrations, made easy to include by means of software developed at the Center] has made our work easier and more pleasant to read and understand.

Part I of this book is suitable for use as a graduate level introduction to the theory of automatic groups; we have gone to some trouble to keep the amount of prerequisite material in this part of the book down to a minimum. Part II gives an account of research by Epstein and Thurston where a more general background in mathematics is assumed; this is suitable for more advanced graduate students.

As mentioned above, the whole project had its seeds in the preparation of the grant application for the Geometry Supercomputer Project, directed by Al Marden. Subsequently this project provided Epstein with the conditions in Minneapolis which enabled the daunting task of writing the book to be completed. Financial support provided by the NSF and the SERC have been of crucial importance. The SERC's Computational Science Initiative has been particularly helpful, and the SERC's Mathematics Committee has also lent support. The SERC and NSF have provided the computers on which the book was written and the programming done, and financed visits which allowed the authors to discuss their ideas face to face, resulting in much more rapid progress. Finally, the University of Warwick and its Mathematics Department provided the congenial environment in which most of the book writing was carried out, and a year of leave (1990–91) for Epstein to complete the work.

Proposal for Automata and Groups Team

by **David Epstein**, **Derek Holt**, **Robert Gilman**, **Stuart Margolis**, and **Charles Sims**.

The following draft proposal is loosely based on the successful model of the **Minimal Surface Team** led by **Jean Taylor** and **Fred Almgren**. The details of this proposal will change as it is further refined, before submission to the Board of Governors of the Geometry Center, to be evaluated in competition with other proposals for the use of Center resources and facilities.

We propose the creation of an **Automata and Groups Team** that would meet periodically, at least once a year, at the Geometry Center. Meetings would be devoted to programming and discussions, with emphasis on programming. Participants would include the five of us and other researchers, possibly different ones each time.

David Epstein and Derek Holt of Warwick University are two of the founders of the theory of automatic groups. In collaboration with **Sarah Rees** (now of Newcastle University), they developed a successful software package for automatic group computations. Charlie Sims of Rutgers University has written a package that efficiently implements the Knuth-Bendix Algorithm. Bob Gilman of Stevens Institute has long been concerned with computer programs for investigating infinite groups, and has written papers investigating the connection between groups and formal languages. He is interested in the development of new data structures to facilitate computations in infinite groups. **Stuart Margolis** of the University of Nebraska is developing programs that can effectively investigate finite state automata and various associated semigroups. These programs are based on work originally carried out by **Jean-Marc Champarnaud** in Paris in 1985. It is also probable that **Gilbert Baumslag** of CUNY, an expert on infinite groups, will be involved with this project. He is working on computer implementations of procedures to investigate such groups.

Epstein and Holt will apply to the UK funding agency SERC for financial support for the UK aspect of this work. We hope that SERC will contribute to the travel and living expenses of UK participants at meetings at the Geometry Center.

Margolis is Director of the Center for Communication and Information Science at the University of Nebraska, Lincoln. This Center is likely to be able to contribute some funding to support members of the Automata and Groups Team visiting the Geometry Center from Nebraska. Details will need to be negotiated. Those connected with Margolis include **Mark Sapir**, **John Meakin**, **Jean-Camile Birget** and programmers and graduate students.

Gilman has indicated his readiness to approach his department and to look for other sources of funds, and Sims may do the same.

A primary reason for establishing the Team at the Geometry Center is that each of us needs time away from our home institutions to develop software. The advice of experts at the Center will make a big difference to how easy it will be for others subsequently to use our code. We are all familiar with the problems of code which only the author can get to work. A related point is many of the software issues which concern us have been intensively investigated at the Center. We know of no other institution where an equivalent level of expertise in software, computer science and mathematics flourishes side by side.

A second reason is the possibility of interesting Center staff or postdocs in our research. We will try to conduct our work without imposing on the staff, though systems help will certainly be necessary. If staff or postdocs become involved in our research and contribute to it within a framework laid down by the Board of Governors and the Director, that would greatly strengthen our work.

A third reason is that the Center can make available to us the requisite number of workstations. This could be a problem elsewhere.

Some specific early goals are:

(a) Integrating Sims' state-of-the-art Knuth-Bendix code with the Warwick automatic groups code.

(b) Integrating or interfacing the Warwick and Nebraska software. There is some overlap, and many important gaps on both sides, but there are also large areas of complementarity. For example, an efficient implementation has been written at Warwick of the computation of growth functions for a regular language, starting from the finite state automata. This could well be included in the Nebraska code. The Warwick code could benefit from a more space-efficient implementation of finite state automata, which already exists in the Nebraska code.

(c) Developing common file formats and common computer languages to control our programs. We also hope to develop code which can be used by other research workers in their own programs. Although programmers regularly write programs which are used by others, this is much less frequent among mathematicians than it should be. As a team, we will have the advantage of seeing what difficulties there are in compiling and running our code in many different environments and on many different architectures. The result will be code that will be easy to port and easy to use. We also intend to pursue the much more rarely attained goal of producing data types and procedures that other mathematicians and computer scientists actually use.

Three-Manifolds, Hyperbolic Geometry and Knot Theory

Knot theory. The program **Snappea**, by **Jeff Weeks**, has led to several important new results in knot theory (page 96). The Center also distributes a relatively small knot manipulation called Linktool (page 146). Experiments at the Center on optimal configurations of knots are described on page 101.

Hyperbolic Geometry. Center postdoc **Oliver Goodman**, building on work of **Silvio Levy**, released a Mathematica program **Hyperbolic.m** to do computations and graphics in **hyperbolic geometry**. The package understands all the common models and works in arbitrary dimension. Its primary objects are hyperbolic subspaces and vectors. It supports translations, rotations, reflections, elements of $\mathrm{PSL}_2(\mathbf{C})$, compositions of such operations, and tilings. A related program by Goodman, **CirclePack.m**, computes circle packings with overlaps on arbitrary surfaces. This can be used to compute polyhedra in hyperbolic three-space with given dihedral angles. See also page 50.

Hyperbolic.m was used to compute the shapes of hyperbolic tetrahedra which can tesselate hyperbolic three-space by reflections through the faces. In his thesis (U. of Warwick), **Ian Redfern** uses the associated groups of isometries to provide interesting examples of limit sets of Kleinian groups. Until now nearly all limit set computations were of quasifuchsian groups or limits of quasifuchsian groups. Redfern's thesis provides many examples which are far from being quasifuchsian. These computations, some done during Redfern's visits to the Center, have been used in connection with automatic group theory to provide tesselations of the two-sphere self-similar under the Möbius group, tying in with work by Thurston and **Rick Kenyon** on self-similar tilings of the plane. Experimental data arising from these tilings of the two-sphere display a number of phenomena which are not yet understood from the theoretical point of view. The main new advance in Redfern's work is the generalization of automatic group theory to the theory of automatic cosets, but the easy production of examples starting from Hyperbolic.m was a major stimulus in this research.

The Combinatorial Riemann Mapping Theorem. Proofs of various versions of what they have called the combinatorial Riemann mapping theorem (for shingled and tiled rectangles, and for annuli) have been given by **Jim Cannon**, **Walter Parry** and **Bill Floyd**. These combinatorial versions of the classical Riemann mapping theorem [Cannon] examine recursive structures on the space at infinity of hyperbolic manifolds. A main impetus for this work is **Thurston's Geometrization Conjecture**. A lot of this work was carried out during visits to the Geometry Center.

Cannon and his group are developing algorithms to recognize and study three important classes of discrete groups: automatic groups, negatively curved (or Gromov word-hyperbolic) groups, and groups of hyperbolic three-manifolds. The techniques employed examine the geometry of the Cayley graph of the group, the associated space at infinity, and the recursive structures that may be used to construct each. An algorithm due to Perry and Cannon, and implemented on the Suns by **Ling Yi**, **Paul Brewster**, and Cannon, calculates the solutions to the finite Riemann mapping problem exactly. They hope their results will supply a new method, both theoretically and in practice, for recognizing hyperbolic groups.

Center computers, here and at Brigham Young University, have been used to implement this and related algorithms.

Snappea

by **Jeff Weeks**

In the mid 1970s, **Bill Thurston** and others revolutionized the study of three-manifolds with their discovery that geometrical techniques could be used to answer previously intractable topological questions. Specifically, they found that most three-manifolds can be given a metric locally isometric to hyperbolic three-space, and that this hyperbolic structure reveals the properties of the underlying topological manifold. My computer program Snappea is the practical embodiment of this geometrical approach to three-manifolds.

Consider, for example, the application of geometrical techniques to knot theory. Thurston proved that a typical knot or link complement admits a hyperbolic structure. The hyperbolic structure can be used to solve the equivalence problem (given two link projections, decide whether they represent the same link) and the symmetry problem (given an arbitrary link complement, compute the link's symmetry group). Snappea realizes this potential and puts it at the service of the working mathematician. The mathematician draws two link projections with a mouse, and within seconds Snappea computes a hyperbolic structure for each and thereby determines whether the links are equivalent. Similarly, Snappea uses the hyperbolic structure to quickly compute the symmetry group of each link.

Knot theorists have made extensive use of Snappea, but its applications are more general. It works with all hyperbolic three-manifolds of finite volume, whether compact or not. Users may access Snappea's data bases of low-volume closed and cusped manifolds, they may create their own manifolds, and they may use the "drilling and filling" feature to create new manifolds. Snappea can compute various numerical, algebraic and graphical invariants for each manifold, such as the volume, the fundamental group, the first homology group, the Dirichlet domain, the horoball packing, the Ford domain, and the length spectrum.

The drilling and filling algorithms invented by Weeks represent important theoretical advances of the last two years. Specifically, Weeks has found algorithms to (a) create an ideal triangulation for a partially Dehn filled multicusp manifold, and (b) drill out geodesics to create new cusps. The error checking in these algorithms is of some mathematical interest, because it is so simple. The algorithms succeed unless the intended manifold contains an essential annulus or compressing disk, in which case the manifold is not hyperbolic. The algorithms test for essential annuli or compressing disks simply by computing an Euler characteristic.

Weeks began Snappea as a graduate student in the winter of 1985. During the years 1987–92 he continued to develop Snappea in informal association with the Geometry Supercomputer Project and then the Geometry Center. By 1992 it was clear that Snappea was suffering many defects characteristic of evolving software (and evolving mathematics). Its internal design was not strong enough to support continued development, so the Center hired Weeks as a paid consultant to begin a complete rewrite. The rewrite, Snappea 2.0, is now about two-thirds complete, and is realizing its design goals. For details, please see the Snappea Technical Report at the Snappea display table.

Snappea has become a standard tool among researchers in low-dimensional topology, as evidenced by the many papers which cite it (see **References** below). But even beyond the explicit citations, many mathematicians have told us that they use Snappea as an exploratory tool in the early stages of their investigations, to study examples, test conjectures, and rule out fruitless approaches to finding a proof. (See the user statements of Snappea at the Snappea display table.)

Snappea stores three-manifolds as triangulations. The combinatorial information needed to specify a triangulation can be encoded in an extremely concise format developed by **Martin Hildebrand**, Thurston and Weeks. Vast databases of manifolds can be stored using only a modest amount of disk space. Hildebrand made a census of all combinatorially distinct ideal triangulations up to a certain measure of complexity; Weeks then used Snappea to figure out which ones were homeomorphic. **Craig Hodgson** and Weeks subsequently used the database of cusped manifolds to generate a database of low-complexity closed hyperbolic three-manifolds. The resulting atlases have become important repositories of information for researchers.

Among other research results, Weeks discovered with **Makoto Sakuma** of Osaka University a triangulation for two-bridge knot and link complements which they conjecture to be the canonical one (dual to the Ford domain). Sakuma and Weeks have also found a simplified and generalized proof of the Tilt Theorem which allows the computation of canonical triangulations.

With **Joe Christy** of MSRI, Weeks discovered a five-component link with a surprisingly large symmetry group. The canonical triangulation of the link complement consists of ten regular ideal tetrahedra. Every possible mapping of one tetrahedron to another extends to a global symmetry of the link complement, for a total of $10\times 4! = 240$ symmetries.

Hodgson and Weeks have found an algorithm which rigorously computes the complex length spectrum (with multiplicities) of a hyperbolic manifold. This algorithm has been implemented in Snappea, and provides the missing link needed to compute symmetry groups of closed hyperbolic three-manifolds. (Previous versions of Snappea computed symmetry groups only for cusped manifolds.)

References. The following is a selection of published papers for which Snappea played a significant role in the research. A more extensive Snappea bibliography is available at the Snappea display table.

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- Bleiler, S. A. and Hodgson, C. D. Spherical space forms and Dehn surgery. *Proceedings of the International Conference on Knot Theory and Related Topics, Osaka, 1990*, 1992:425–433.
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- Neumann, W. D. and Reid, A. W. (1993). Rigidity of cusps in deformations of hyperbolic 3-orbifolds. *Mathematische Annalen*, 295:223–237.
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Progress toward the proof of Thurston's Geometrization Conjecture

by **David Epstein**

The most important current line of development in three-dimensional topology is the attempt to solve Thurston's Geometrization Conjecture. This conjecture implies the Poincaré Conjecture, and has the advantage that it can be broken up into more accessible problems that one can tackle in turn, each time building on what is already known. **David Gabai** of Caltech is a leader in this effort, and has solved several of these intermediate problems. One of his significant results is this:

Theorem 1. If a closed irreducible three-manifold is homotopically equivalent to a hyperbolic three-manifold, there are finite covers of the manifolds such that the homotopy equivalence can be lifted to a map homotopic to a homeomorphism.

Gabai has shown that if there is a simple closed geodesic in the hyperbolic manifold which does not trilink (a concept he defines), then the above theorem could be strengthened to the following result.

Conjecture 1. If a closed irreducible three-manifold is homotopically equivalent to a hyperbolic three-manifold, then the homotopy equivalence is homotopic to a homeomorphism.

The Thurston Conjecture would imply this result, and this result clearly implies Theorem 1. Conjecture 1 seems to be an essential step on the way towards the proof of the Thurston Conjecture.

A geodesic can be proved not to trilink if it has a sufficiently large tube radius, namely $\eta = \frac{1}{2}\log 3$. Thus the theorem would follow from a positive solution to the following "tube radius" question for Kleinian groups with two generators. Does the shortest geodesic in a torsion-free, parabolic-free two-generator Kleinian group have a tube of radius η? Actually the answer to that question is no—Jeff Weeks has used **Snappea** (page 96) to find a manifold for which the tube radius of the shortest geodesic is less than η. (But Snappea also shows that the shortest geodesic does not trilink.)

In summary, Gabai's work reduced Theorem 1 to:

Conjecture 2. All but a sparse subset of a compact subset of the six-dimensional parameter space (parametrizing the two-generator Kleinian groups) have thick tubes, and the groups in this sparse subset nevertheless satisfy the no trilinking condition.

This opened the way to a possible computer proof of the theorem. At this stage Gabai enlisted the help of **Bob Meyerhoff**, who had studied tubes intensively in connection with his work on low-volume hyperbolic manifolds. Meyerhoff produced a computer program to investigate the situation. Despite encouraging results, it became clear that formidable computational problems would have to be overcome, especially near one edge of the parameter space. Later **Bob Riley** was encouraged to modify his longstanding computer program PNCRE to investigate this problem. Again, his results, though very promising, pointed to the considerable technical problems that had to be faced.

Meanwhile, conversations involving at various times Gabai, Meyerhoff, Epstein and **Al Marden** led to a visit by Meyerhoff to the Geometry Center, where he explained the problem to the staff. Apprentice **Nathaniel Thurston** was immediately fired up by this problem, whose solution seemed to need many of his diverse skills. Nathaniel has been developing a program to tackle Conjecture 2, basing his work on the progress already made by Gabai, Meyerhoff and Riley. His approach depends on computing over whole regions at once, using interval arithmetic. The program divides Gabai's compact region into pieces, for each of which it checks certain assumptions. If the assumptions are correct, this piece of parameter space provides no counterexample to the conjecture. If the assumptions are false, the region is subdivided. (Thus the division algorithm is dynamic, otherwise there would be a prohibitive number of pieces.) The hope is that eventually the program will either dispose of all pieces or generate, for certain pieces, examples where the shortest geodesic does not have a thick tube. The program may also prove that there are only finitely many such examples. They can then be checked by hand to look for a simple geodesic that does not trilink. Such examples may well repay individual study, as they may have other unusual properties.

Nathaniel's program has gradually been refined with a number of ingenious ideas that move the calculation into the realm of the possible, lowering the estimated required time from over 10 million CPU days to some 100 CPU days. The program is not guaranteed to work (even if Conjecture 1 is true), but if it fails, the way it fails may still lead to a complete proof, by enabling a theoretical attack on a very small part of parameter space. At the time of this writing (April 27) it is unclear whether an answer will have been found by the time of the site visit; an oral update will be given then. In any case the computer program is very likely to yield important new results: see items (b)–(d) on page 100.

If the program succeeds and Conjecture 2 is settled in the affirmative, it will perhaps be the most striking example of a theorem proved by computer since the four-colour problem. One difference is that the four-colour problem is clearly a combinatorial problem, long ago reduced to a finite but huge verification, whereas here the problem is one of dealing with a compact subset of parameter space. (Such investigations had previously been carried out during Lanford's proof of the Feigenbaum conjectures; but here the parameter space is really huge compared with Lanford's.) Another difference is that the four-colour problem was a bit of a cul-de-sac, as it seems not to lead to any new mathematics, whereas this work is a significant milestone on the main highway of mathematical research into the twenty-first century.

Lanford has discussed the various theoretical problems that arise when using a computer in this way. One is how to verify the correctness of the program, in view of the fact that large and complicated programs always have bugs. (It worries mathematicians less that large and complicated proofs also always have bugs, varying from misprints to gaps to nontrivial errors. The classification of simple groups still has nontrivial gaps 14 years after it was first "completed".) Nathaniel's program is too complicated to be completely convincing to a sceptic. He proposes to deal with this by using the facts discovered by his main program as input data for a new program. The new program will be short and simple and easy to check.

Several byproducts can be foreseen from this research:

(a) If Conjecture 2 is true, Mostow's Rigidity Theorem can be improved to show that two hyperbolic structures on the same hyperbolic three-manifold are isotopic rather just homotopic, a statement that is false in higher dimensions and that is not implied by the Thurston Geometrization Conjecture.

(b) A successful investigation of even a small portion of parameter space would yield huge improvements over the best current state of research on discreteness criteria for Kleinian groups (for example Jorgensen's inequality and the work of Gehring and Martin in understanding two-generator Kleinian groups).

(c) The investigation has already produced one interesting new compact three-manifold and will produce others.

(d) The computation should yield an enormous improvement on the estimate of the smallest possible volume of a compact hyperbolic three-manifold. The conjectured best value is about .94. Currently the best estimate is around .00135. This work will almost certainly improve that estimate 100-fold, even if it is not crowned with the jewel of complete success, namely the proof of part of the Thurston Geometrization Conjecture.

    (Improving the volume lower bound is a substantial industry, on which several mathematicians with international reputations are currently working. This work is regarded by the mathematical community as sufficiently important to be the basis for awards of several NSF grants and the award of tenure at famous American universities. Theorists reviewing such grant proposals and making such tenure decisions may not always have realized the role played in the work being reviewed by the **atlas** of three-manifolds produced with Geometry Center backing—see page 97.)

The collaboration between Gabai, Meyerhoff and Nathaniel Thurston is an excellent example of the indispensable contribution of the Geometry Center. Where could Meyerhoff have gone to discuss this problem, if not to the Geometry Center? What other institution would have been likely to have produced someone with such appropriate skills? The computation is very demanding in terms of CPU hours; not many other institutions could have provided the CPU hours necessary for the computation, which is being carried out in parallel on the Geometry Center's workstations. The computation also demands very large amounts of disk space: although such disk space may always be bought, projects are more often successful when the infrastructure is in place when it's needed.

In sum, where else can questions about hyperbolic geometry, about complex linear interval arithmetic, data visualization, and the logistics of distributed computing be answered promptly? What other institution has all of the tools needed for this project (g++, gdb, perl, Mathematica, Geomview and distribute), installed and cooperating smoothly? The Geometry Center is unique in its computing environment, unique in its culture encouraging a graduate student to use substantial resources to attack a huge problem, and unique in that the inevitable difficulties that arise when tackling such a large problem have already been seen and dealt with.

Optimal Geometries

by **John Sullivan**

One of the Center's main research areas has been in optimal geometries and geometric evolution processes. Within this area there have been several projects. **Jean Taylor** (Rutgers) and **Fred Almgren** (Princeton) established the **Minimal Surface Team**, devoted to collaborative studies on optimal geometries, recently emphasizing crystal growth problems. **Ken Brakke** (Susquehanna University), who was a sabbatical visitor to the Center in 1992–93 and a frequent visitor at other times, has developed the **Surface Evolver** (page 105), a computational tool for general problems in geometric optimization. Former Center postdoc **John Sullivan** (University of Minnesota, see also page 47) has studied Willmore surfaces, knots minimizing Möbius energies, and many other problems, using the Evolver and other computational tools.

Two workshops held at the Center are related to this area of research, and were designed to produce interaction between people doing theoretical and computational work—see (2) and (7) on page 9. Several of the U of M faculty (page 58) spending time at the Center have worked in optimal geometries.

Crystal Growth. The **Minimal Surface Team** has assembled at the Center in Minnesota for each of the past six summers. In early years, the team produced the Geometry Supercomputer Project's first scripted video, "Computing Soap Films and Crystals", which has been shown around the world, and recently inspired researchers in Ireland to find a solution to Kelvin's conjecture on equal-volume foams (see page 106).

A central project of the team in more recent years has been a theoretical and computational study of **growth phenomena** such as the creation of a crystal from melt, solution, or vapor. It is a grand mathematical challenge to try to come to grips with the spontaneous pattern formation which occurs under conditions favoring dendritic growth (as in snow flakes). A theory predicting growth speed, feature size, side-branching behavior, etc., under these diffusion-controlled conditions would also be useful for the prediction and control of the microstructure of metal alloys and similar substances. The team studying these problems has included **Rob Almgren** (University of Chicago), **Nelson Max** (UC Davis and Lawrence Livermore), and **Andy Roosen** (NIST), in addition to Fred Almgren, Brakke, Sullivan, and Taylor.

The interaction between computation and theory has been very important in this work. The team's first mathematical models, when implemented computationally, did not produce dendritic growth. More realistic theoretical models were developed only after consideration of why these first computations failed. The theoretical existence results for these models are more difficult, but in turn shed light on possible computational methods.

The models considered for crystal growth take the interface velocity to be a mobility function times a forcing term due to bulk energies (from the temperature field) plus surface terms (the weighted mean curvature). In 1991, Rob Almgren implemented such a model for smooth interfaces in the plane; his heat-flow code was also used by Roosen, who implemented a crystalline version of crystal growth in two dimensions. The next summer, the emphasis was on extending these programs to work in three dimensions, where there are new difficulties with data structures and with visualization of temperature fields.

Computationally, we must diffuse heat for a certain time interval, and then update the interface position. This suggests a variational approach to crystal growth problems, which has met with surprising theoretical success. The new interface position minimizes some functional of the previous position and the new temperature field. A pivotal first estimate in the theory is to show that sequences of such discrete evolutions converge to something reasonable as the time step approaches zero.

[Almgren et al. 1993a] studied this new mathematical approach to the study of time evolutions of solids moving by weighted mean curvature. The resulting theory works for smooth or crystalline surface energies; the weighted mean curvature in either case is the initial rate of change of surface energy with volume swept out. The "flat curvature flows" are defined as limits of sequences of solutions to variational problems in which a sum of surface and bulk energy is minimized among all possible new solids. The analysis is formulated in the context of integral currents and sets of finite perimeter in order to utilize powerful techniques from geometric measure theory. It is shown that the flows agree with classical smooth flows when the data is smooth and elliptic in any dimension; they also coincide with motion by crystalline curvature for polyhedral curves in the plane [Almgren and Taylor]. Both of these papers were prepared at the Center in response to the computational results, and were first issued as Center preprints.

[Almgren and Wang] introduces a geometric evolution process which incorporates many features necessary for a realistic model of several types of crystal growth in nature. One novel feature of the analysis is the use of a Monge–Kantorovich metric in studying heat distributions; this is the first time such methods have been used in parabolic problems. Minimizing this metric is related to linear-programming problems studied earlier by Sullivan.

[Roosen and Taylor 1992] gives the basic outline of the crystalline method applied to dendritic crystal growth and Ostwald ripening, and shows examples of computations. Full details appear in Roosen's thesis (January 1993). Roosen's work at the Center led to a postdoctoral fellowship at NIST, where he continues his work at the interface between mathematics, computation, and materials science. He is now proposing creation of a new center for computational materials science, and says that the Geometry Center is an important model for the structure of the new center (page 41).

Research with the Surface Evolver. The **Surface Evolver** developed by Ken Brakke (page 105) has been an extremely useful tool for those interested in optimal geometries, from both theoretical and practical viewpoints. Here we will only mention how it has been used for mathematical research at the Center.

The Evolver is useful because it is easy to modify it to work with new energies. Although the program was originally written to minimize surface area (and is used in this mode, with added terms for gravity or contact angles, for most real-world applications), it is easy to define new mathematical energies to be minimized. If a researcher writes code to compute the energy, the Evolver's facilities for conjugate gradient minimization, interactive control of triangulations, graphics, and so on are automatically available.

With **Rob Kusner** (UMass) and **Lucas Hsu** (IAS), **John Sullivan** studied **Willmore surfaces** (surfaces minimizing the integral of mean curvature squared) with the Evolver [Hsu et al. 1992]. Since this energy is infinite for polyhedral surfaces, it is an interesting question how best to discretize it for the polyhedral approximations to smooth surfaces used in the program. With Brakke's help, several discretizations were implemented for comparison. Numerical minimization experiments gave support to the existing conjectures about minimizing surfaces of each genus, and also led to the discovery of several previously unknown critical points; the authors are now working on proving their existence theoretically. Further experiments with the Evolver have given Kusner the understanding necessary to lead to a proof (now almost completed) of the existence of minimizers in every genus.

More recently, Kusner and Sullivan have used the Evolver to study knot energies, including the one shown by Freedman, He, Wang to be Möbius invariant. (While this work computes optimal geometries for knots, it is also related to the Center's other work on knot theory—see page 95—and it was encouraged by participants at the Knots workshop held in March 1993 at the Center.) In [Kusner and Sullivan 1994] they computed the (presumed) energy-minimizing configurations of all knots and links up through eight crossings (entered using Linktool—see page 20). They also showed that several examples of tangled unknots all evolve to the round circle under energy minimization. Many had conjectured earlier these examples would get caught at some other local minimum. Again in this problem there are interesting questions about discretizations. Note that since the Evolver uses the Center's Geomview package for graphics, it is possible to apply Möbius transformations interactively, or to view the knots in the three-sphere. (This capability was added to Geomview by Sullivan, building on modifications postdoc **Oliver Goodman** had made for viewing conformal hyperbolic surfaces.) This aids immensely in understanding the (nonintuitive) Möbius invariance.

[Kusner and Sullivan 1994] also developed an analogous energy for embedded surfaces, which again prevents crossings and is Möbius invariant. Sullivan added a discretization of this to the Evolver, and is doing some experiments; **Dennis Roseman** (U. Iowa) has also used this energy in the Evolver to study knotted spheres and tori in four-dimensional space. Again, the Evolver's Geomview graphics allow easy viewing of these four-dimensional objects.

[Sullivan 1994b] studied the optimal constant for the Besicovitch covering theorem. This can be reduced to a question about sphere packings, which can be answered exactly in dimension two. In dimension three, upper and lower bounds are known, but the exact constant is only estimated from computer experiments, again using the Surface Evolver, not on curves or surfaces, but merely on repelling point charges with hard-shell potentials. In [Kusner and Sullivan 1994] the energies of Hopf links are related to similar questions about repelling point charges. Here, the problem was to find all critical points, so a new energy, the norm of the gradient of the previous potential, was added to the Evolver, and used to find the critical points.

Recently, there has been interest in evolution equations for curves and surfaces which resemble mean curvature flow, but which are affine invariant. **Allen Tannenbaum** and **Peter Olver** (both U of M) have been working with Sullivan and Guillermo Sapiro (a prospective postdoc at the Center) to model these flows in the Evolver (see page 58).

Sullivan is now working with **Carlo Séquin** (UC Berkeley) on finding mathematical energies (involving curvatures) which will lead to pleasing forms for computer-aided design. These energies are being incorporated into the Evolver to aid in studying their mathematical and aesthetic properties.

Johannes Nitsche (U of M) has used his release time at the Geometry Center using the Evolver to study surfaces minimizing various quadratic functions of curvature (see page 60). With Brakke, he explored such surfaces both with fixed boundaries and triply periodic boundary conditions.

Other work on optimal geometries. **John Sullivan** investigated with **Steven Altschuler** (IMA) the behavior of the curve-shortening flow on space curves, and in particular self-similar or soliton solutions. They developed an external module for Geomview to show these solutions interactively. Sullivan has recently extended this program to compute also self-similar solutions to the smoke-ring and affine-curvature flows.

[Sullivan 1994a] extended the results of his Ph.D. thesis on finding area-minimizing surfaces to the case where there is also a bulk energy term. He also wrote software to implement the algorithm described there (see page 109).

Ken Brakke, while on sabbatical at the Center, developed a new mathematical model of soap films, viewing them as minimal cuts living in a cover of space branched along the boundary curves [Brakke 1994]. This model allows natural representation of nonorientable films and films with triple junction. Brakke's program Polycut illustrates the covering spaces involved.

The Surface Evolver

by **Ken Brakke**

Description. The **Surface Evolver** is an interactive program for the study of surfaces shaped by surface tension and other energies. A surface is implemented as a simplicial complex, that is, a union of triangles. The user defines an initial surface in a datafile. The Evolver evolves the surface toward minimal energy by a gradient descent method. The evolution is meant to be a computer model of the process of evolution by mean curvature.

The energy in the Evolver can be a combination of surface tension, gravitational energy, squared mean curvature, user-defined surface integrals, or knot energies. The Evolver can handle arbitrary topology (as seen in real soap bubble clusters), volume constraints, boundary constraints, boundary contact angles, prescribed mean curvature, crystalline integrands, gravity, and constraints expressed as surface integrals. The surface can be in an ambient space of arbitrary dimension, which can have a Riemannian metric, and the ambient space can be a quotient space under a group action. The user can interactively modify the surface to change its properties or to keep the evolution well-behaved. The Evolver was written for one- and two-dimensional surfaces, but it can handle higher-dimensional surfaces with some restrictions on the features available. Graphical output is available as screen graphics and in several file formats, including PostScript.

The program is in the public domain, and is available on the **Internet** by anonymous ftp from the Geometry Center. The Center also prints hardcopies of the user manual in its preprint series from time to time.

The source code for the Evolver may be compiled and run on any system with a C compiler, but there are some special versions for certain systems. There is a Macintosh version, a 32-bit version for DOS, a NeXT version, and a version for multi-processor Silicon Graphics machines.

History and Center Role. The Evolver wouldn't exist if it weren't for the Center. When I first heard of the Geometry Supercomputing Project in 1988, I wanted to be a part of it, and the Evolver was the proposal I used to join the **Minimal Surface Team** (led by **Fred Almgren** and **Jean Taylor**). The idea for the program had been in my mind for several years, but I probably never would have started it if it weren't for the Project. I only had access to DOS computers elsewhere, and they just weren't fast enough, big enough, or good enough at graphics to make such a program feasible. I produced the first working version during a month-long visit to the Project in Minneapolis during the summer of 1988. Development proceeded during similar visits the following summers. During the 1992–93 year, I spent a **sabbatical** at the Center.

The Center has been essential for the development of the Evolver. Even if I had started the Evolver on my own, it would probably not have gone beyond the toy stage if it had not been used by other people. The early versions were used by **Sullivan**, Almgren, Taylor, and others my first summer at the Project, and that gave me the motivation to continue with the program. Without our group being together physically, there would have been no seed group of users to start the spread of the Evolver.

The Center has provided much support in miscellaneous ways. I have used the **video equipment** to make several tapes that have been in published collections, which helped publicize the Evolver. The Evolver Manual has been published in the Center's **preprint series**. The Center distributes Evolver by its anonymous ftp facilities. None of this kind of support is available at my home institution, which is a small liberal arts college.

The Center has computers that far outclass what I have elsewhere, particularly in terms of graphics. It also has a variety of systems (SGI, Sun, NeXT, Mac) so that I have been able to make the Evolver available to a wide audience. The **Geomview** program (and Minneview before it) are far superior to any other display programs available to me for showing the surfaces the Evolver produces.

The continual stream of visitors to the Center gives me opportunities to work with others with the Evolver. Since visitors can be here a week or longer, it is possible to do much more than at a brief conference somewhere. This also provides opportunities to publicize the Evolver to groups who otherwise might not come across it (such as the participants at the Knot Workshop at the Center in March 1993).

The early versions of the Evolver were mainly for my personal use, and for the Minimal Surface Team. In the summer of 1989, I was persuaded to make the Evolver publicly available, so I wrote a user manual and put together an ftp package. The technical history of the Evolver can be followed in the History chapter of the Manual. In January 1993, I started a **Surface Evolver Newsletter** to keep users updated on latest Evolver versions, news from other users, bibliography, and such. The newsletter is distributed by email, and seven issues have appeared so far, coinciding with new Evolver versions.

Applications to Aerospace Engineering. I had two consulting jobs with aerospace companies in 1993, modeling spacecraft tanks. The first, with David Frank of **Lockheed** during April and May, was to model the liquid helium tank on the Gravity Probe B mission, which is to be launched in 1997 to test Einstein's General Theory of Relativity. The tank is cylindrical with a rod down the middle. It is rotating on the rod axis in weightlessness, and partially full of liquid helium, which perfectly wets all surfaces. The goal is to rotate the tank fast enough so that the empty bubble forms a torus around the rod instead of a spherical bubble off to the side. My task was to find the rotation rate needed to force a sphere into a torus, and to find the lower limit of rotation for the torus to be stable against breaking up into a sphere. An informal report on the results in available at the Evolver display table.

The second job, with Jim Tegart of **Martin Marietta**, involves modeling the fuel tanks on the Cassini spacecraft to Saturn, which will be launched in 1997. Here, the tank is non-rotating in weightlessness, and the fuel perfectly wets all surfaces. The goal is to keep the fuel in a partially full tank in a known location near the fuel outlet. This is done with eight metal vanes in the half of the tank with the fuel outlet. My task was to set up models of the tank, so that Jim Tegart can use the Evolver to simulate the effects of various forces. A similar tank flew on a Shuttle mission recently, and I also included a model of that tank.

Lord Kelvin's Conjecture Disproved. In 1887, **Lord Kelvin** posed the problem of finding the partition of space into equal volume cells minimizing the interface area. He suggested a partition which is basically the Voronoi cell for a BCC lattice. Robert Phelan and Denis Weaire of Trinity College, Dublin, have found a structure using two types of cells that has 0.3% less area than Kelvin's. They used the Evolver to compute the areas involved.

Other Applications. The Evolver has been used in much of the **optimal geometry** research at the Center (page 101). In particular, many various knot energies and quadratic functions of curvature were added to the code at the request of Center researchers and participants in the Knot Workshop.

Xavier Michalet in France is using Evolver's squared mean curvature energy to model the elastic membranes of cell vesicles.

Tim Singler of SUNY Binghamton is using the Evolver to model liquid solder on microcircuits, particularly the "bridging problem" of unwanted mergers of droplets on neighboring pads. Scott Deering at MIT is also doing liquid solder.

Jeremy Ackerman, a student in the 1993 undergraduate **Summer Institute** at the Geometry Center, used the Evolver to model vibrating soap films.

Undergraduates at the REU at Smith College in the summer of 1993 used the Evolver to study knot energies and minimal foams.

More applications can be found in the newsletters available at the Evolver display table.

Future directions. The Evolver continues to grow as users request features and I think of more things I want it to do. I have arranged the internal structure of the program so that new types of energy can be added in a systematic way. The command language is developing more and more into a general purpose programming language and database query language. The ultimate goal is to provide the user with complete control and information. Also under development are versions for several parallel machines.

Additional materials. Copies of these are available at the Evolver display table.

- Brakke, K. (1992). The surface evolver. *Experimental Mathematics*, 1.
- Brakke, K. (1992–1993c). Surface evolver newsletters. Distributed electronically.
- Brakke, K. (1993b). Surface evolver manual. Geometry Center Preprint GCG55.
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- Michalet, X., Fourcade, B., and Bensimon, D. (1994). Fluctuating vesicles of nonspherical topology. *Phys. Rev. Lett.*, 72.
- Tegart, J. (January 1994). Cassini propulsion module subsystem.
- Weaire, D. and Phelan, R. (February 1994). A counter-example to Kelvin's conjecture on minimal surfaces. *Phil. Mag. Letters*.

Computational Geometry

The Geometry Center is having a growing impact in **Computational Geometry**. The Center is known in the community through the research of the Center postdocs. Drawing on its special strengths, the Center is promoting the production of efficient computational geometry software and the communication of algorithms through computer animation, two rapidly growing areas.

Software. Computational geometry has produced many "theoretical" algorithms with good asymptotic behavior (as the size of the input data goes to infinity), a classic example being the brilliant linear-time triangulation algorithm for simple polygons by **Bernard Chazelle** that appeared in *Discrete and Computational Geometry* 6 (1991), pp. 485–524 (and which the Center claims no credit for). There has been a recent trend, however, towards developing algorithms which are simple and efficient in practice. This is driven both by the development of paradigms for asymptotically efficient but simple algorithms for broad classes of problems (primarily using randomization), and by pressure from outside the field to produce software as well as algorithms.

But in fact there are still very few examples of usable computational geometry software. Center postdoc **Brad Barber's** convex hull program, qhull, is one. The basic algorithm, due to Barber and **David Dobkin**, uses good heuristics and intelligent storage management. The initial version of qhull, written together with undergraduate programmer **Hannu Huhdanpaa**, solves big problems very efficiently in practice. The recently released version 2.0 also computes approximate hulls; this is useful for imprecise input data and much more efficient than any other convex hull software on inputs containing clouds of co-planar or nearly co-planar points. Qhull is one of the most popular items in our ftp directory. Low-dimensional qhull output can of course be viewed with Geomview.

The growing imperative to produce this kind of useful software has led Center postdoc **Nina Amenta** to organize a workshop on computational geometry software for January of 1995. Working with co-organizers Chazelle and Dobkin, Amenta has secured the participation of most of the leading computational geometers, some prominent researchers from other areas which use computational geometry, and a select group of young researchers and students involved in the production of computational geometry software. This workshop will provide an opportunity for the Center to exercise some leadership in this growing effort. In addition, through the organization of the workshop we plan to collect all publicly available computational geometry software, and make it available as an archive via ftp and the World-Wide Web. Amenta's research and publications are covered on page 52.

The program **vcs** for computing Voronoi diagrams in three dimensions, which **John Sullivan** wrote at the Geometry Supercomputer Project, has continued to be used and updated. Although it is probably no longer state-of-the-art (compared to qhull) for the basic problem it has special features that make it attractive for certain applications: it can compute Voronoi cells in a three-torus, and can move the sites to optimize the diagram in several ways. It has been used by researchers in at least eight countries, including Phelan and Weaire in the discovery of their new equal-volume foam (page 106).

See also the section on Animations below for implementations of new algorithms due to Center postdoc Leonidas Palios.

Animations. Computational geometry is one mathematical discipline in which the exposition of results using computer graphics and video is becoming increasingly popular; there has been a refereed video review associated with the ACM Symposium on Computational Geometry for the past three years. The mathematical visualization tools developed at the Center are ideal for this purpose.

Leonidas Palios and staff member **Mark Phillips** have used **Geomview**, custom software, and the video facilities, to animate the algorithm due to Palios and Chazelle for the decomposition of a three-dimensional polyhedron into tetrahedra. Palios augmented an existing C implementation of the algorithm (which he developed in graduate school) with statements to produce graphical output readable by Geomview. The new implementation was applied to an example polyhedron, whose decomposition was controlled by a C program written by Mark Phillips and animated using Geomview. The animation led to the construction of the video "Tetrahedral Break-up" that was included in the 1992 ACM Symposium on Computational Geometry video review.

Recently, Palios implemented in C an early version of his algorithm to tetrahedralize the space between a convex polyhedron and a convex polygon; the output of the program can be piped to Geomview, yielding an interactive animation of the algorithm. Moreover, a video has been created to illustrate how the algorithm works on a particular pair of polygon and polyhedron.

Linus Upson and **Adrian Mariano**, undergraduates in the 1993 Summer Institute, produced a video that was accepted for the 1994 ACM Symposium on Computational Geometry video review. Their video illustrates their original observation that the shadows cast by a polygonal light source shining on a single polygon in a three-dimensional scene are the projections of four-dimensional polytopes. Since shadow boundaries are a hot topic in both computer graphics and computational geometry, the video has generated some excitement. This project is interesting both as an example of undergraduate research fostered by the unique Center environment, and as perhaps the first example of a research result communicated primarily by video rather than text.

John Sullivan wrote code implementing the algorithm described in his Ph.D. thesis. This algorithm computes the absolutely area-minimizing hypersurface on a given boundary, by means of linear programming. In fact, this work can also be viewed as giving a geometric interpretation of the usual algorithm for the minimum-cost circulation problem. The implementation he created at the Center is animated, with a graphical interface using Geomview. Sullivan's short video showing this algorithm animation was included in the published proceedings of the Crystal Growers Workshop (see page 62).

Research. **Leonidas Palios** created an algorithm, which, for a given set of points contained in a rectangle R, computes the maximum number of these points that can be connected to the boundary of R by means of non-intersecting (horizontal or vertical) line segments; this finds applications in the production of fault-tolerant VLSI/WSI array processors, where spare processing elements (located around the array processor) are used to replace faulty ones. Except for a new method to compute many similar partial solutions together, the algorithm uses a new data structure that is interesting in its own right. The work is described in detail in the preprint GCG56; a shorter version has been accepted for presentation in the 1994 Scandinavian Workshop on Algorithm Theory.

Palios has also worked on computing upper and lower bounds. Given a polyhedron P and a direction λ, the λ-extrema of P are the local minima and maxima of P with respect to λ. He was able to establish a tight upper bound of $2r + 2$ on the number of extrema of a polyhedron with respect to any direction, where r denotes the number of reflex polyhedron edges. The work appears in the preprint GCG68, and has also been submitted to the 1994 Canadian Conference on Computational Geometry.

Palios also computed upper and lower bounds on the minimum cut size of partitions of the nodes of a tree into two sets of prespecified cardinalities; the bounds are only a factor of 2 away, and the proof of the upper bound is constructive. Finding optimal partitions of trees or graphs in general has very interesting applications, such as finding the optimal way to distribute a computation over a network of computers. The results are presented in preprint GCG47, and a paper has been submitted to the 1994 European Symposium on Algorithms.

Palios' newest algorithm solves the problem of tetrahedralizing the 3D region between a convex polyhedron and a convex polygon. Given a convex polyhedron and a convex polygon (whose supporting plane does not intersect the polyhedron), the problem asks for a tetrahedralization of the 3D region "between" the polyhedron and polygon, that is, the convex hull of their union minus the polyhedron. The problem arises naturally from a similar problem involving two convex polyhedra that do not intersect, proposed by **M. Bern**. A paper on this work has been submitted to the 1994 Canadian Conference on Computational Geometry. A preprint is under preparation.

Future plans. More computational geometers will be involved with the Center next year. Prospective postdoc Ioannis Emiris works on the production of usable software, focusing on robustness in the face of degenerate input, while prospective postdoc Ayellet Tal works on the animation of computational geometry algorithms. A new computer science faculty member at the University of Minnesota, Shang Hua Teng, works on computational geometry and scientific computation. The opportunity to collaborate with the Geometry Center is part of what drew him to Minneapolis.

The increasing prominence of the Center in the field is also evidenced by recent visits of eminent computational geometers previously unconnected with the Center: **Franco Preparata** (Brown University), **Michiel Smid** (Max Plank Institute, Germany), **Peter Shor** (AT&T Bell Labs), **Leo Guibas** (Stanford University), and soon **Marco Pellegrini** (King's College, Great Britain) and **Jack Snoeyink** (University of British Columbia, Canada). As the Center becomes more well-known, we expect that, first, it will become a repository for computational geometry software and knowledge about computational geometry software development, and second, that Center graphics and video facilities will be used by computational geometry visitors as well as in-house participants to make images, interactive programs, and videos explaining computational geometry. We have developed concrete proposals in both these areas.

Dynamical Systems

by **Rick Wicklin**

Bifurcation theory. Center visitor **Bruce Peckham** and **Richard McGehee** have investigated bifurcation phenomena for two-parameter families of maps on $\mathbf{R}^2$. They have paid particular attention to *resonance regions*, the set of parameter values for which there exists a periodic orbit of a specified rotation number.

Geometrically, the resonance regions are best understood as the projections onto the parameter space of surfaces living in the (four-dimensional) product of parameter space and state space. These surfaces are defined as the set of points $(x, \mu) \in \mathbf{R}^2 \times \mathbf{R}^2$ for which the point x in the state space is a periodic orbit with a certain rotation number for the map with parameter value μ. One can show that, generically, this set is a two-dimensional surface embedded in the four-space. Much of the complicated structure of the boundary of the resonance regions can be understood in terms of the singularities of the projection from $\mathbf{R}^4$ to $\mathbf{R}^2$ of the corresponding resonance surface.

Using the four-dimensional visualization software at the Geometry Center, Peckham and McGehee made a video illustrating the global structure of some resonance surfaces for a certain class of forced oscillators. Their work has led to new insights into bifurcations and to new computational approaches. For example, preliminary work indicates that surfaces analogous to resonance surfaces can be defined for homoclinic orbits, a fact that can be used to develop new techniques for computing some so-called "global bifurcations" such as homoclinic tangencies.

The desire to compute bifurcation surfaces provided much of the motivation for a new software project at the Geometry Center. The numerical techniques used by McGehee and Peckham required detailed knowledge of the map they were studying. The surfaces were computed in patches, which were then pieced together in order to reassemble the global surface. Software that could compute the entire surface at once—and without *a priori* knowledge—would be extremely useful to the dynamics community. Such software is being developed as part of the Pisces project (page 140).

Symplectic Maps and Celestial Mechanics. The problem of determining the stability of a periodic orbit of a mechanical system remains largely unresolved despite centuries of study. Typically, one studies the dynamics of solutions near the periodic orbit by studying iterations of the Poincaré map, and the question of the stability of the periodic orbit is reduced to determining the stability of the corresponding fixed point of the map. For conservative classical mechanical systems, the map is *symplectic* and the possibly stable fixed points are called *elliptic*.

For elliptic fixed points of a certain class of symplectic maps on $\mathbf{R}^2$ (known as *near-integrable*), the Kolmogorov–Arnold–Moser (KAM) theory states that generically there exists invariant circles arbitrarily close to the fixed point and surrounding it. Since points starting inside one of these circles can never escape: the fixed point is stable. For maps on $\mathbf{R}^{2n}$, KAM establishes the existence of invariant n-dimensional tori near the fixed point, but since n-dimensional tori no longer separate $\mathbf{R}^{2n}$ for $n > 1$, this does not settle the stability question. It is known that an instability called *Arnold diffusion* commonly occurs in a neighborhood of elliptic fixed points, although the geometry of this diffusion is poorly understood.

McGehee has been studying the stability question in $\mathbf{R}^4$ using numerical simulations and visualization techniques. Recent developments using variational techniques to find orbits of symplectic maps can be modified to produce algorithms for numerically computing homoclinic and heteroclinic orbits, the existence of which form the basis for the proof of the existence of Arnold diffusion. These homoclinic and heteroclinic orbits are the intersections of stable and unstable manifolds. **Eduardo Tabacman**, a graduate student working under McGehee and supported by the Center, designed and implemented algorithms for the visualization of these stable and unstable manifolds. He also designed and implemented algorithms for the computation of homoclinic and heteroclinic orbits, based on variational techniques developed by Aubry and Mather. Tabacman's methods appear to work better than previous methods on the standard examples and to converge even in cases where previous methods are known to fail. Furthermore, the same ideas appear to produce a new proof of the existence of homoclinic points in many important examples. The Center preprint series provided early dissemination of some of these results.

This research stimulated and was stimulated by the work of other researchers who came to the Geometry Center for a workshop on the visualization of invariant sets of 4D symplectic mappings. This workshop occurred while **J. Moser** was visiting the Center. Moser talked with many people during his stay, from graduate students to full professors. He even gave a talk on the mathematics of billiards to high school teachers. During his visit, Moser seemed particularly impressed with some of the computational and visualization tools that were demonstrated during the workshop.

At about the same time, Center visitor **Alessandra Celletti** implemented her algorithm for the computation of KAM tori for symplectic maps in 4D. This algorithm is a technique for computing the Fourier expansion of an embedding of an invariant two-torus in four dimensions, taking advantage of special properties of the embedding. **Wen-Xiong Liu**, a grad student supported by the Center, was able to visualize the embedded tori by combining the results of Celletti's computations with Center software. Inspired by the pictures, Celletti is continuing her attempts at a computer-assisted proof of the existence of KAM tori in particular examples. Meanwhile, at the Geometry Center, the visualization efforts of Tabacman and Liu provided the early impetus for a Geomview module for visualizing objects imbedded in high dimensional space. This module is now distributed as part of Geomview.

Center visitor **Antoni Susin** studied with McGehee and Rick Moeckel (School of Mathematics) the computation of connecting orbits on the collision manifold for the planar **three-body problem**. Properties of orbits passing close to triple collision in the classical three-body problem can be determined by studying the saddle connections on the collision manifolds; the problem is to understand how the connections depend on the masses of the bodies. This problem can be reduced to the study of the linking of two circles embedded in a three-sphere. Using Center tools, Susin was able to illustrate the changes in the linking and to establish the existence of new saddle connections in the collision manifold.

Noninvertible Maps. McGehee and two of his graduate students are studying the dynamics of noninvertible mappings. Although noninvertible maps are extensively studied in complex dynamics, a systematic mathematical theory of global structures for noninvertible maps in real Euclidean spaces has not been developed. Noninvertible maps arise in a number of applications, including biological systems, control theory, and numerical analysis, and there have been many numerical studies of their dynamics. The numerical studies indicate a rich structure of phenomena which do not occur in the invertible case.

McGehee and **Ioannis Kevrekidis** are organizing a workshop on noninvertible maps to be held at the Geometry Center in the spring of 1995. It will bring together both mathematicians and engineers to discuss noninvertible maps: the rapidly expanding numerical observations, the as-yet immature theory, and the applications. Meanwhile, McGehee and his students are working on numerical techniques for the computation of invariant manifolds for noninvertible maps.

Computer Graphics

The mathematical visualization challenges faced by the technical staff lead to original research in computer graphics. See [Hanson et al. 1994] for a discussion of current interactive graphics research which includes the contributions of the Center.

Hyperbolic Shading and Lighting. For example, in commercial graphics packages like RenderMan, the standard built-in illumination computations are implicitly Euclidean; In a non-Euclidean geometry, correct rendering of surface shading requires custom software shaders that use alternative inner products for computing distances and angles. Such shaders were implemented and shown in by **Charlie Gunn** in [Gunn 1992b] to impose only moderate performance penalties in comparison to the built-in Euclidean ones.

Hyperbolic Motion. The projective models of non-Euclidean geometry can be represented by 4×4 real matrix transformations on homogeneous coordinates that are serendipitously supported by today's computer graphics transformation hardware and software. In [Phillips and Gunn 1992] **Mark Phillips** and Gunn discuss hyperbolic motion as implemented in Geomview (page 133).

Maniview. Staff member **Charlie Gunn** (now at the Technical University of Berlin) wrote a Geomview module to depict a scene in a Euclidean, hyperbolic or elliptic space (manifold or orbifold) M. When M is not simply connected or the curvature is positive, light rays following different paths may nonetheless start and end at the same point, with the result that multiple images of the same object are seen. Maniview takes a geometric description of an object immersed in the space, and replicates it according to the holonomy of the space (specified by **Snappea**, for example), so that when the data is rendered by Geomview in the universal cover, the result is what the object would look like to an observer living inside M. Some of the most striking scenes in the animation *Not Knot* are of this type, and were painstakingly generated. Maniview grew out of Gunn's work on the movie, and is a general tool for interactive navigation of discrete groups, which can be specified either by 4×4 matrices or in automatic group format. This novel method for of three-manifold visualization was the subject of Gunn's 1993 SIGGRAPH paper [Gunn 1993].

Geomview. The interactive 3D viewer Geomview [Phillips et al. 1993], written by Center staff, is the core of much of the graphics research here. Among its novel features are the builtin support for non-Euclidean geometries and a command langugage that allows extensibility by external modules. See page 133 for a full discussion of Geomview.

Documentation: Staff and Infrastructure

The Geometry Center Technical Staff

In many ways this entire book documents the work of the Center staff. They are crucial to nearly every aspect of the Center. This section gives some additional personal information about each staff member.

Senior Staff

Scott Bertilson (May 1991 – present). Scott is the Center's computer systems administrator. He was with the Geometry Project when it became the Center in 1991. Many visitors comment that the computing environment at the center is one of the most integrated and stable ones they've used; this is in large part due to the work and dedication of Scott. In particular, he

- handles the purchasing and maintenance of all our computers, networks, and video equipment;
- keeps abreast of technological advances, suggesting new products and finding the most economical way of achieving a desired end;
- handles software purchases and coordinates software installation and upgrades.

Charlie Gunn (May 1991 – February 1993). Charlie was with the Geometry Project when it became the Center in 1991. He worked as a designer, programmer, and consultant for many Center projects. While at the center, he

- started, together with **Pat Hanrahan**, the project that led to the development of **Geomview** (page 133);
- was one of the directors of *Not Knot* (page 32);
- worked with visitors to the Center on issues of software design, computer systems, video animation, and education—in many cases this involved substantial collaborations resulting in software and video productions;
- supervised a number of apprentice and student projects.

Silvio Levy (May 1991 – present). Silvio received his Ph.D. in Mathematics at Princeton University in 1985 under the direction of **Bill Thurston**. He has been the Center's Director of Technology since December of 1993; in this role he makes decisions concerning the direction of software development, allocation of technical resources, hiring of technical staff, and so on. Prior to that he was a member of the technical staff, having been with the Geometry Project when it became the Center in 1991.

- developed software such as **Cweb** (page 145), **Mathematica** packages, and a program that eventually became incorporated into **Pisces** (page 140);
- was one of the directors of *Outside In* (page 32);
- edited Thurston's *Three-Dimensional Geometry and Topology*, edited *The Mathematica Journal* (1989–91), co-authored *Word Processing in Groups* (page 91), and still edits *Experimental Mathematics* (Dec 1990 – present).

Stuart Levy (May 1991 – present). Stuart was with the Geometry Project when it became the Center in 1991. He works as a designer, programmer, and consultant for many Center projects. He

- was one of the three original designers and authors of Geomview (page 133);
- wrote a number of programs to facilitate interactive viewing and editing of images, which are used frequently to prepare animations and high-quality prints for publication;
- works with Scott Bertilson on computer systems administration tasks, and in setting up and maintaining the Center's video production facility;
- had an important role in the creation of *Not Knot* and *Outside In* (page 32);
- did the computations for **"Building a New World"** (page 34), and, with Tamara Munzner, designed and set up the museum exhibit (page 31);

Tamara Munzner (June 1991 – present). Tamara has been a part of the technical staff of the Geometry Center since June 1991, when she received her B.S. in Computer Science from Stanford. Her first year at the Center was spent as an apprentice, and since June 1992 she has been one of the senior technical staff. In 1990, she was a summer student in the Geometry Supercomputer Project undergraduate research program, which later evolved into the Geometry Center Summer Institute. As a senior technical programmer, she

- is responsible, with Stuart Levy and Mark Phillips, for design, implementation, documentation, distribution, and maintenance of Geomview (page 133), its modules, and other mathematical software;
- was one of the directors of *Outside In* (page 32);
- with Stuart Levy, designed and set up the **museum exhibit** (page 31);
- provides user support for Center computational environment, helps visitors and staff make many mathematical animations, high-level consulting on graphics and **user interface issues**;
- coordinates the **tour** program (page 37), and gives many of the tours;
- created a **mentoring** program for four female high school students during the 1991–1992 school year (see page 28 and the Education and Video display tables).

Mark Phillips (July 1992 – Present). Mark has been a senior staff member of the Geometry Center since July 1991. He received his Ph.D. in Mathematics from the University of Maryland under the direction of Bill Goldman (1990). As a senior technical programmer, he

- is one of the three original designers and authors of **Geomview** (page 133);
- is working with postdocs Rick Wicklin and Davide Cervone on **Pisces** (page 140), focusing on the program's internal design, implementation of algorithms, and user interface;
- has as written much software documentation, including the Geomview manual, and prepared a set of software development guidelines for use within the Center;
- worked with Jeff Weeks on the Unix version of **Snappea** (page 96), and on the design of the version 2.0;

- maintains the Center's Internet ftp server for software distribution;
- wrote the part of Geomview that deals with hyperbolic geometry, and worked with postdoc Oliver Goodman on the **Hyperbolic.m** Mathematica package (page 95);
- supervises a number of apprentice and student projects;
- co-organized a special session on Computer Visualization at the Heidelberg joint meeting of the AMS and the German Mathematical Society in October of 1993.

Stuart Levy, Tamara Munzner and Mark Phillips will teach the AMS **Mathfest** mini-course on Basic Issues in Computer-Aided Visualization in August 1994 (page 35).

All three, and also Silvio Levy, work with visitors to the Center on issues of software design, computer systems, video animation, and education. In many cases this has involved substantial collaborations resulting in software and video productions.

Staff Publications

The following papers and books have been published by staff members since the Center's inception. For talks given by the staff, see page 125.

- Gunn, C. (1992a). Remarks on mathematical courseware. In *Interactive Learning Through Visualization*, pages 115–129. Eurographics, Springer Verlag.
- Gunn, C. (1992b). Visualizing hyperbolic geometry. In *Computer Graphics and Mathematics*, pages 299–313. Eurographics, Springer Verlag.
- Gunn, C. (July, 1993). Discrete groups and visualization of three dimensional manifolds. *Computer Graphics*, 27:255–262. Proceedings of SIGGRAPH 1993.
- Gunn, C. and Epstein, D. (1991). *Supplement to "Not Knot"*. Jones & Bartlett Publishers, Boston.
- Hanson, A., Munzner, T., and Francis, G. (1994). Interactive methods for visualizable geometry. *IEEE Computer*. To appear.
- Levy, S. (1992a). LaTeX labels in mathematica graphics. *The Mathematica Journal*, 2(4).
- Levy, S. (1992b). Automatic generation of hyperbolic tilings. *Leonardo*, 25:349–354.
- Levy, S. (1992c). The Geom style for LaTeX. Available by request from admin@geom.umn.edu.
- Levy, S. (1993). Literate programming and Cweb. *Computer Language*, 10:67–70.
- Levy, S. and Knuth, D. (1993). The Cweb system of literate programming. To be published by Addison-Wesley.
- Palios, L. and Phillips, M. (1992). Tetrahedral breakup. In *Animation of Algorithms: A Video Review*, publicaiton 87a. DEC Systems Research Center, Palo Alto, CA. Video.
- Phillips, M. and Gunn, C. (1992). Visualizing hyperbolic space: Unusual uses of 4×4 matrices. In *Proceedings of the 1992 Symbosium on Interactive 3D Graphcis*. ACM SIGGRAPH.
- Phillips, M., Levy, S., and Munzner, T. (1993). Geomview: An interactive geometry viewer. *Notices of the American Mathematical Society*, pages 985–988. Computers and Mathematics Column.
- Seroul, R. and Levy, S. (1991). *A Beginner's Book of TeX*. Springer-Verlag.

Apprentices

An **apprentice** is a highly gifted and motivated student in math, computer science, or related subject who will work full time at the Center for at least six months, typically during a leave of absence from college or grad school, or between college and grad school or a corporate career.

The apprentice program started under the Geometry Supercomputer Project with summer appointments, and has blossomed under the Geometry Center. Ten apprentices have enjoyed the Center since its creation in 1991.

Steve Anderson (May–September 1991). Mathematics major at Caltech 1989–1993.

Steve was an apprentice at the Geometry Project when it was converted to the Center in 1991. During that time he worked on a number of projects, from software development to organizational work to physically moving computers around and getting the new Center set up.

He had a hand in the development of Geomview, which was being completely rewritten. He created the geom_help software, which served as a quick reference, by subject and alphabetically, to the Center's software. He aided the presentation of the first **Geometry and the Imagination** course, and took the class himself. He alse made a set of slides of the movie *Not Knot*, which has been widely shown and distributed.

Steve now works at Minerva Software, a Minneapolis computer graphics firm led by **Toby Orloff**, a former Geometry Supercomputer Project staff member.

David Ben-Zvi (Summer 1992; March–September 1993). Mathematics major at Princeton; will graduate June 1994. Westinghouse Science Talent Search Finalist in 1990; Andrew Brown Prize (Princeton Math Department) in 1993; Barry M. Goldwater Scholarship 1993–94.

David got involved with the Center through contact with **Silvio Levy** at Princeton; he worked in the creation of *Outside In*, presented a draft of it at the **MSRI Visualization Conference** in October 1992 and the **Smith Regional Geometry Institute** in July 1993, and explained the mathematics behind it to countless visitors and conference participants.

David helped a lot in the 1993 **Summer Institute**, in particular by preparing a collection of project ideas for the undergraduates. He will return in the summer of 1994 to play a similar role. The Teichmüller Navigator program by **Deva van der Werf**, which we have put up on the **World-Wide Web** (see page) is an example of this interaction.

He also did research on the cubic connectedness locus (the Mandelbrot set for cubic polynomials), a topic that led to his senior thesis, "The Geometry of Hyperbolic Components in Complex Dynamics: The Case of Cubics". These efforts also resulted in his posting a series of articles to the **Internet**, on topics such as exponential function, hyperbolic geometry, Teichmüller theory, the connectivity of the Mandelbrot set and its relation to the Riemann mapping theorem, the Cubic Connectedness Locus, etc. Anyone who has read the Internet for a while can appreciate the value of such lucid expository postings.

David learned an extraordinary amount of mathematics at the Center. Although he probably would have learned just as much at Princeton, he found that the contacts he made with visitors at the Center expanded his horizons and speeded up his maturation process. He said in his final report:

> One of the great joys of a place like the Geometry Center is the opportunity to be taught and to teach others. I have gained considerably both from having many intriguing subjects explained to me and from attempting to convey my limited understanding of various topics to others. The latter has provided me with both invaluable teaching experiences and a far clearer, more organized understanding of the subjects I explained.

David will start as a Harvard math Ph.D. student next fall, having been awarded a three-year NSF Graduate Fellowship.

Adam Deaton (September 1993 – present). AB in Mathematics, Princeton University, June 1993 (High Honors).

Adam's senior thesis at Princeton was completed under the supervision of former Center faculty members **John Conway** (Math) and **David Dobkin** (CS). It consisted in a program, **Lafite**, designed to help visualize the hyperbolic geometry of two-orbifolds by creating patterns based on their symmetry groups. Lafite uses an algorithm described by Bill Thurston to decompose an arbitrary two-orbifold into simple pieces for which a hyperbolic structure can be computed. See figure on page 122.

Since coming to the Geometry Center, Adam has worked on a number of projects. He has continued to work on Lafite in response to the comments of a number of visitors to the Center who saw Conway's demonstration of the program at the **Smith College RGI** in July 1993. Recent modifications have centered on making the program more stable and user-friendly in order to make it ready for a wider distribution, and on putting it on the **World-Wide Web** (page 29).

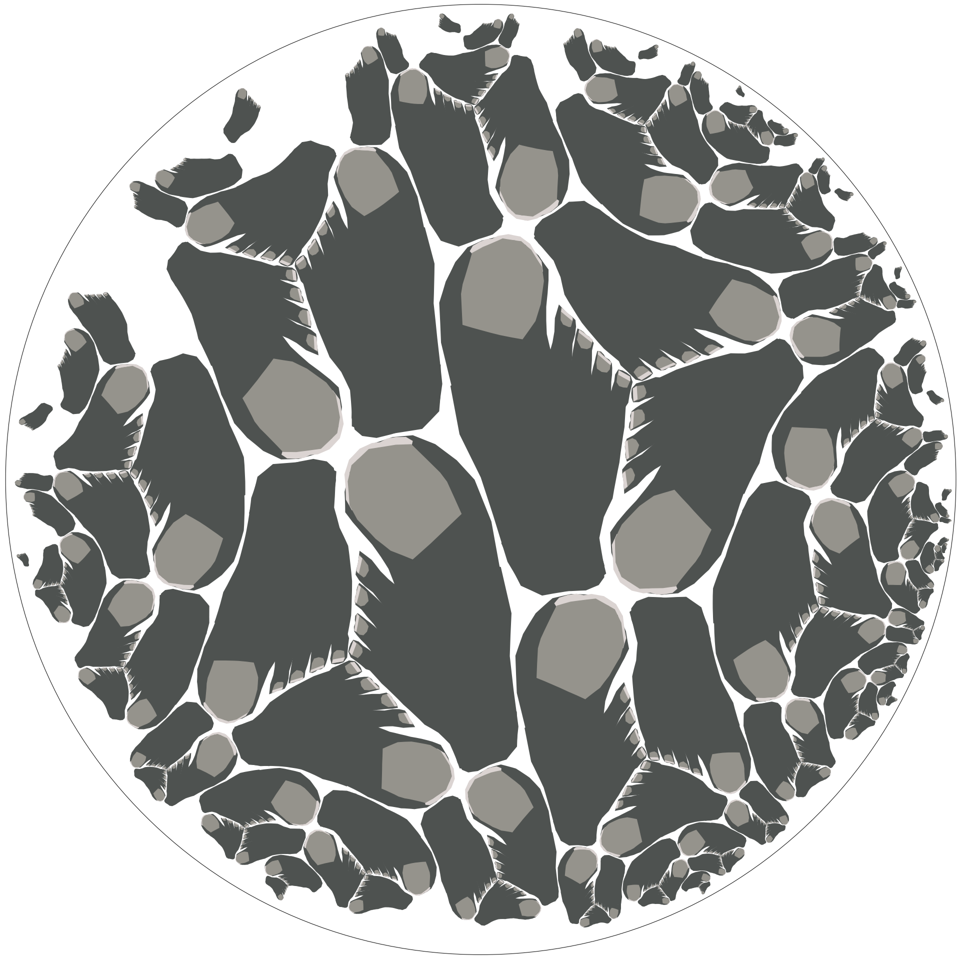
In addition he has continued working on a project begun at Princeton with the New York environmental artist Agnes Denes. She is creating a man-made mountain covered with 10,000 trees in Finland, and needed help in creating a pattern of trees that would satisfy both horticultural and artistic constraints. Adam worked with Conway and **Tim Hsu**, a graduate student at Princeton, to create software and models for the problem. He is currently working with a firm of surveyors in Pittsburgh who are converting data computed at the Geometry Center into a form usable by the satellite surveying technology which will position the trees on the mountain.

Adam created the credit sequence for *Outside In*. In order to create high-quality text for the movie, he wrote a general-purpose tool called Labeler that can convert PostScript text into a geometric object. This tool can also be used to make labels within **Geomview**, and has been distributed in a preliminary form to researchers at Cornell and in Germany.

Adam will enter the Ph.D. program in computer science at Harvard in September 1994.

Celeste Fowler (June 1992 – March 1993). Computer Science major at Princeton; will graduate June 1994. While at the Center, Celeste wrote several support programs and modules for **Geomview**, such as anytooff, corners, crayola, drawbdy, pssnap, sweep, tackdown, transformer, and warp (see page 146 for brief descriptions).

She also worked on the development of Geomview proper, on video editing (together with **Charlie Gunn** she compiled the 1992 summer student works into a com-



Sample output from the Lafite program written by Adam Deaton

mercially distributed tape), and on video creation (draft of *Shape of Space*, some work on *Outside In*.) Finally, she had a major role in the Center's mentoring program.

After Celeste left the Center, she took a summer job at **Silicon Graphics**, developing computer graphics software, and will return to full-time job in the same area after graduation. She has coauthored a paper with Princeton postdoc Seth Teller, Bell Labs researcher Tom Funkhouser, and **Pat Hanrahan** that was accepted for publication in SIGGRAPH '94.

Olaf Holt (June 1992 – present). Olaf graduated with a major in Mathematics from Swarthmore College in June, 1992. He has worked with **Jeff Weeks**, the author of **Snappea**, on the problem of computing the canonical triangulation of cusped hyperbolic 3-manifolds. Olaf also made a beautifully artistic film which introduces the Platonic and Archimedean solids to a general audience. Most of Olaf's efforts, however, have focused on visualization of higher dimensions. He has designed and implemented a **Geomview** module called **NDview** for visualizing objects in higher dimensional spaces. NDview is now available as part of Geomview via anonymous **ftp**.

In connection with his writing higher dimensional visualization software, Olaf made a film about 4 dimensional rotations with Dr. Paul Humke of St. Olaf College, and wrote a program for Dr. **Tom Berger** of the University of Minnesota to use in his complex analysis course.

Olaf will be going to graduate school in computer science at the University of Minnesota in the fall of 1994.

Stephanie Mason (May – December 1993). Mathematics major at Virginia Polytechnic Institute and State University; will graduate in May 1995.

While at the Center, Stephanie worked on several projects. She wrote programs for exploring two- and three-dimensional cellular automata. In collaboration with **Tony Phillips** she wrote a program to generate sounds representing patterns in ocean tides. With former Center faculty member **Jean Taylor** and Center postdoc **Paul Burchard**, she wrote software and refined an algorithm for motion of curves by crystalline curvature. She also assisted visitors and students in using the Center's computer system, and served as a frequent guide for groups coming to the Center for tours.

After leaving the Center, Stephanie returned to school at Virginia Tech, where she has continued her work on Lindenmayer Systems and music, begun while taking part in the Geometry Center's **Summer Institute** in 1992. She has written a paper on this with Michael Saffle at Virginia Tech.

Daeron Meyer (May–December 1992; June 1993 – present). Daeron Meyer first became an apprentice at the Geometry Center in May of 1992. He took a semester off to complete his BS in Electrical and Computer Engineering from the University of Wisconsin, Madison before continuing on with the apprenticeship.

During his first stay at the Center Daeron put his programming talents to work designing Ginsu, an external module for **Geomview** which provided users with the capability to slice apart any 3-dimensional object with an arbitrarily positioned slicing plane. He also created 4dview, another extension to Geomview which allowed users to manipulate 4 dimensional objects and slice them with arbitrary hyperplanes. Using 4dview, Daeron assisted **Dennis Roseman**, a visiting professor to the Center, in the creation of a videotape about 4 dimensional knots.

He also spent a lot of time working with summer students on their projects which inspired several more projects/programs of his own. One of these projects was Pathmake, a program which allowed users to interactively generate smooth, linearly interpolated paths for an object to move along as well as smooth interpolations for the orientation of that object. Another program Daeron developed during the summer was Animate, an external module for Geomview that let users flip through a sequence of objects in any order, in effect animating that sequence.

During his second stay at the Center Daeron has been working on several larger programming projects. First, he designed a user interface builder based on **Motif**, which lets one drag and drop user interface elements onto a blank canvas (similar to NeXT Interface Builder). This program, called "mib", was used by two summer students to create interfaces for their summer project programs.

Second, Daeron used mib to create the user interface for an **X11** version of Geomview. Geomview had previously only been available on the SGI and NeXT platforms, but this, along with conversion of the rendering libraries Geomview used, made it possible to port Geomview to all platforms that supported X windows.

Also, Daeron spent a good deal of time working on scene 5 of *Outside In*. This involved writing of set of perl scripts to control the sequence of events in parts of the scene.

Finally, he worked on setting up the Geometry Center's **World-Wide Web** server, which is proving to be a new an effective way of distributing information about the Center to a world wide audience.

Tamara Munzner (June 1991 – June 1992).

Tamara came to the Center as an apprentice, and after approximately one year she joined the permanent staff; see the section on the senior technical staff, page .

Nathaniel Thurston (January 1993 – present). Mathematics graduate student at Berkeley since September 1991; NSF graduate fellowship.

Nathaniel played a major role in the production of *Outside In*, and wrote several programs closely related to the movie, including the program that parameterizes the eversion, a program for interactively moving the "camera", and the scripting system.

He is currently working with **Dave Gabai** and **Rob Meyerhoff** on a computer-assisted proof of a major theorem in hyperbolic geometry (page 98). Last year, he investigated the geometry of lens spaces, and experimented with a system to rate bridge players based on their statistical performance.

Linus Upson (September 1992 – March 1993). Mathematics major at Princeton University.

Linus was a participant in the **Summer Institute** during the summers of 1991 and 1992; in September of 1992 he returned for a 6-month stay as an apprentice, during which time he continued work on several projects he started during the previous summers. One of these was FlyHyperbolic a hyperbolic "flight simulator", by request from **George Francis** of the University Illinois. Linus also developed a program called Brot for visually exploring 2D maps. It provides services such as transparent distribution of computation, and a user interface for navigating the computed images. Fellow students **David Ben-Zvi** and **Christine Heitsch** used Brot to explore the Cubic Connectedness Locus. Images they produced appeared in the October 1993 issue of **Scientific American**. Linus's also developed a small library of Objective-C objects for doing simple geometric computations.

In addition to the above, Linus was involved in a number of other activities at the Center including contributions to *Outside In*, the **mentoring program**, and routine administration of the Center's network.

Since leaving the Center Linus has been working at NeXT Computer, Inc., where he applies what he learned as an apprentice to real-life software product development.

Talks Given by Geometry Center Staff

Silvio Levy February 1991
Contour tracing by piecewise linear approximation
GANG, University of Massachusetts at Amherst

Charlie Gunn June 1991
Representation of numerical data in visual form
National Center for Supercomputer Applications, Urbana, Illinois

Charlie Gunn July 1991
Rendering in hyperbolic space
"Frontiers in Rendering" course, SIGGRAPH, 1991, Las Vegas

Charlie Gunn July 1991
Volume rendering the cubic connectedness locus
"Advanced Topics in Fractals" course at SIGGRAPH, 1991, Las Vegas

Charlie Gunn July 1991
Visualizing the cubic connectedness locus
ACM SIGGRAPH

Charlie Gunn July 1991
Visualizing hyperbolic space (short course)
ACM SIGGRAPH

Mark Phillips October 1991
Several presentations
Workshop on Scientific Visualization Environments/IEEE Visualization Conference (San Diego)

Charlie Gunn January 1992
Computers and mathematical intuition
Symposium on Computers and Geometry, Institute of Technology, Tokyo, Japan

Charlie Gunn January 1992
Not Knot, hyperbolic geometry, and computer graphics
NICOGRAPH weekly seminar, Tokyo, Japan

Mark Phillips March 1992
Visualizing hyperbolic space
ACM SIGGRAPH Symposium on Interactive Computer Graphics (Boston)

Mark Phillips March 1992
Visualizing hyperbolic space
Mathematics Seminar, University of Massachusetts at Dartmouth

Charlie Gunn June 1992
The making of Not Knot
Prix Ars Electronica, Linz, Austria

Mark Phillips, Leonidas Palios June 1992
Film: "Tetrahedral Breakup"
ACM Symposium on Computational Geometry (Berlin)

Mark Phillips, Stuart Levy, Tamara Munzner October 1992
Presentations on Geomview
MSRI Workshop on Visualization of Geometric Structures (Berkeley)

Mark Phillips, Al Marden November 1992
Communicating geometry: ideas and tools
NCTM meeting (Minneapolis)

Mark Phillips April 1993
Life in non-Euclidean space
St. Olaf College, Northfield, MN

Mark Phillips April 1993
Visualizing non-Euclidean geometry
DEC Systems Research Center, Palo Alto, CA

Tamara Munzner July 1993
Introduction to Geomview
Regional Geometry Institute, Smith College, Amherst MA

Mark Phillips October 1993
Geomview: an interactive geometry viewer
Experimental Mathematics seminar at University of Bonn, Germany.

Mark Phillips October 1993
Presentation of Geomview
Joint meeting of AMS and German Mathematical Society, Heidelberg, Germany; Phillips co-organized, with George Francis and Norbert Quien, a special session on "Interactive Real-Time Software for Mathematical Visualization".

Tamara Munzner November 1993
Mathematical visualization using Geomview
Gustavus Adolphus College, St. Peter MN

Tamara Munzner November 1993
Mathematical visualization using Geomview
St. John's College, Collegeville MN

Olaf Holt (apprentice) March 1994
Skiing the Himalayas: Interactive Computer-Aided Visualization of Complex Functions
Mathematics colloquium, St. Olaf College, Northfield, MN

Stuart Levy April 1994
Demonstration of Geomview and showing of *Outside In*
Visualization seminar, University of Wisconsin, Madison Mathematics Department

Tamara Munzner, Al Marden April 1994
Visualizing the invisible
Minnesota Council of Teachers of Math 1994 Spring Meeting, Brainerd, MN

Tamara Munzner, Al Marden April 1994
Visualizing the invisible
Weisman Art Museum, University of Minnesota, Minneapolis, MN

Hardware and Video Production Facilities

by **Tamara Munzner**

The productivity of the staff, visitors and summer students is made possible by a first-class hardware environment. This hardware is a resource to the mathematical community that is not available at most departments. This is particularly true of the video production facilities.

The Center has around thirty NeXTs, fifteen SGIs, fifteen Suns, six HP Geckos, in addition to Macs and PCs. Fifteen additional Macs are available for the **summer course**. About half of the machines were purchased in the last two years, and are naturally significantly faster that the earlier ones. All the computers are heavily used during the summers, during workshops, and during peak periods — for instance, in the rendering of movies or in the solution of research problems such as the one being tackled by **Nathaniel Thurston** (page 98).

The Center as a Bellwether. We believe that the computer graphics facilities available to math departments will continue to improve, as workstation prices drop and graphics programming environments become more standardized, and as demand within the mathematical community increases. As a result, hardware supporting the kind of work we do here will become common. For example, IBM PC's now run **X** windows, the environment that we use on the Suns; the recently purchased HP Geckos are inexpensive and will run NeXTstep, the programming environment of the NeXTs. SGI now sells the low-end Indy workstation for only $2500.

Center visitors almost universally ask the technical staff for advice on hardware purchases. The Center's influence is sometimes definitive; **Gene Klotz**, for example, selected an SGI Indy with video hardware for Swarthmore. **Jean Taylor** set up a video production facility at Rutgers based on the Center's model, and MSRI has also used the guidance of the Center staff in purchasing video equipment. The Center pioneered the use of a high-quality Tektronix color printer, and other departments such as UC Berkeley followed suit.

The computational resources of the Center are also available over the Internet. This is useful for people who need a powerful machine for some computationally intensive task, usually with the **Evolver**. **Dennis Roseman**, **John Sullivan**, **Rob Kusner**, **Lucas Hsu**, and **Bob Meyerhoff** have all used the Center's machines for remote computation.

Computer Animations. We have produce eight hours of video animation at the Center, in comparison with 104 minutes under the Geometry Supercomputer Project. We have produced video with a wide range of intent, from relatively rough non-narrated clips for conferences to our flagship pedagogical videos *Not Knot* and *Outside In*, aimed at the general public (page 32).

Less ambitious animations are sometimes created in hours or days. Many of these collaborations between staff and visitors are intended to be shown at mathematical conferences as a kind of "video overhead" during a talk by the visitor, often in lieu of a logistically difficult interactive computer demo. The AMS Selected Lectures in Mathematics video series includes a compilation of video from the **Computational Crystal Growers Workshop** organized by **Jean Taylor** (page 62): six of the segments were made here during the conference itself.

Between these two extremes are the animations on 3D and 4D knots made by **Dennis Roseman** (University of Iowa), and those produced by the undergraduates in the **Summer Institute** documenting their research projects (page pagerefSummer Institute). The "Animated Geometry" compilation tapes from the past two summers are being distributed by Media Magic of Nicasio, CA.

Our hardware allows single-frame recording, or live recording from a NeXT or an SGI. In either case the resulting quality is much higher than shooting off the screen with a camera.

Staff Support. Technical staff support is a crucial part of the video production environment of the Center. Experienced technical staff can guide a visitor through the process of making a video far more quickly and with far better quality than would result from the attempts of a novice. In addition to the entirely nontrivial task of operating the hardware, there are many video production issues completely foreign to the uninitiated. The following list of considerations only scratches the surface:

Saturated colors, which are easy to specify and look temptingly vivid on a computer monitor, usually "bleed" all over the television screen. Colors that are clearly distinct on a computer screen often look indistinguishable on video. Two television screens hardly ever have the same color balance. The portion of the picture actually shown on a television screen varies greatly between monitors: only roughly 75% of the screen area is "TV-safe" and guaranteed to be visible. One-pixel wide horizontal lines result in an annoying shimmer effect because TVs interlace the even and odd scan lines.

This excerpt from the 1993 visit report of **Charles Peskin** illustrates the difference that technical staff support can make, even to people with access to video hardware:

> A secondary purpose of the visit was to make a videotape showing the latest results from the heart project. This was done primarily by my colleague David McQueen, but I also participated. McQueen and I have previously been unable to capture images on videotape of similar quality to those that we see on the workstation screen. This difficulty was overcome with the help and advice of Stuart Levy, who showed us how to make the best use of the limited resolution available in the NTSC standard.. . . Some of the software that we used during this process was created by Levy himself, and he was very generous with his time in helping us use it. Because of the complete control that his system provides, the film that we produced is in some respects better than what we can see directly on the workstation screen! In particular, the pace of the film is faster, and . . . independent of the complexity of the individual frames. These are important features in displaying a computation of a beating heart. The film that we produced is so realistic that I can hear the heartbeat when I watch it run! (Some day, we'll make a soundtrack, so other people can hear it, too.)

Finally, technical staff member Stuart Levy has created a great deal of software to make video production easier and faster, including a program, called Heave, that allows single-frame recording of one minute of video tape in ten minutes instead of seven hours!

List of Videos, February 1991 – March 1994

The Geometry Center has generated over eight hours of mathematical computer animations. There are also 72 hours of live footage from summer courses.

Outside In	21.25 minutes
visitor	190.75 minutes
staff	107.50 minutes
student	166.25 minutes
total	485.75 minutes

The tables below summarize finished pieces gleaned from the 162 tapes made here. Collaborations between staff and visitor are counted under visitors.

min	tape	VISITOR
3.5	S-51	**Bayard Johnson, Robert Sekerka**, Anisotropy in Diffusion Limited Aggregation (DLA) and in Modified DLA
4	S-51	**Elizabeth Holm**, Potts Model Domain Growth Simulations
3	S-51	**Rob Almgren**, Simulations of Dendritic Solidification
7	S-51	**Andy Roosen**, Simulation of Crystal Crowth Using Fully-Faceted Interfaces
5.5	S-51	**Jean Taylor**, Geometric Crystal Growth in 3D via Faceted Interfaces
3.5	S-51	**Ken Brakke**, Evolution of 100 Voronoi Grains
8.75	S-51	**Clarence Lehman**, Phyllotaxis: Music of the Leaves
2.5	S-52	**S. Aletta and Stuart Levy**, Julia Sets on the boundary of the Mandelbrot Set
2.5	S-52	**Ryo Kobayashi**, 3D Crystal Growth
6.5	M-53	**Richard Morris**, A Parade of Surfaces
11	M-53	**John Maddocks**, MC^2: Multiplier and Constraint Continuation
8	M-53	**Dennis Roseman**, Viewing Knotted Spheres in 4-Space
8	M-53	**Alfred Gray**, Strange Surfaces
10	M-53	**Bruce Peckham**, Resonant Surfaces
3	M-53	**John Lowengrub**, Vortex Sheet With Surface Tension
6	M-53	**Charlie Peskin and Steve McQueen**, Heart animation
15.5	S-58	**George Francis**, Apéry's version of Boy's surface flythrough
6	S-58	**George Francis**, Lawson's Snail
19.5	S-59	**Dennis Roseman**, Twisting and Turning in 4 Dimensions
2.5	S-70	**Tony Robbin**, Quasicrystal Sculpture for Denmark's COAST
3	S-81	**Craig Carter**, Capillary Surfaces and Forces Attached to Three Spheres
6.5	S-86	**John Sullivan**, Knots Minimizing a Moebius-Invariant Energy
28	S-86	**George Havas**, Algorithm Animations
17	S-87	**Alfred Gray**, Surfaces IV

min	tape	STAFF
17	M-11	**John Sullivan**, Minimum Surfaces via Maximum Flows
6.5	M-28	**David Ben-Zvi and Matt Headrick**, Sphere Eversion
6	S-45	**Tamara Munzner**, Museum Proposal
6	M-51	**Olaf Holt and Paul Humke**, Voyager from the Fourth Dimension
10.5	M-53	**Charlie Gunn**, Exploring Hyperbolic Space Using Geomview
9.5	M-53	**Leonidas Palios and Mark Phillips**, Tetrahedral Breakup
13	M-53	**Celeste Fowler**, Shape of Space
3	M-53	**Olaf Holt**, Dance of the Platonic and Archimedean Solids
5.5	S-61	**Mark Phillips**, Hoops in R3 Prelim
10	S-87	**Stephanie Mason**, TideSounds
9.5	S-87	**Leonidas Palios**, Tetrahedralizing the 3D Region Between a Convex Polygon and a Convex Polyhedron: The Movie
11	S-87	**Stuart Levy and Tamara Munzner**, Triangle Tiling

min	tape	STUDENT
18	M-08	**91 Summer Student Compilation:**
		Henry Rowley, Rolling cube intro
		Adrian Mariano, Parallel Curves and Caustics
		David Broman, Demonstration of a Linktool.
		Adam Halvorsen, A Quick Dissection of the Fourth Dimension.
		Henry Rowley, Isometric Deformations of Surfaces
		Arek Goetz, Midpoint Transformation of Polygons.
		Kenneth Bromberg, Bisection Envelopes.
		Linus Upson, Flying in Hyperbolic Space.
		Mike Huberty, CEM-ECM Logo and Curve Shrinking.
67	M-30	**92 Summer Compilation**
		Kate Jenkins, Lindenmayer Systems and Plant Development
		Stephanie Mason, Lindenmayer Systems and Space-Filling Music
		Thomas Colthurst, Fractal Polytops and Equivalent IFS's
		Nick Coult, Special 3-Body Systems
		Jacques Friedman, The 3-Ring Circus: A Conservative Dynamical System
		Mark Meloon, Nonlinear Control and the Inverted Pendulum Problem
		Linus Upson and Adrian Mariano, Penumbral Shadows
		Ken Bromberg, A Minimal Surface in Hyperbolic Space
		Gary Gutman, Rolling Up a Two-Holed Torus
		David Broman, Linktool 2.0
		Chris Cianflone, Experiments in Computer Music
		Craig Sutton, SO(3): It's Fundamental
		Jennifer Ellison and Sherry Scott, Groups of Symmetry
		Prem Janardhan, Karen Olsson, Carol Sohn, The Shape of Space

min	tape	STUDENT
8	S-40	**Alex Gamburd**, Archimedean Solids, Whythoff Symbols, Soap Bubbles, Summer 91
55	M-60	**Summer 93 Compilation**
		Ackerman, Jeremy, Vibrating Evolved Minimal Surfaces
		Kirsten Bancroft, Circulatory System Animation
		Elizabeth Callaghan, The Diving Simulator
		Sang Chin, Visualizing the Weierstrass p-function
		Nick Coult, The Solar System Viewer
		Danek Duvall, Clouds
		Patrick Friel, Mazes and Meanders
		Christine Heitsch, Slicing the Cubic Connectedness Locus
		Kate Jenkins, Binary Trees and Ray Tracing
		Heather Leonard and Timothy Rowley, Northern Exposure
		Jing Li, Tonality Circles
		Brian Meloon, The Complex Henon Map
		Millie Niss, The Sphere Scribbler
		Hiren Parekh, Conformal Embeddings in 3-Space
		Michael Sullivan, Singular knots for Linktool
		Deva Van der Werf, The Teichmller Navigator
		Dan Wade, Spiderwebs
		Brendan Dunn, Networks of Neurons
14	M-53	**1992 High School Mentoring Compilation**
		Maria Nagan, School
		Elizabeth Callaghan, Jaws
		Karin Holt, Graphics, Lights, Motion: SoftImage
		Ahna Reza-Girshick, The Fibonacci Sequence
4	S-71	**Bronwyn Collins**, Mouse, High-School Mentoring 92–93
.25	S-81	**Dan Porter**, High-School Mentoring 92–93

min	tape	COMPILATION
23	M-61	**Quick Cuts from the Geometry Center:** Excerpts from Staff, Student and Visitor Work 1989–1993

hrs	LIVE VIDEO
24	Geometry and the Imagination Summer Course (Conway, Doyle, Gilman, Thurston)
24	Chaos and Fractals Summer Course (Branner, Felsager, Hubbard, Vedelsby)
24	Visualization and Graphics Summer Course (Dobkin, Hanrahan, Sorensen, Souvaine)

Software Development

In order for a computer program to be useful to mathematicians, it must be easy to get mathematical information into and out of it. This means that the program must be easy to use and must produce output in a useful format. Although many mathematicians do some sort of programming in the course of their research, their programs usually remain part of their private work and are not distributed to or usable by others.

The Center supports the development and exchange of mathematical software. We focus on making programs usable and distributable via the **Internet**. This involves major projects such as **Geomview** and **Pisces** as well as many smaller programs which are frequently the result of Center **staff members**, including **apprentices**, working with **visiting researchers**. The Center maintains an **anonymous ftp** archive on the Internet which has serviced over 25,000 requests for downloading software since May of 1991.

Frequently it is necessary to use the output of one program as input to another. Because different algorithms internally use different data structures to represent abstract mathematical concepts, communications between different programs, and between people and programs, can become very difficult—sometimes even making communication a dominant issue in software design.

One approach to solving the communication problem is the **standardization of data formats**, and the Center has worked in this direction, with consultation with interested parties from other institutions. For example, by bringing together a large number of developers of knot software, the Center's 1993 **knot workshop** proved to be a catalyst for discussions of standardization. One of the successes of the workshop was the agreement on a simple, standard format for communicating planar knot diagrams.

A diverse group began discussion at the 1992 **MSRI Conference on Geometric Visualization** about the need for a powerful common multidimensional geometric data format, which continued via an email mailing list after the end of the workshop. The multinational group included all the Geomview developers, GRAPE developer **Konrad Polthier** of Bonn, **AVS** users **Ulrich Pinkall** of Berlin and **Ivan Sterling**, **Evolver** developer **Ken Brakke**, Unigrafix developer **Carlo Séquin** of Berkeley, and **Jim Hoffman** and **David Hoffman** of the Amherst **GANG** group.

Because of the level of abstraction in mathematical concepts, the communication problem in mathematical software goes deeper than standardization of data formats. The development of good forms of **interprogram communication** is therefore crucial to the advancement of mathematical software. The Center is sponsoring **Henry Cejtin** and **Igor Rivin** under an Exploratory Research grant for the creating of a new prototype of interprogram communication. The prototype will involve a number of independent modules, whose behavior is completely described by an external specification, and a communication substrate which will allow these modules to communicate in a flexible way. In addition, it will be possible to run different modules on different computers, possibly of differing architecture.

We make a substantial effort in programs developed at or supported by the Center to allow easy interaction with other programs. Formats are open and, when possible, standard. When there is no one clear standard, programmers try to cater to a variety of formats by means of filters, or develop local standards in consultation with faculty members or other specialists.

Geomview

Introduction. Geomview is an interactive 3D viewing program written by Geometry Center staff. Its simplest use is to view and manipulate a single static object. It reads the data in a simple file format, displays a window with an image of the object in it, and lets user translate, rotate, and scale the object by moving the mouse in the window.

Geomview can handle any number of objects and allows both separate and collective control over them. The data can come from a file or from another program that is running simultaneously. In the second case, as the other program changes the data, the Geomview image reflects the changes.

Geomview can work in conjunction with other software such as **Mathematica**, **Maple**, or special purpose programs, serving as a display for geometric objects which these programs create.

In general, Geomview's purpose is to handle the *display* and *interaction* aspects of the presentation of geometric data. The idea is that many aspects of the display and interaction parts of geometry software are independent of the geometric content and can be collected together in a single piece of software that can be used in a wide variety of situations.

Geomview's strongest points are its interactivity and the fact that it is easy to use. Many Geomview users have commented on how quickly they are able to learn to use it to view objects from their area of study.

Geomview now runs on a wide variety of **Unix** workstations: Silicon Graphics workstations, NeXT workstations, 486 PCs running the NeXTStep operating system, and any graphics workstation running the Unix operating system and the **X window system** (this includes Sun workstations).

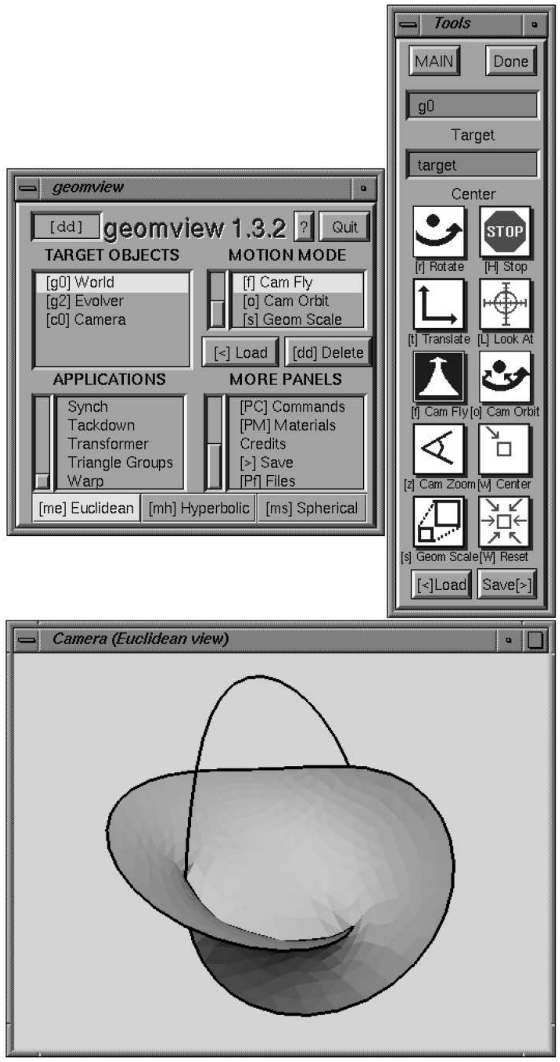
Who uses Geomview? Nearly everyone who comes to the Geometry Center uses Geomview in some capacity. In addition to use within the Center, we have made 11 public releases of Geomview for distribution on the **Internet**. At the time of this writing over 5000 sites from around the world have downloaded Geomview from our Internet server.

Every week we hear via electronic mail from new Geomview users on the Internet. Although we developed Geomview with the needs of mathematicians in mind, its generality has led to its use in a wide range of disciplines, including robotics, CAD, astrophysics, and computer graphics. The user testimonials at the end of this section give a flavor of the uses Geomview has been put to.

Many people use Geomview to view simple polygonal models or surfaces, frequently generated with Mathematica or Maple. Others write their own software for generating objects of interest, and use Geomview to display the objects. Geomview also interfaces well with video recording equipment; many users have made videotapes for showing at conferences using Geomview.

We were extremely pleased recently to note that Indiana University has been evangelizing the use of Geomview as the default 3D viewer for **World-Wide Web**. This was completely without the involvement of anyone from the Geometry Center.

We know of one commercial product based on Geomview: the Liverpool Surface Package, developed by **Richard Morris** at the University of Liverpool, England.



A typical Geomview session. The object in this picture is a soap film whose boundary lies on part of a trefoil knot; it was computed with Ken Brakke's Surface Evolver (page 105.)

The **Triangle Tiling** museum exhibit at the Science Museum of Minnesota in St. Paul is based on Geomview. Geometry Center staff worked in collaboration with the staff of the Science Museum to create this interactive exhibit which runs on a Silicon Graphics workstation. Roughly 10,000 people see this exhibit every week; for more details on this see page 31.

Geomview figured prominently in two **workshops** on mathematical visualization: the October 1992 workshop on "Visualization of Geometric Structures" at the **Mathematical Sciences Research Institute** in Berkeley, and a special session on "Computer Visualization" at the October 1993 joint meeting of the AMS and the German Mathematical Society in Heidelberg, Germany. Center staff member **Mark Phillips** was co-organizer of the latter of these; at both of these workshops speakers from the Center as well as from other institutions used Geomview in interactive presentations of their work.

Geomview can be regarded as infrastructure. Most people take operating systems, text editors, and 2D image viewers for granted: they are simply a ubiquitous and expected part of the computational environment. In the future, interactive 3D graphics will be just as ubiquitous, and Geomview is a step in that direction.

What does Geomview do? Geomview displays objects in three-space and lets you move them around, view them from different angles, and adjust other aspects of the display such as color and lighting. Its strongest points are its interactivity and the fact that it is easy to use.

There are commercial products on the market which provide some of these same capabilities—most notably **AVS**, IRIS Inventor, and IRIS Explorer. These are large systems, however, which require a substantial amount of time to learn. They also provide many additional computational capabilities. We have found that many people just want a simple way to view their 3D data, and Geomview fills this gap nicely. Moreover, one AVS license, for example, costs several thousand dollars, while Geomview is distributed freely.

Geomview also offers the advantage of flexibility. We have built many features into it which have risen out of the needs of the mathematics community, for example the ability to display and manipulate objects in hyperbolic and spherical space. By writing "external modules" (see the section on Geomview Modules below), users have been able to extend Geomview to deal with a wide variety of applications.

There are some things that Geomview does not do. In particular, Geomview does not do any numerical or symbolic computations, and the quality of the pictures that it produces is not as good as with photorealistic rendering systems. Building these features into Geomview would have detracted from the main goal of a fast, easy to use, interactive viewing program. Instead, we chose to make Geomview able to communicate with other software which does these jobs very well. Geomview interfaces easily with **Mathematica** and **Maple** for numerical and symbolic computations, and with **RenderMan** (a commercial product by Pixar, Inc.) for photorealistic imaging. We made much of the video *Outside In* using this connection between Geomview and RenderMan: Geomview provided a fast an easy way to adjust the viewing angles and positions of the objects in each scene, and then produced data for RenderMan to use in creating the final, high-quality images.

Geomview Modules. One of Geomview's most powerful aspects is its ability to support "external modules," which are separate programs that rely on Geomview for their display. The main point of this arrangement is to allow the person writing the module to focus on a particular subject matter, leaving to Geomview the (frequently complicated) details of how to display 3D data.

Many people at the Geometry Center and elsewhere, from **visiting researchers** through **apprentices** and **summer students**, have written Geomview modules for a wide variety of applications including viewing 4D objects, exploration of three-manifolds, visualizing symmetry groups, interactive clipping, creating surfaces of revolution, animation, editing of motion paths, visualizing hyperbolic pleated surfaces, and more.

History and Current State. During the summer of 1989 at the Geometry Supercomputer Project, **Pat Hanrahan** and staff member **Charlie Gunn**, began work on a simple interactive viewing program called MinneView. Many sophisticated CAD/CAM and other specialized programs existed which included 3D viewing capabilities, but they were not very well suited for visualizing mathematics. They were frequently expensive, hard to use, and could not be extended. MinneView was an immediate hit because it was so easy to use. Over the next three years many people contributed to MinneView, including staff member **Stuart Levy**, apprentices **Tamara Munzner** and **Steve Anderson**, and **students** Mark Meuer, Todd Kaplan, and Mario Lopez.

As MinneView grew, structural limitations became apparent. The program also depended on specific aspects of the computer system at the Geometry Center, such as the existence of certain files, and hence was difficult to use outside the Center. Since experience with it clearly indicated that such a tool was very useful, the Center decided to rewrite it with a new design that would be more flexible and portable and which would include a better user interface.

In the fall of 1991 staff members Stuart Levy, Tamara Munzner (who was then an apprentice), and **Mark Phillips** began work on the new version, which became Geomview. The first public release became available in early 1992. Since that time there have been ten additional public releases, with contributions from **Daeron Meyer**, **Nathaniel Thurston**, and **Celeste Fowler**, postdocs **Oliver Goodman** and **John Sullivan**, and students **Daniel Krech** and **Scott Wisdom**.

At present, Geomview is relatively stable and is not under major development. Apprentice Daeron Meyer has been working under the direction of the senior technical staff writing and updating the **X** version, which we are distributing in preliminary form.

In addition to helping people who are physically at the Center use Geomview, we provide electronic mail support to the **Internet** community of Geomview users; each of the staff spends two or three hours per month answering questions, assisting users, and making minor bug fixes.

User Testimonials and Applications

Bill Bruce, University of Liverpool, England

The Liverpool Singularities Group has been awarded three substantial SERC grants in support of their computing activities. We are currently putting together a general purpose package using homegrown C programs, **Maple**, Singular and, crucially Geomview, which provides our main geometric tools.

In most applications of **singularity theory** to geometry, **computer vision and robotics**, one needs to classify singularities, determine their unfoldings and bifurcation sets, and the geometric form of the map in the components of the complements of the bifurcation sets. Such calculations quickly become too complicated to do by hand. The facilities that Geomview provides, e.g. rotation, rescaling, and its ease of us, have greatly aided these investigations. Indeed our package has highlighted mistakes in the literature, e.g. in results of Arnold concerning 1-parameter families of Lagrangian maps. They have also recently led to the discovery of new results (now proved rigorously) concerning the structure of asymptotic curves at flat umbilics, and the geometry of focal sets. It is indeed true that Geomview has provided us with stunningly beautiful pictures for which we offer no apology. They inform, entertain, and also inspire graduate students and researchers alike.

In conclusion, Geomview has been of immense benefit to our group. The geometric tools provided by Maple, Mathematica and other commercial packages are by comparison of poor quality, difficult to use, and of course expensive.

Visitors from Brazil and Japan have also been using Geomview.

Andrew Glassner, Xerox PARC (glassner@parc.xerox.com)

Hi! Thanks for the new version—I've just brought it over. I use Geomview to look at the results of new interactive modeling algorithms, currently designed around shape grammars and programmed developmental simulations. . . .

There is no question in my mind that Geomview has helped me in several important ways: I'm more productive (I don't have to write all that viewing code), my work is better (I can concentrate on my work, not the system support), and my results look better (I can tune the point of view, lighting, materials, and other parameters for the most effective display of my results).

I support Geomview and its continued development strongly.

Greg Griffin, CMU Physics Dept, Center for Astrophysical Research in Antarctica (ggriffin@cmbr.phys.cmu.edu)

Geomview is really marvelous. We are using it in conjunction with AutoCad to help visualize the design of a new telescope (we study microwave background radiation at the south pole).

Elisha Sacks, Computer Science Department, Princeton University (eps@Princeton.edu)

I use Geomview to animate mechanisms as part of a **CAD** project. Sample output appears in the paper: Sacks, Elisha and Joskowicz, Leo, *Automated Modeling and Kinematic Simulation of Mechanisms*, in **Computer-Aided Design**, Vol. 25 No. 2, 1993, pp 106–118.

Ioannis Z. Emiris, Computer Science Division, University of California, Berkeley (emiris@cs.Berkeley.edu)

My work concerns the solution of systems of sparse polynomial equations by means of the sparse resultant and their applications to **robotics, vision** and molecular **biology**.

Our algorithms. . . rely on computing the mixed subdivision of the respective Newton polytopes. We have implemented an algorithm to compute the mixed subdivision; its output format is compatible with Geomview, which helps visualizing the interaction of polytopes that defines "mixedness". The partial Mixed Volumes are also available from this subdivision and this information can be encoded by coloring the different cells accordingly.

Wilfried Trump, Institut für Betriebs- und Dialogsysteme, Universität Karlsruhe, Germany (trump@ira.uka.de)

We are working in Computer Aided Geometric Design and **Computational Geometry**. We are using Geomview to visualize results in research and educational projects. Geomview increased our productivity because it decreases the efforts for visualization of our results very much. Without Geomview our students had to write their own visualization software. It's a really nice and useful tool. The X version of Geomview is a very useful progress for us because we have only a few Iris but a lot of Suns.

Jim Tegart, Martin Marietta Corporation, Denver CO (jtegart@den.mmc.com)

I am using **Surface Evolver** and Geomview (just got the new version today). Evolver is being used to calculate low gravity gas-liquid interface shapes in spacecraft propellant tanks. My only use of Geomview thus far is the display of those interfaces. The current application is the propellant tank for the Cassini spacecraft for NASA Jet Propulsion Lab, that will go to Saturn. Evolver has been essential to verifying that this tank will function as designed. And Geomview makes a very impressive presentation of the results from Evolver.

Hartmut Chodura, Technische Hochschule, Darmstadt, Germany (cultur@nlp.physik.th-darmstadt.de)

I am a student in **physics** and at the moment I'm working on my Diplom graduation at Technische Hochschule Darmstadt (30 km south to Frankfurt)

My job is to visualize 3D Data which comes from scanned holographic plates. These plates contain 3D pictures of cavitation bubble fields. Cavitation is the phenomenon of gas bubbles in fluids and are produced for instance by an ultrasonic fields in water. We got series of these pictures and I'm searching the correlation in time in these fields. For the visualization I am using Geomview because of an easy to use interface and easy handling with external modules. My machine is an SGI-Indy and the speed of your program is fast enough to show up to 500 shaded bubbles (which are bounced by a cube).

Colin Cryer, Institut für Numerische Mathematik, Muenster, Germany (cryer@goedel.uni-muenster.de)

I have been using Geomview to view a model of the heart, which I had plotted using **Mathematica**.

I found Geomview excellent, and was particularly pleased that it could be installed without the slightest difficulty.

Software Distributed with Geomview

The Geomview distribution includes the following software modules (the platform column refers to availability on SGI, NeXTSTEP, and **X** versions):

Module	Platforms	Description/Author
4dview	S	4-dimensional slicing & rotation; Daeron Meyer, apprentice
CellularAutomata	S	Cellular automata animation; Stephanie Mason, apprentice
animate	SNX	Flip through a sequence of objects; Daeron Meyer, apprentice
clipboard	S	Cut, copy and paste geometric objects; Daeron Meyer, apprentice
corners	S	Create vector skeleton of object; Celeste Fowler, apprentice
crayola	SN	Interactively color objects; Celeste Fowler, apprentice
drawbdy	SNX	Compute and draw the boundary of an object; Celeste Fowler, apprentice

Module	Platforms	Description/Author
flythrough	S	Interactive version of "Not Knot" hyperbolic flythrough; Charlie Gunn, staff, Tamara Munzner, staff
ginsu	S	Interactively slice objects; Daeron Meyer, apprentice
graffiti	SN	Draw line segments on objects; Mark Phillips, staff
gvclock	SNX	3D clock, demonstrates real-time motion; Celeste Fowler, apprentice
hinge	S	Hinge copies of a polyhedron around its edges; Mark Phillips, staff
maniview	S	3-manifold viewer; Charlie Gunn, staff
NDview	S	n-dimensional viewing controls and demonstration; Olaf Holt, apprentice, Stuart Levy, staff
NDdemo	S	n-dimensional viewing demonstration; Olaf Holt, apprentice
nose	SN	Demonstrates picking; Mark Phillips, staff
pssnap	SNX	Generate PostScript snapshot; Celeste Fowler, apprentice
stereo	S	Hardware, crosseyed, red/cyan stereo (beta version); Stuart Levy, staff
sweep	SN	Generate objects of rotation from line segments; Celeste Fowler, apprentice
tackdown	S	Redefine an object's "home" position; Celeste Fowler, apprentice
transformer	S	Explicitly control an object's transformation matrix; Celeste Fowler, apprentice
trigrp	S	Explore triangle symmetry groups; Charlie Gunn, staff
warp	SN	Interactively deform an object; Celeste Fowler, apprentice

The Geomview distribution includes the following auxiliary programs:

Module	Platforms	Description/Author
anytooff	SNX	Convert any OOGL object into OFF format; Celeste Fowler, apprentice
bdy	SNX	Compute the boundary edges of a geom as a VECT; Celeste Fowler, apprentice
geomstuff	SX	Pipe your program's OOGL data to geomview via a pipe; Stuart Levy, staff
math2oogl	SNX	Convert Mathematica graphics object to OOGL format; Tamara Munzner, staff, Stuart Levy, staff, Nils McCarthy, student
offconsol	SNX	Consolidate duplicate vertices in an OFF file; Celeste Fowler, apprentice
oogl2rib	SNX	Convert OOGL to RenderMan RIB; Scott Wisdom, student, Tamara Munzner, staff
togeomview	SNX	Pipe GCL commands or geometry to a copy of geomview, invoking geomview if necessary; Stuart Levy, staff

Pisces: A Platform for Implicit Surface Computation and Exploration of Singularities

Project Description and Goals. Pisces is a project that is investigating numerical solutions of the following mathematical problem: Given a function $f\colon \mathbf{R}^{n+k} \to \mathbf{R}^n$ and $c \in \text{Range}(f)$, find the set of points mapped onto c by f. For a simple example, note that the function $f\colon (x, y) \to x^2 + y^2 - 1$ maps the unit circle to 0, so that the function f and the value 0 implicitly define the unit circle. One tangible result of the Pisces project is a software package that provides a variety of state-of-the-art algorithms for computing implicitly defined curves and surfaces. This software is under development (see Current Status, page 141).

When the Jacobian of f is full rank at every point in the preimage of c, then the Implicit Function Theorem guarantees that the preimage is a manifold that is as smooth as f. The literature on this problem is extensive, but there are no publicly available programs that can compute the preimage for general functions. (For functions from $\mathbf{R}^{n+1} \to \mathbf{R}^n$, AUTO, HOMPACK, PITCON and SCOUT can compute curves of preimages. For functions from $\mathbf{R}^2 \to \mathbf{R}$ or from $\mathbf{R}^3 \to \mathbf{R}$, both **Maple** and **Mathematica** have very simple algorithms that attempt to compute the implicitly defined curve or surface, or to compute a set of contours along a surface.)

When the Jacobian of f is not full rank, the inverse image is no longer a smooth manifold and most continuation algorithms break down or give incorrect results in a neighborhood of singular (or near-singular) points. We do not know of any publicly available software that correctly computes singular preimages, although codes written by A. Geisow and **R. Morris** take a first step in this direction for *polynomial* functions $\mathbf{R}^2 \to \mathbf{R}$ and $\mathbf{R}^3 \to \mathbf{R}$, respectively. Geisow's program is not publicly available; Morris is trying to market his code commercially.

The goal of this project is to create software that contains:

- a set of known continuation algorithms for computing implicitly defined curves and surfaces. This will include both piecewise-polynomial methods and also predictor-corrector methods;
- a command language for batch processing and high level programming;
- a graphical interface for choosing and controlling algorithms;
- new algorithms for computing the structure of singular surfaces.

In addition to being a useful tool to researchers and educators who need to compute implicitly defined surfaces, the Pisces program will serve as an environment that can be used to study the way that algorithms fail and to develop algorithms that perform better.

Potential Users. Pisces will be potentially useful to:

- researchers in dynamics (bifurcation surfaces, invariant manifolds), differential geometry (surfaces, curves, singularities), algebraic geometry (real and complex algebraic varieties), differential equations (DEs on manifolds, generation of finite element meshes), mathematical physics (equipotentials, caustics), mathematical biology (nonlinear models in physiology, epidemiology, neurology), etc.;

- educators in ODEs, dynamics (level sets, elementary bifurcations, normal forms), differential geometry (surfaces, curves, envelopes), algebraic geometry (real algebraic varieties);

In addition, we foresee industrial applications in, for example, computer-aided geometric design, and the study of solution manifolds for families of nonlinear equations that include parameters.

Current Status. Researchers at the Geometry Center have contacted several leading researchers in the field of **continuation theory** (the name given to computing implicit surfaces) and **singularity theory**. These researchers have agreed that there is a need for "a robust, portable set of tools, with a friendly interface that can be used interactively or in the background. . . " (R. Mejia, NIH). Lengthy discussions with **J. Guckenheimer** (Cornell) and **E. Doedel** (Concordia University) have indicated a need for a tool that can compute singular structures.

Structurally, the "platform" nature of Pisces is developing. It is straightforward for users to add their own functions to the current library of functions that define implicit surfaces and curves. The code contains a "data manager" (adapted from DsTool, written at Cornell University) so that algorithms can be added to the core code in a modular fashion. To date, we have incorporated three algorithms into the code, and additional algorithms are under development, including a new adaptive mesh algorithm designed by Center postdoc **Davide Cervone**. One of the algorithms (by **Silvio Levy** and **Allan Wilks**; coded by Levy) can compute k-dimensional surfaces in n-dimensional space.

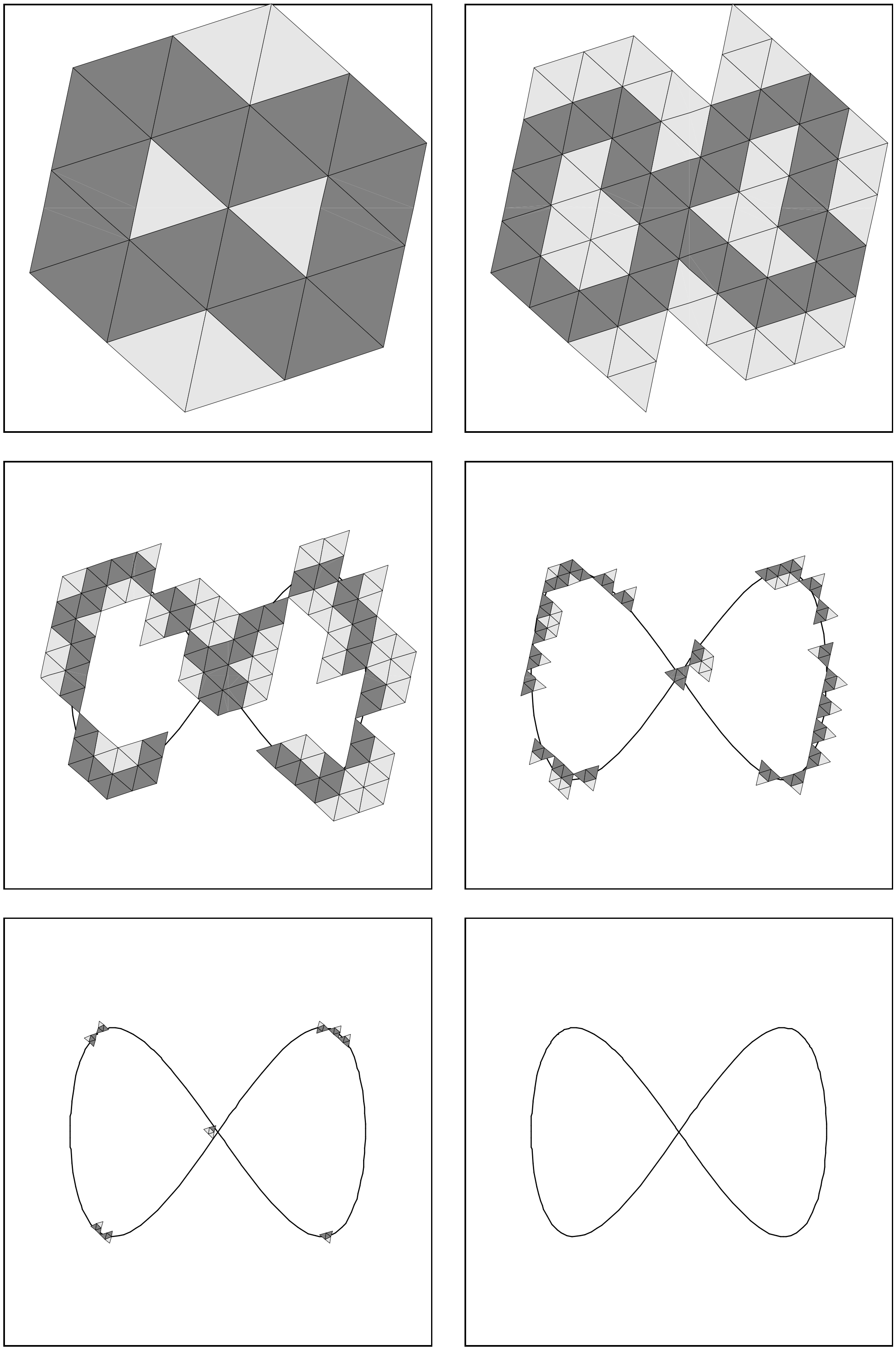
We have recently begun work on implementing a command language (using Tcl) and a graphical user interface (using Tk).

The Pisces project is being directed by **Richard McGehee**; Center postdoc **Frederick Wicklin** is the team leader. Although still in its infancy, Pisces has already proved useful to researchers. **Mark Phillips** has rewritten portions of Levy's code and is making it available for public distribution via anonymous **ftp** from the Geometry Center. Researchers not at the Geometry Center have used Levy's code to study complex algebraic varieties in $\mathbf{C}^2$. Wicklin (together with **Arlie Petters**, Princeton) has used Pisces to compute caustics for equations that model gravitational lensing. The numerical experiments suggest the existence of previously unsuspected caustics when there are multiple lenses between the source and the observer. Wicklin has also successfully computed bifurcation surfaces known as "resonance regions" that are important in understanding the dynamics of coupled oscillators.

Examples of Pisces algorithms. The following pages give examples of output from Pisces and compare them to other standard nonadaptive methods of computing implicit curves and surfaces.

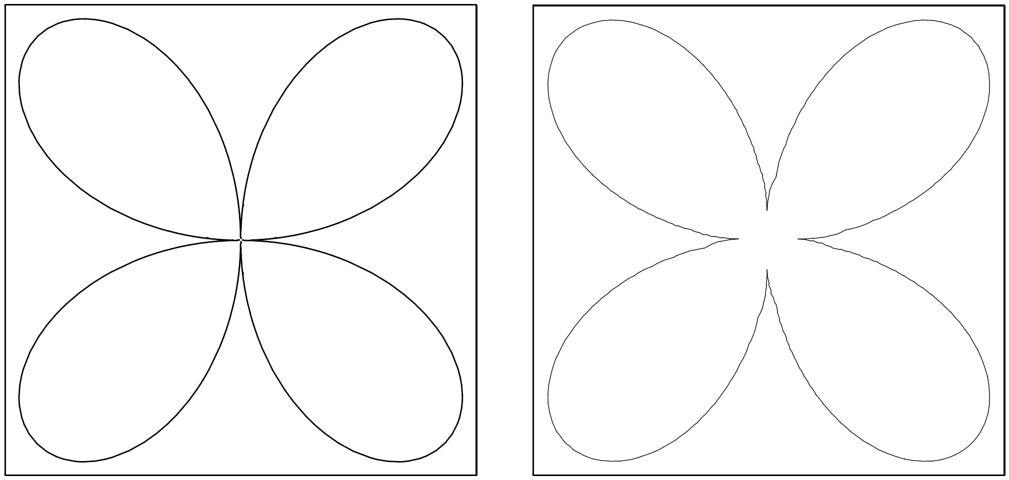
On page 142, we see successive iterations of an adaptive-mesh algorithm developed by postdoc Davide Cervone. At each step, some triangles are subdivided, some are discarded as not likely to contain solutions, and others are replaced by linear approximations to the level set. The singularity in this example is detected and properly represented by the algorithm.

On page 143, the output of the adaptive mesh algorithm is compared to the output generated by Maple, for three algebraic functions. (Mathematica produces results similar to those of Maple.) The number of subdivisions for the Maple runs was

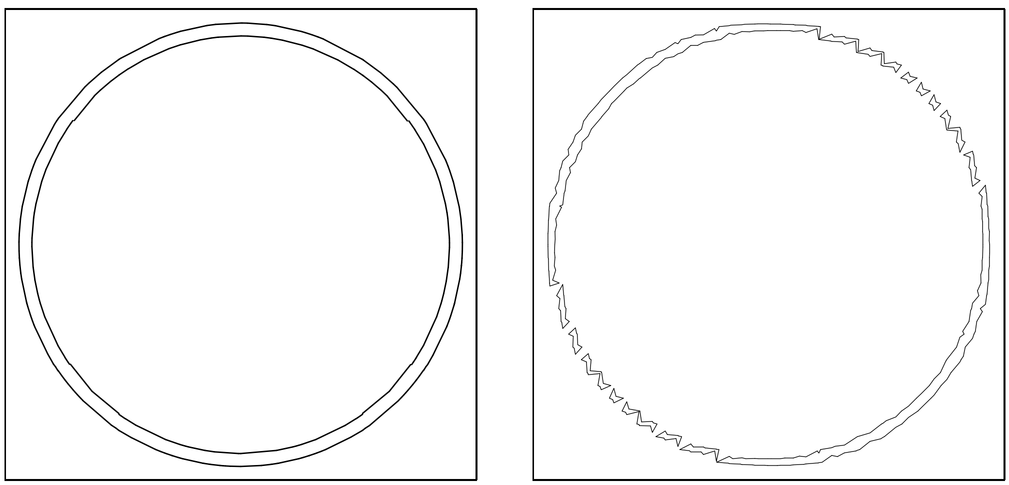


$$-x^4 + 2x^2 - y^2 = 0$$

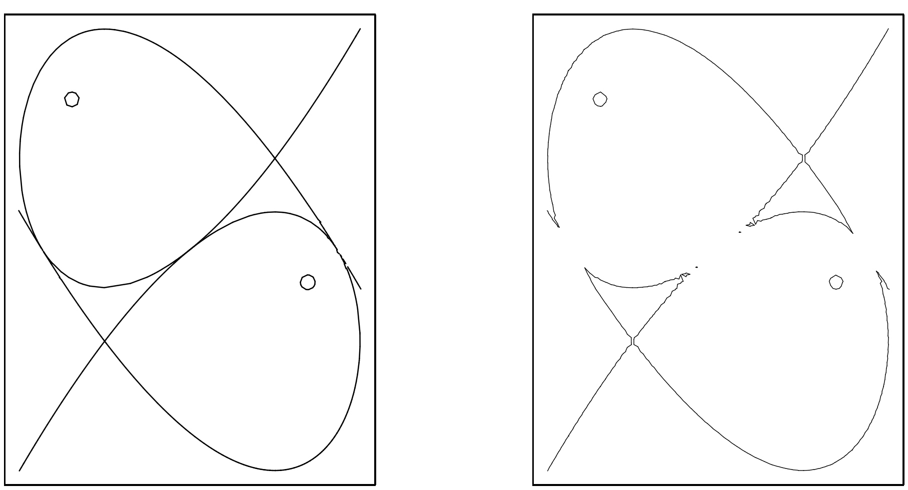
Adaptive Mesh Algorithm Standard Algorithm (Maple)



$$(x^2 + y^2)^3 - 4x^2y^2 = 0$$

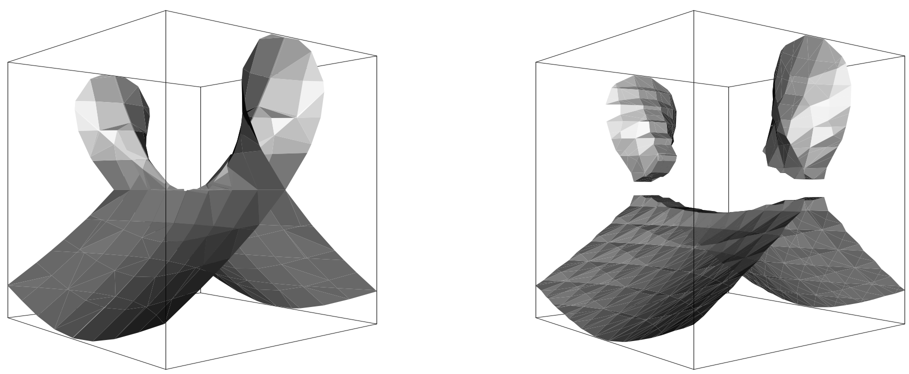


$$(x^2 + y^2 - 1)^2 - .0036 = 0$$



An exercise for the reader (Hint: it's a tenth-degree polynomial)

Adaptive Mesh Algorithm (Morris) Standard Algorithm (Maple)



chosen so that the total number of function evaluations made by each algorithm would be approximately the same for each example. These comparisons show the typical behavior of the standard algorithms in the neighborhoods of singularities or multiple solutions that are close together, and shows that the adaptive-mesh algorithm can handle these situations in a more robust fashion.

Finally, the figure at the top of this page compares an adaptive-mesh algorithm due to **R. Morris** to the standard algorithm used by Maple in the case of a singular algebraic surface. Note that the adaptive-mesh algorithm generates the correct topology for the surface, and handles the singularity well.

Future Directions. The three algorithms currently in Pisces are piecewise-linear methods. We will soon be implementing examples of the second major class of continuation algorithms, predictor-corrector methods. Although programs like AUTO and PITCON implement predictor-corrector algorithms for functions from $\mathbf{R}^{n+1} \to \mathbf{R}^n$, it is an area of current research to develop robust algorithms for functions from $\mathbf{R}^{n+k} \to \mathbf{R}^n$, $k > 1$. Algorithms such as W. C. Rheinboldt's "moving frame" algorithm can compute small patches of k-dimensional surfaces in n-space, but more research is needed in order to create predictor-corrector algorithms that can correctly compute a simplicial approximation to compact surfaces.

We also intend to start looking closely at the behavior of algorithms in the neighborhood of singularities and develop algorithms that behave more robustly in those neighborhoods. To help us in this difficult task, the Geometry Center intends to invite an expert in singularity theory to be a long-term **visitor** at the Center during the 1994–95 academic year. We are also looking into the possibility of holding a **workshop** on continuation methods.

In the realm of programming, there are numerous ways to make Pisces more useful, including implementing additional algorithms, coding utilities to manipulate, query, and filter previously computed data, allowing animation of implicit surfaces as a parameter is varied, and porting Pisces to other platforms, including **personal computers.**

W3Kit

The **World-Wide Web** (WWW) global hypertext system offers an appealing platform for the distribution of mathematical software applications (see page 29 for more information on WWW).

Developing a graphical interface for a WWW application from scratch is a complex task: the Web is a highly unusual software environment, in which application programs do not run continuously during interactive sessions with the user, moreover, there is no direct support for high-level 2D or 3D graphics, only for simple bitmap images.

The **W3Kit toolkit**, designed by Center postdoc **Paul Burchard**, simplifies the interface development process for Web applications involving 2D and 3D graphics. It provides both the user and developer with the illusion that they are dealing with an ordinary, self-contained, continuously running graphical application that has powerful 2D and 3D graphics capabilities at its disposal. W3Kit accomplishes this by acting as "software glue" which ties together the operation of a Web client, Web server, display server, and 2D and 3D graphics libraries.

The 3D graphics technology in W3Kit is the same as that in **Geomview**, since it is based on the **OOGL** libraries that underlie Geomview; this increases the return on the Center's original development efforts. In particular, W3Kit takes essential advantage of the wide portability which has recently been achieved for Geomview.

W3Kit has been employed in the development of WWW interfaces for at least six different mathematical applications originally written by a variety of authors associated with the Center. The initial development of the toolkit and applications was accomplished in approximately three person-months. W3Kit is portable to a variety of server platforms, and has now been tested under Solaris, SunOS, Irix, and NeXTSTEP.

As with nearly all the software developed at the Geometry Center, W3Kit is available for free to the world network community. It is already becoming known to the Web software developers, and with the success of our own Web applications, we expect to see a rising interest in the W3Kit toolkit.

Cweb and literate programming

The expression **literate programming** was introduced by **Donald Knuth** to designate a type of programming where documentation and code are intertwined in the same source file, and evolve in parallel. In 1987, using Knuth's Web system as a basis, Silvio Levy created **Cweb**, a literate programming tool for C. Cweb was adopted by many C programmers worldwide—for example, the "automata" programs, developed by **David Epstein** and his collaborators under Center sponsorship, are written in Cweb. Knuth subsequently adopted Cweb as his language of choice, and has been using it for such projects as the forthcoming volumes of *The Art of Computer Programming* and the extensive Stanford GraphBase library. In 1992, Levy and Knuth revised Cweb to provide support for the language's widely increasing user base. Their resulting program, Cweb 3, is described in [Levy and Knuth 1993].

Software Distributed by the Center

`geomview` Retreivals: 7680
Stuart Levy, Tamara Munzner, Mark Phillips, staff, et al
Description: See page 133.

`evolver` Retreivals: 2460
Ken Brakke, Susquehanna University, Center supported
Description: See page 105.

`qhull` Retreivals: 894
Brad Barber, postdoc, and *Hannu Huhdanpaa*, grad student
Description: A general-dimension convex hull and Delaunay triangulation computations program. See page 108.

`geom_utils` Retreivals: 631
Mark Meyer, student, *Mark Phillips*, staff, *Nina Amenta*, postdoc
Description: This package contains the following geometric utility programs:
poly2tri – break arbitrary polygons into triangles
poly2area – compute polygon areas
poly2xy – transform polygons in 3-space to lie on the xy-plane
arc2list – converts arcs into polylines

`geom.sty` Retreivals: 586
Silvio Levy, staff
Description: This style package for LaTeXprovides the following features, among others:
· inclusion of PostScript figures, and of TeX text within figures
· automatic creation of index entries and cross-references where appropriate
· no need to worry about fragile commands in most situations
· greater versatility in defining theorem-like environments
· proofing aids such as version numbers and a running index

`Snappea` Retreivals: 517
Jeff Weeks, Center supported
Description: See page 96.

`hyperbolic` Retreivals: 374
Oliver Goodman, postdoc
Description: A Mathematica package for hyperbolic geometry computations. See page 95.

`4DSlicer` Retreivals: 309
Adam Halvorsen, summer student 1991
Description: A Mathematica package designed to shed some light on a world of higher-dimensional objects.

`Linktool` Retreivals: 297
David Broman, summer student 1990–92
Description: A NeXT application for manipulation of knots, links, and braids. Allows the user to enter links with the mouse or via Conway notation and redisplay links and tangles, smoothed out with splined curves and with rope-texture.

`automata` Retreivals: 270
David Epstein, Derek Holt, Sarah Rees, University of Warwick
Description: Version 2.0 (November 1990) of the automatic groups programs.

`mathfig` Retreivals: 258
Silvio Levy, staff
Description: A system to include Mathematica files in TeX and LaTeX documents.

`IconBuilderScaleFilter` Retreivals: 234
Linus Upson, apprentice
Description: A loadable filter for scaling bitmaps in IconBuilder.app, one of the standard developer applications bundled with NeXTSTEP 3.0.

`MinneView` Retreivals: 191
Pat Hanrahan, Charlie Gunn, Stuart Levy, Tamara Munzner, et al
Description: The precursor of Geomview (page 133). Still distributed for backward compatibility.

`mppt` Retreivals: 185
Arek Goetz, summer student 1991
Description: A Mathematica package for the transformation of polygons and curves using the Midpoint Polygon Transformation. The package might be a useful tool for various animations, such as the changing of shape of polygons, curves and surfaces to an ellipse, knots unraveling, ordered movement of particles, etc.

`kali` Retreivals: 180
Nina Amenta, now postdoc, and *Tamara Munzner*, staff
Description: Interactive 2D Euclidean symmetry pattern editor for SGI IRISes.

`W3Kit` Retreivals: 164
Paul Burchard, postdoc
Description: A system for building interactive graphical applications for the World-Wide Web. See pages 29 and 145.

`RiTrainer` Retreivals: 160
Linus Upson, apprentice
Description: A small application that allows you to type in RIB commands and instantly have them rendered (with Quick RenderMan).

`DynamicKit` Retreivals: 155
Paul Burchard, postdoc
Description: An Interface Builder palette and Objective-C class library for making highly interactive mathematical applications.

`circlepack` Retreivals: 154
Oliver Goodman, postdoc
Description: A collection of Mathematica packages to calculate and display the circle packing corresponding to a given triangulation of a closed orientable surface. It will do this in whichever geometry is appropriate depending on the genus of the surface and can compute packings with specified overlaps as well as the special case in which all circles are tangential.

`envelopes` Retreivals: 148
Adrian Mariano, summer student 1991
Description: Mathematica packages for plotting parallel curves, evolutes and caustics.

`geometry_games` Retreivals: 138
Jeff Weeks, Center supported
Description: A collection of geometry games (torus chess, 3-manifold flight simulator, hyperbolic MacDraw & hypercube) for the Macintosh.

`SphereDraw` Retreivals: 122
Paul Burchard, postdoc
Description: A simple drawing program on the surface of a sphere. Demonstrates effects of curved space. Supports copy/paste, dragging, ASCII file read/write. (See also spherescribble below. SphereDraw and spherescribble are two different programs by different authors; neither is a port of the other.)

`CRSolver` Retreivals: 117
Paul Burchard, postdoc
Description: An interactive application for experimenting with conformal mapping, complex analytic functions, and meromorphic sections of line bundles on Riemann surfaces of genus one. Uses novel relaxation algorithms. An example of what can be done with DynamicKit. Links with Geomview.

`polycut` Retreivals: 102
Ken Brakke, Susquehanna University, Center supported
Description: A program for visualizing covering spaces of 3D Euclidean space from the inside. Branch curves are various loops and knots. The aim is to illustrate the author's contention that soap films are best viewed as minimal area cuts in covering spaces.

`ODE` Retreivals: 95
Paul Burchard, postdoc
Description: An interactive flow viewer for ordinary differential equations. Allows symbolic entry of the differential equations as well as the projection map from dynamic space to screen space. Supports all DynamicKit interactive features.

`dumppkg` Retreivals: 78
written or modified by Stuart Levy, staff
Description: This is a file system dump package intended for tapes (e.g. Exabytes) which are typically much larger than the file systems needing to be dumped.

`QuasiTiler` Retreivals: 74
Eugenio Durand, grad student
Description: An interactive application for experimenting with Penrose tilings and more. The approach to Penrose tilings presented by this program was developed by N. G. deBruijn.

`spherescribble` Retreivals: 35
Millie Niss, summer student 1993
Description: Another interactive program for drawing on the surface of a sphere; this one for SGI workstations. (See also SphereDraw above. SphereDraw and spherescribble are two different programs by different authors; neither is a port of the other.)

Documentation: Management

Center Management

As noted in the introduction, there have been major changes in the management of the Center. The supervisory structure has changed, as has the governance structure. A new system for promoting the integration of projects is beginning. A new budgetary process provides monthly monitoring of expenditures and budget projections, and gives the Board of Governors the budgetary information necessary to make long range decisions. The financial and programmatic support of the University of Minnesota has increased.

Supervisory and Governance Structure. The supervisory structure has fewer individuals reporting directly to the Center Director, as reflected in the top chart on page 154. The **Director of Technology** systematically monitors the activities of the **technical staff** so that staff effort on projects can be accounted for. Each **apprentice** is assigned to a specific senior staff person to act as a mentor and a supervisor. These modifications have provided for a more efficient operation and for a more effective accounting system to provide the information necessary to make decisions involving the allocation of Center resources.

The bottom chart on page 154 summarizes the new governance structure. This structure is described in detail in the following section, page 156. The two main components of the governance structure are the External Advisory Board and the Board of Governors. The membership of these bodies follows.

External Advisory Board

Forest Baskett, Silicon Graphics
Hyman Bass, Columbia University
James Blinn, California Institute of Technology
Fred Gehring, University of Michigan (Chair)
John Guckenheimer, Cornell University (Vice Chair)
Deborah Hughes Hallett, University of Arizona
Maria Klawe, University of British Columbia
Jill Mesirov, Thinking Machines
Alan Schoenfeld, University of California, Berkeley

Board of Governors

David Dobkin, Princeton University (Chair)
David Epstein, University of Warwick (Vice Chair)
John Franks, Northwestern University
Bill Goldman, University of Maryland
Nelson Max, University of California, Davis
Marjorie Senechal, Smith College
James Sethian, University of California, Berkeley
Jean Taylor, Rutgers University
Bill Thurston, University of California, Berkeley
Allan Wilks, AT&T Bell Labs

Integration of Projects. The **Pisces** project described on page 140 serves as an example of the way many projects will proceed in the future. It was initiated by **Richard**

McGehee (who was at the time a member of the **Center faculty**) and **Rick Wicklin**, a Center **postdoc**. Next, **David Cervone**, another postdoc, **Mark Phillips**, a member of the **senior technical staff**, and **Daniel Krech**, a student programmer, became involved. Phillips began adapting a program by **Silvio Levy**, the Director of Technology, based on a generalization by Levy and **Allan Wilks**, a Center Faculty member, of an algorithm by Dobkin, Thurston, Levy, and Wilks (*ACM Transactions on Graphics*, October 1990). **Joel Roberts**, a faculty member of the School of Mathematics, has become interested in the project. **Audun Holme**, a mathematician from the University of Bergen, Norway, is visiting the Geometry Center partly to provide advice on the project. **Ioannis Emiris** has been offered a postdoctoral position at the Center based partly on his interest in the project. We plan to organize a workshop at the Center with the purpose of assembling individuals interested in the subject, both from the viewpoint of computational algorithms and from the viewpoint of users of the software. More Center visitors will be invited specifically because of their interest in the project. The overall goals are to provide a platform on which researchers can develop algorithms for the computation of implicitly defined curves and surfaces and for the detection and resolution of singularities, and to produce a widely usable and widely distributed software package for general use in the research and education community.

This project will provide a model for the design of future projects. It shows how software projects can be combined with research projects and with the various Center programs: the postdoc program, the visitor program, Center workshops, and student training. Of course, many projects in the past had many elements of this integration, but a consciousness of the advantages to combining the many resources available through the Geometry Center will play a larger role in the future.

Budget. A summary of the 1994 Center budget is shown on page 155. The complete budget is available but is not shown here; it can easily be broken into recurring and nonrecurring expenditures, and shows the points of flexibility. When combined with the data kept on staff effort, the costs of various projects can be tracked, providing the **Board of Governors** with solid accounting information to be used when making budgetary and programmatic decisions.

Institutional Commitment. A summary of the financial commitments of the University of Minnesota to the Center is shown in the table on the next page. In addition to what is shown below, as a match to the original Center proposal in 1991, the University provided the **School of Mathematics** with sufficient funds to hire two new faculty members. The School has hired **John Sullivan** and **Ben Chow**, both of whom have been active in the Center, and funding for those positions is now a permanent part of the department budget.

The University also provided funding for the release from formal classroom duties of faculty members to participate in Center activities. This contribution is shown below in the rows labeled “Faculty Release”, in units of full-time equivalent faculty. Over the first three academic years covered by the grant so far, this release time has been spread among the following faculty from the School of Mathematics: **Ben Chow**, **Robert Gulliver**, **Richard McGehee**, **Johannes Nitsche**, **Peter Olver**, **Vic Reiner**, and **Peter Webb**. Their activities are described starting on page 58.

Programs Committee. Working in conjunction with the Center Director, the Board of Governors will appoint the Programs Committee. The Programs Committee will consist of mathematicians, scientists, and educators who will be involved actively in Center programs during a significant portion of their tenure on the committee. The Center Director will be a permanent member of this committee and will serve as the Chair. Other members of the committee will generally serve a term of two or three years, depending on their involvement in Center activities. Terms will be staggered to insure continuity of programs. Committee members may be members of the Board of Governors or may be selected from outside the Board.

The Programs Committee will appoint subcommittees responsible for specific activities. Subcommittee members are not necessarily members of the Programs Committee; in particular, members of the staff of the Geometry Center and faculty members from the University of Minnesota may be appropriate as subcommittee members.

It is expected that at least two subcommittees will be appointed annually: (1) the Postdoc Selection Subcommittee, which will be responsible for soliciting applications for **postdoctoral positions** at the Center and for selecting the candidates from among the pool of applicants, and (2) the Visitors Selection Subcommittee, which will be responsible for soliciting applications for Center **visitors** and for selecting the visitors from among the applicants. Both of these subcommittees will conduct their searches according to University of Minnesota guidelines.

It is anticipated that organizing committees for Center **workshops** will operate as subcommittees of the Programs Committee, as will the committees for the selection of student participants in **summer programs**.

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