University of British Columbia
CPSC 314 Computer Graphics
Jan-Apr 2013

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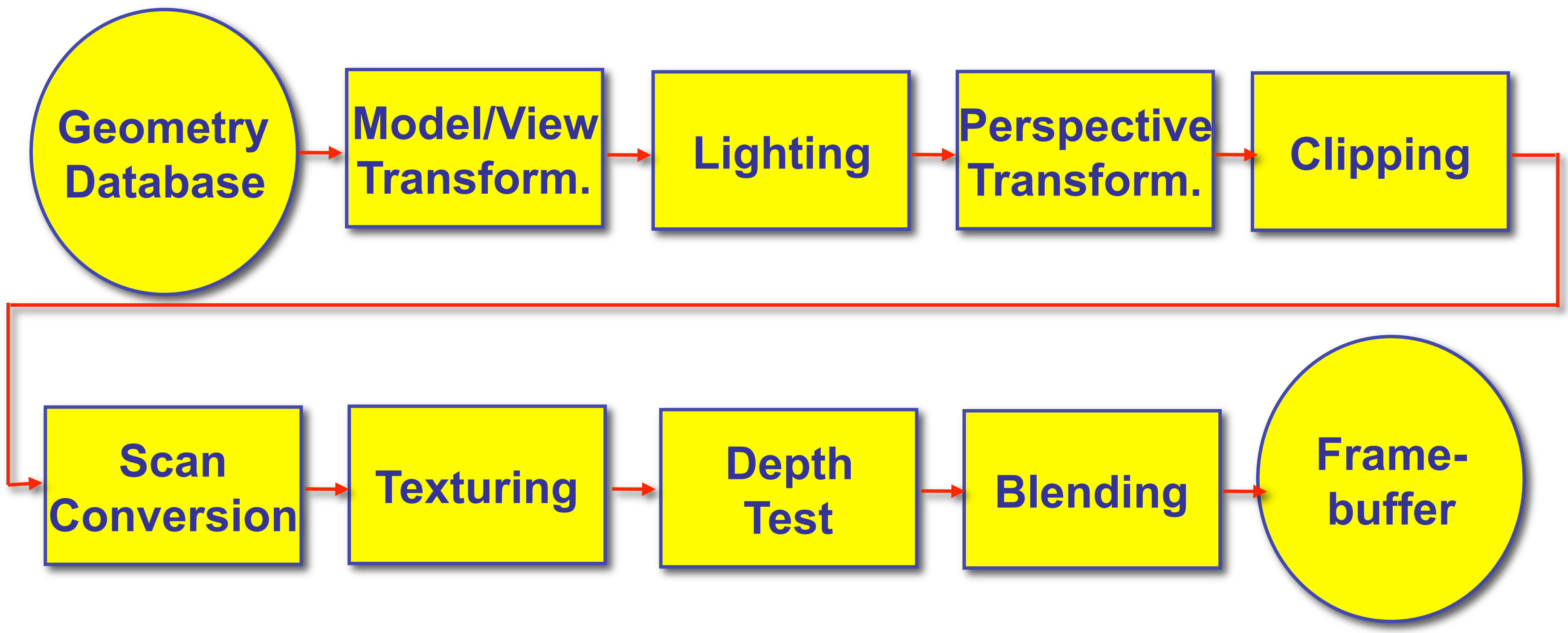
Transformations

http://www.ugrad.cs.ubc.ca/~cs314/Vjan2013

Review: Rendering Pipeline



Review: Graphics State

- set the state once, remains until overwritten
  - glColor3f(1.0, 1.0, 0.0) → set color to yellow
  - glSetClearColor(0.0, 0.0, 0.2) → dark blue bg
  - glEnable(LIGHT0) → turn on light
  - glEnable(GL_DEPTH_TEST) → hidden surf.

Review: Geometry Pipeline

- tell it how to interpret geometry
  - glBegin(<*mode of geometric primitives*>)
  - *mode* = GL_TRIANGLE, GL_POLYGON, etc.

- feed it vertices
  - glVertex3f(-1.0, 0.0, -1.0)
  - glVertex3f(1.0, 0.0, -1.0)
  - glVertex3f(0.0, 1.0, -1.0)

- tell it you're done
  - glEnd()

Review: GLUT: OpenGL Utility Toolkit

- simple, portable window manager
  - opening windows
    - handling graphics contexts
  - handling input with callbacks
    - keyboard, mouse, window reshape events
  - timing
    - idle processing, idle events
- designed for small/medium size applications

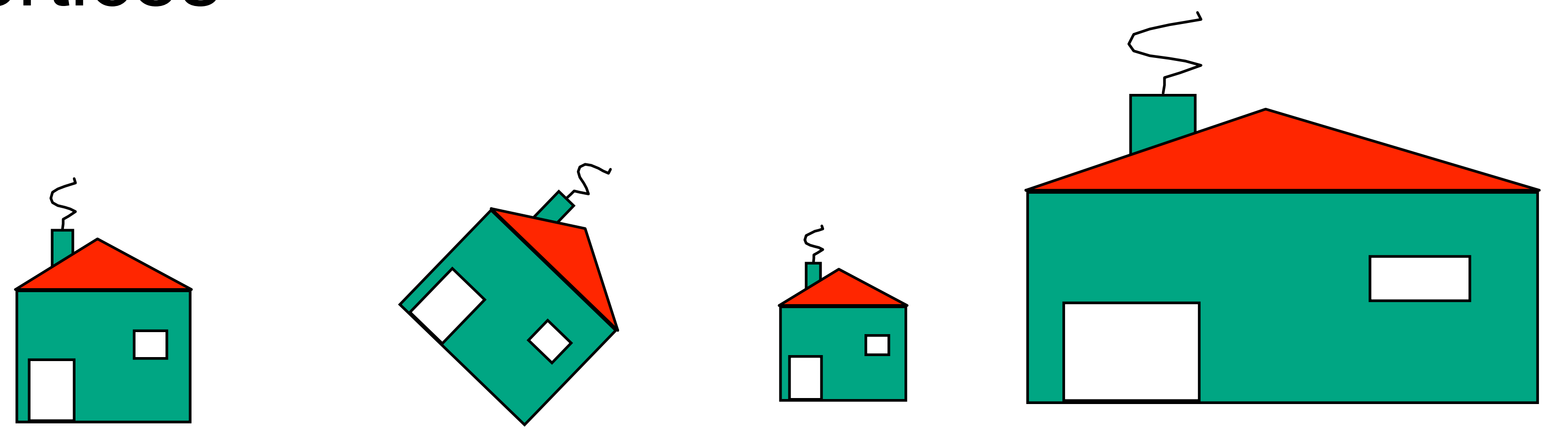
Readings for Transformations I-IV

- FCG Chap 6 Transformation Matrices
  - *except* 6.1.6, 6.3.1
- FCG Sect 13.3 Scene Graphs (3rd ed: 12.2)
- RB Chap Viewing
  - Viewing and Modeling Transforms *until* Viewing Transformations
  - Examples of Composing Several Transformations *through* Building an Articulated Robot Arm
- RB Appendix Homogeneous Coordinates and Transformation Matrices
  - *until* Perspective Projection
- RB Chap Display Lists

2D Transformations

Transformations

- transforming an object = transforming all its points
- transforming a polygon = transforming its vertices



Matrix Representation

- represent 2D transformation with matrix
  - multiply matrix by column vector $\Longleftrightarrow$ apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = ax + by$$
$$y' = cx + dy$$

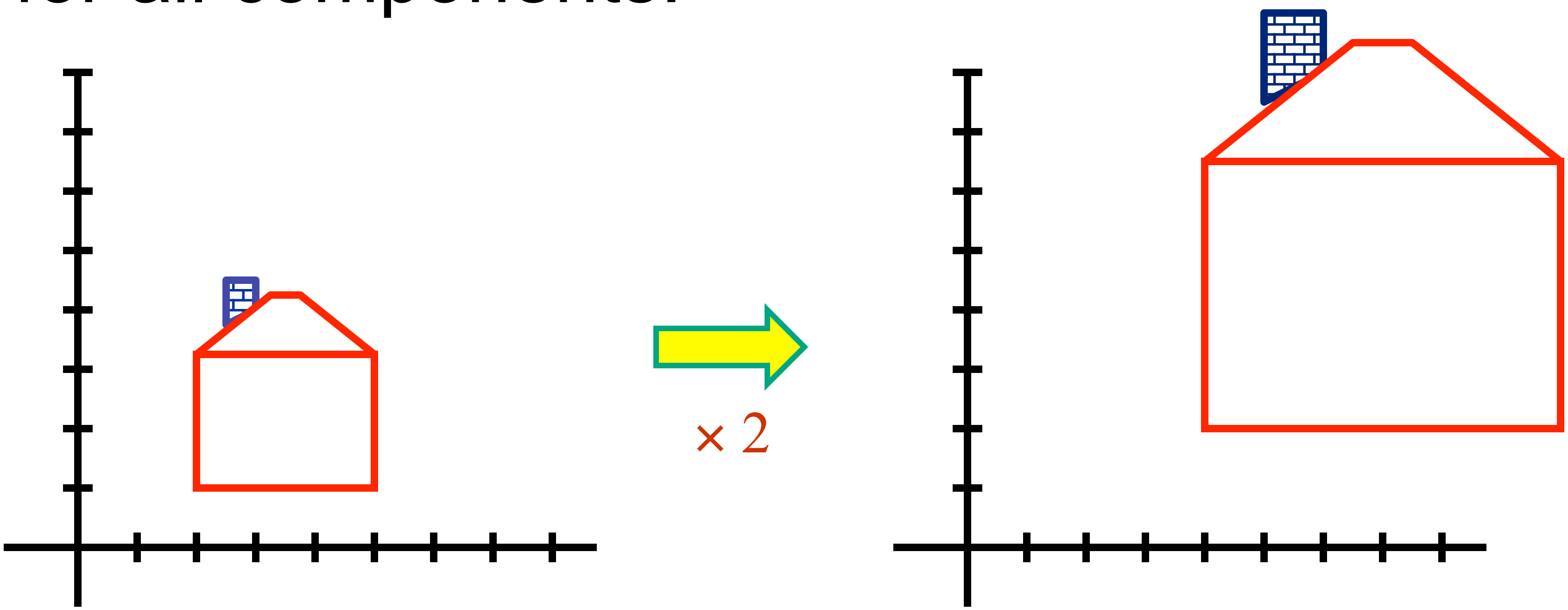
- transformations combined by multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & e \\ f & g \end{bmatrix} \begin{bmatrix} h & i \\ j & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

  - matrices are efficient, convenient way to represent sequence of transformations!

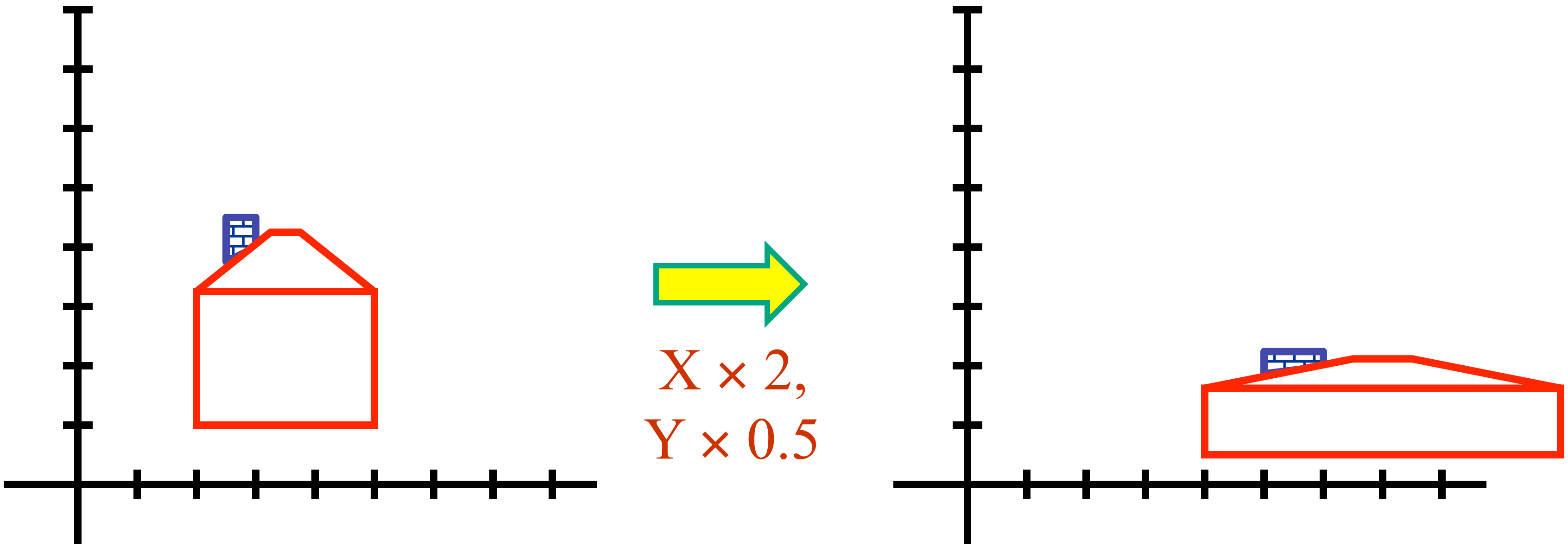
Scaling

- scaling a coordinate means multiplying each of its components by a scalar
- uniform scaling means this scalar is the same for all components:



Scaling

- non-uniform scaling: different scalars per component:



- how can we represent this in matrix form?

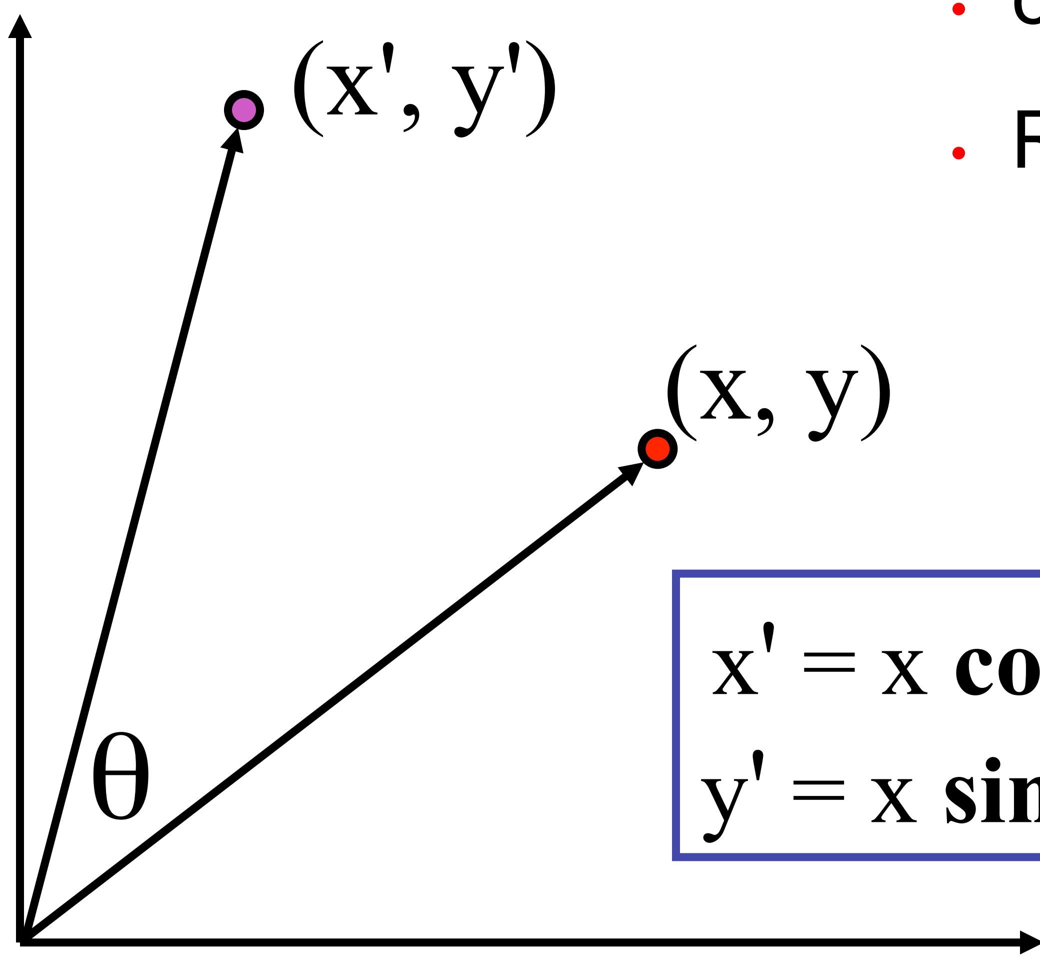
Scaling

- scaling operation: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$

- or, in matrix form: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\textit{scaling matrix}} \begin{bmatrix} x \\ y \end{bmatrix}$

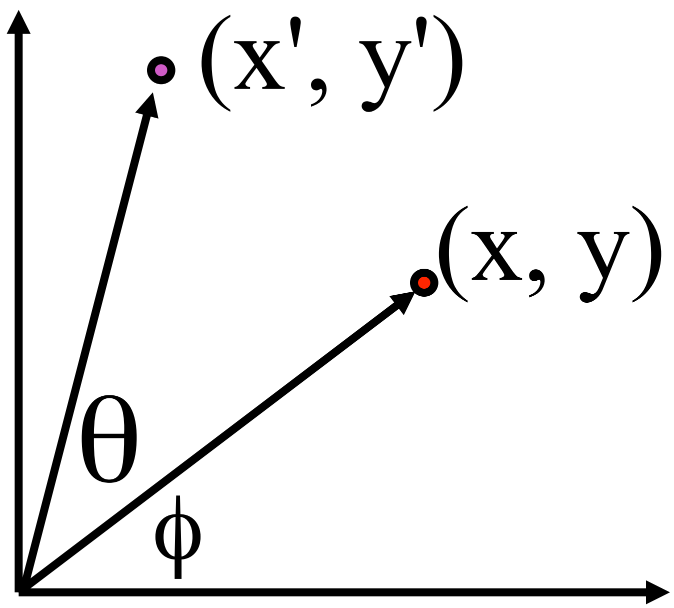
2D Rotation

- counterclockwise
- RHS



$$x' = x \cos(\theta) - y \sin(\theta)$$
$$y' = x \sin(\theta) + y \cos(\theta)$$

2D Rotation From Trig Identities



$x = r \cos(\phi)$

$y = r \sin(\phi)$

$x' = r \cos(\phi + \theta)$

$y' = r \sin(\phi + \theta)$

Trig Identity…

$x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)$

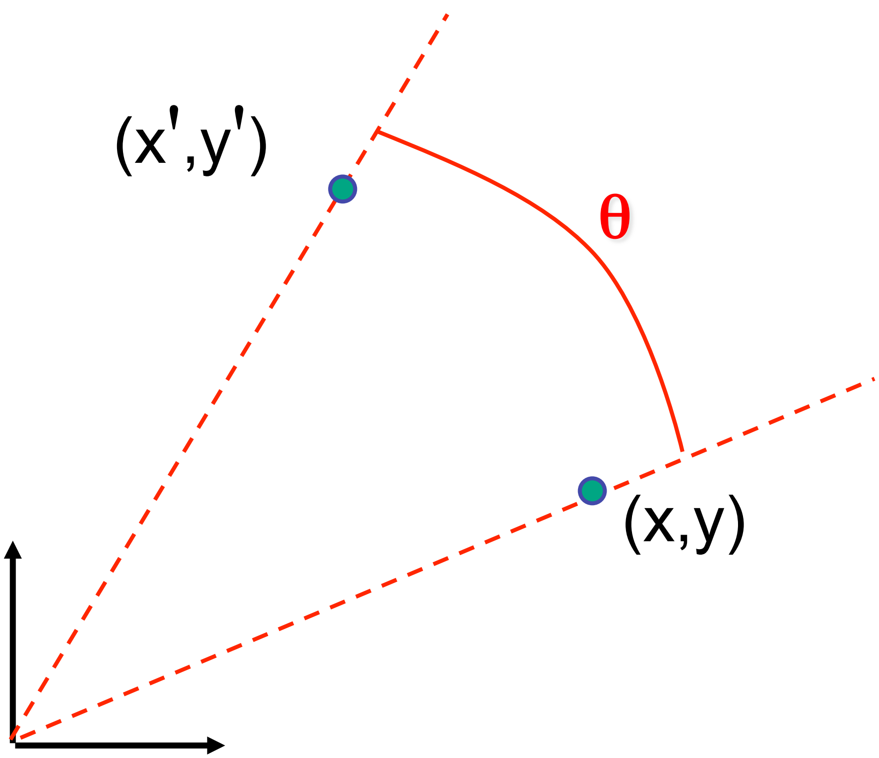
$y' = r \sin(\phi) \cos(\theta) + r \cos(\phi) \sin(\theta)$

Substitute…

$x' = x\, \mathbf{cos}(\theta) - y\, \mathbf{sin}(\theta)$

$y' = x\, \mathbf{sin}(\theta) + y\, \mathbf{cos}(\theta)$

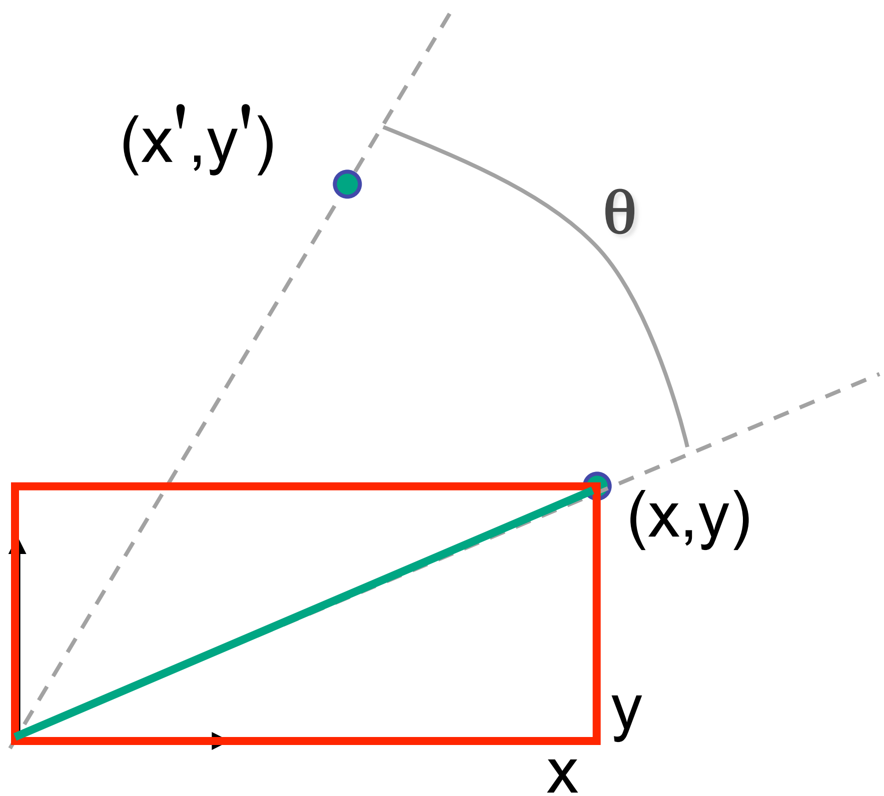
2D Rotation: Another Derivation



$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

2D Rotation: Another Derivation



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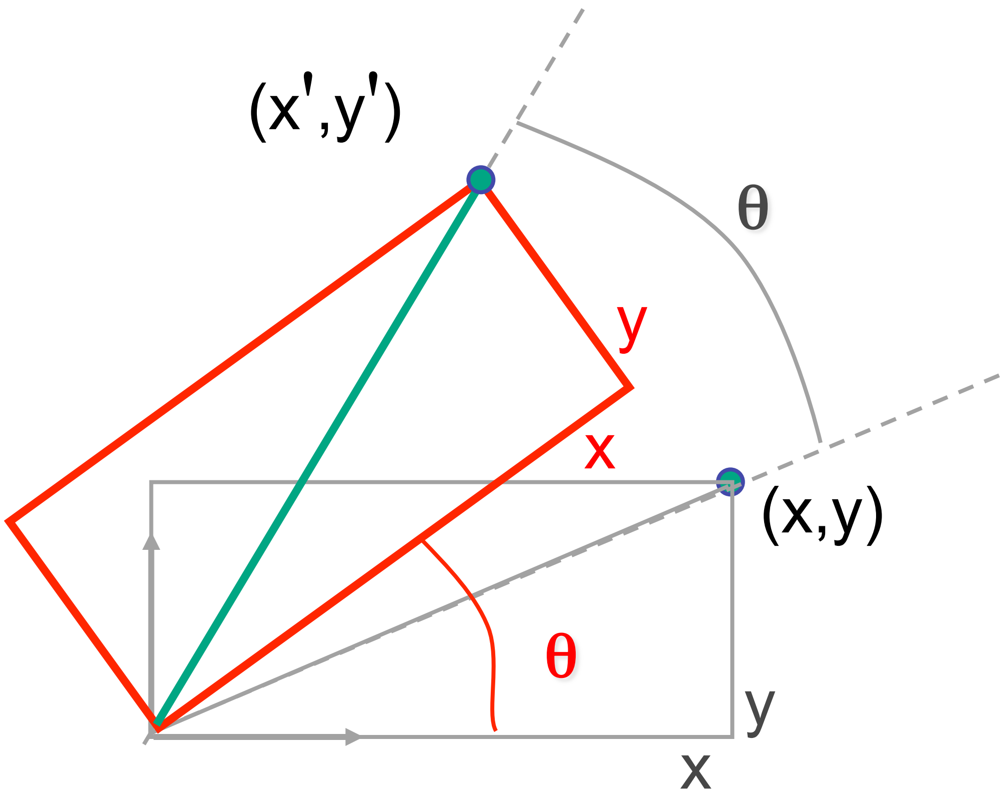
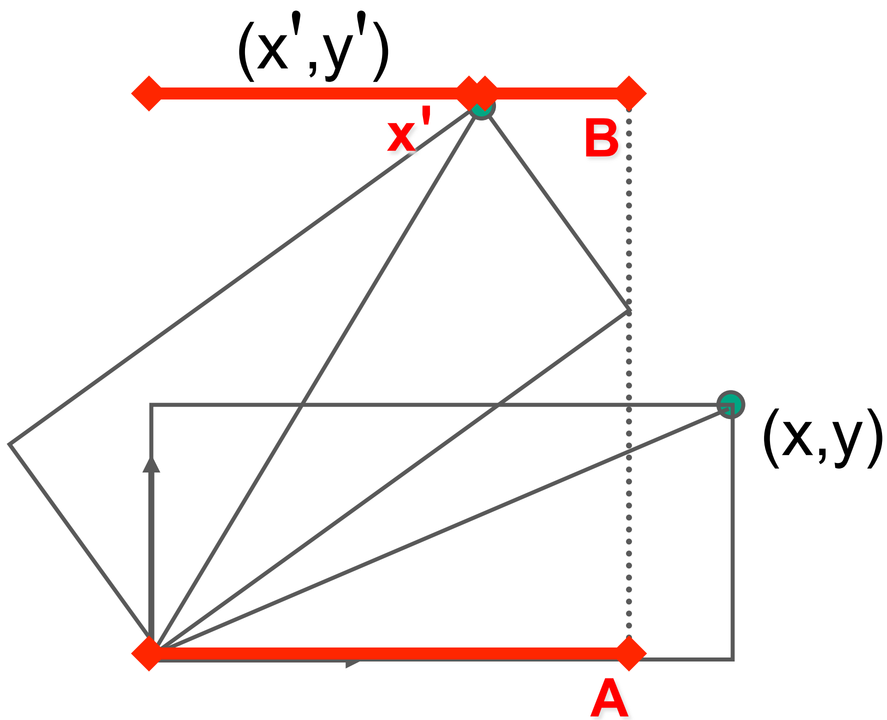


$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

2D Rotation: Another Derivation

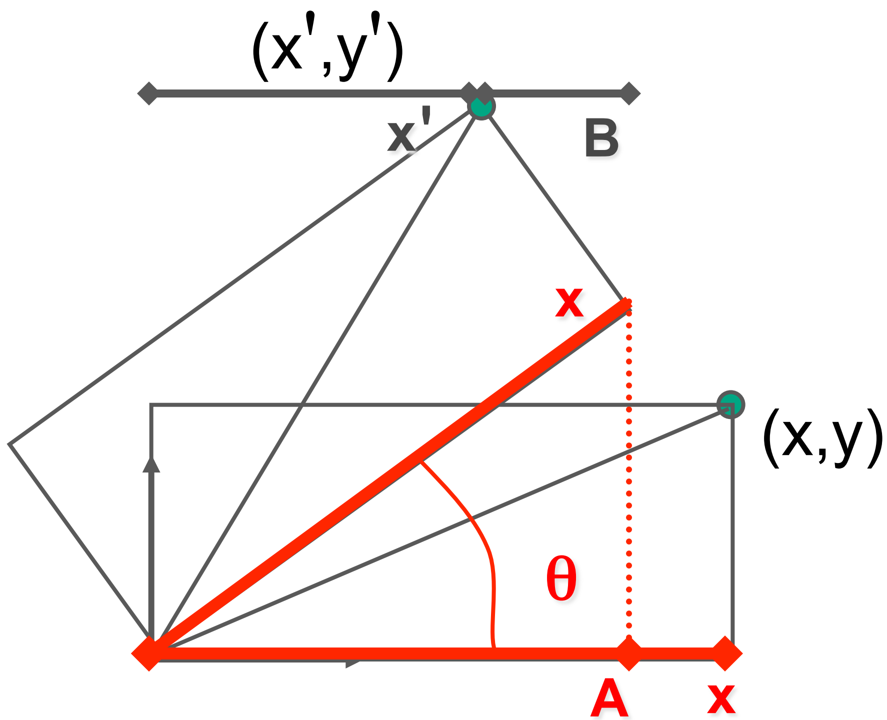


$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

$$x' = A - B$$

2D Rotation: Another Derivation



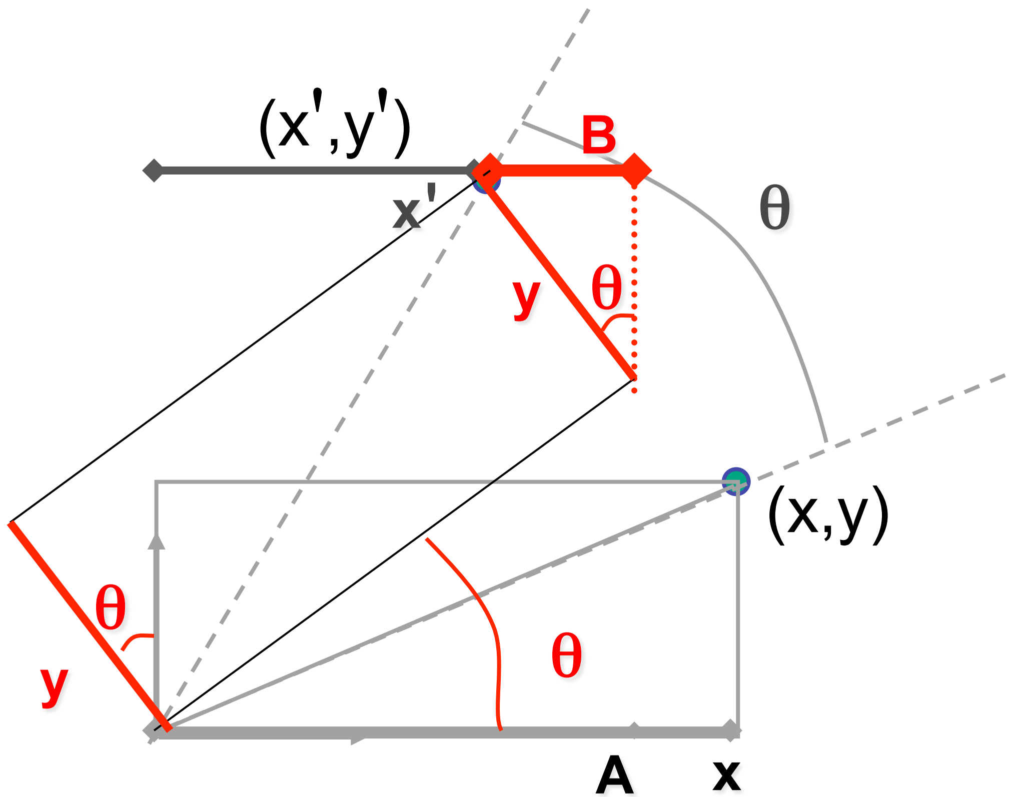
$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

$$x' = A - B$$

$$A = x\cos\theta$$

2D Rotation: Another Derivation



$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

$$x' = A - B$$

$$A = x\cos\theta$$

$$B = y\sin\theta$$

2D Rotation Matrix

- easy to capture in matrix form:

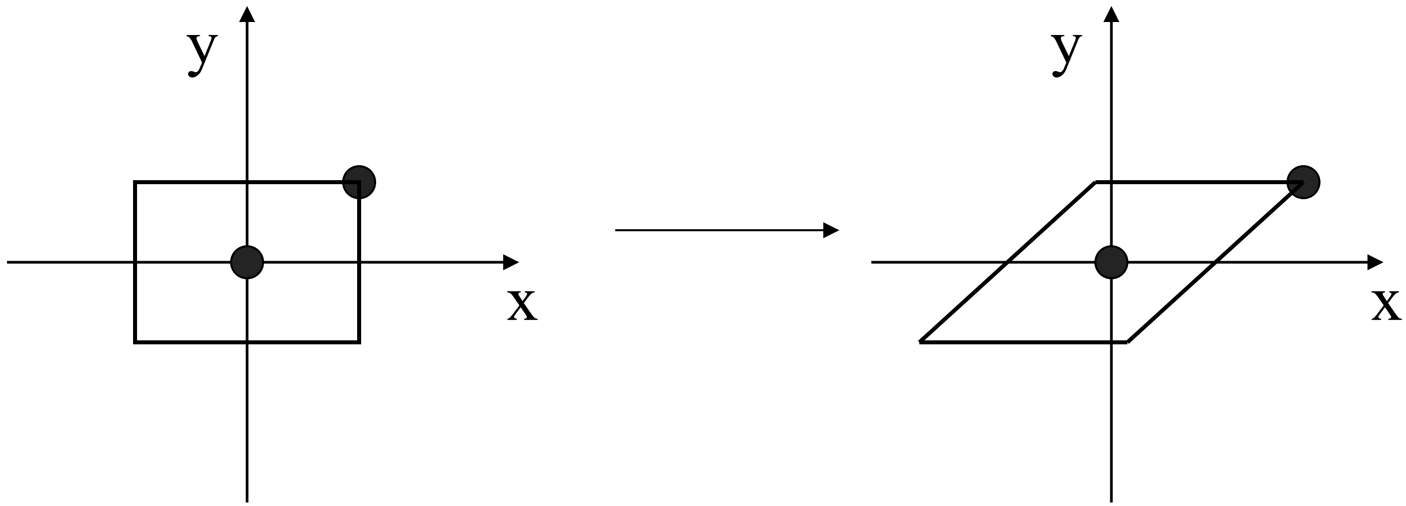
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- even though sin(q) and cos(q) are nonlinear functions of q,
  - x' is a linear combination of x and y
  - y' is a linear combination of x and y

Shear

- shear along x axis
  - push points to right in proportion to height

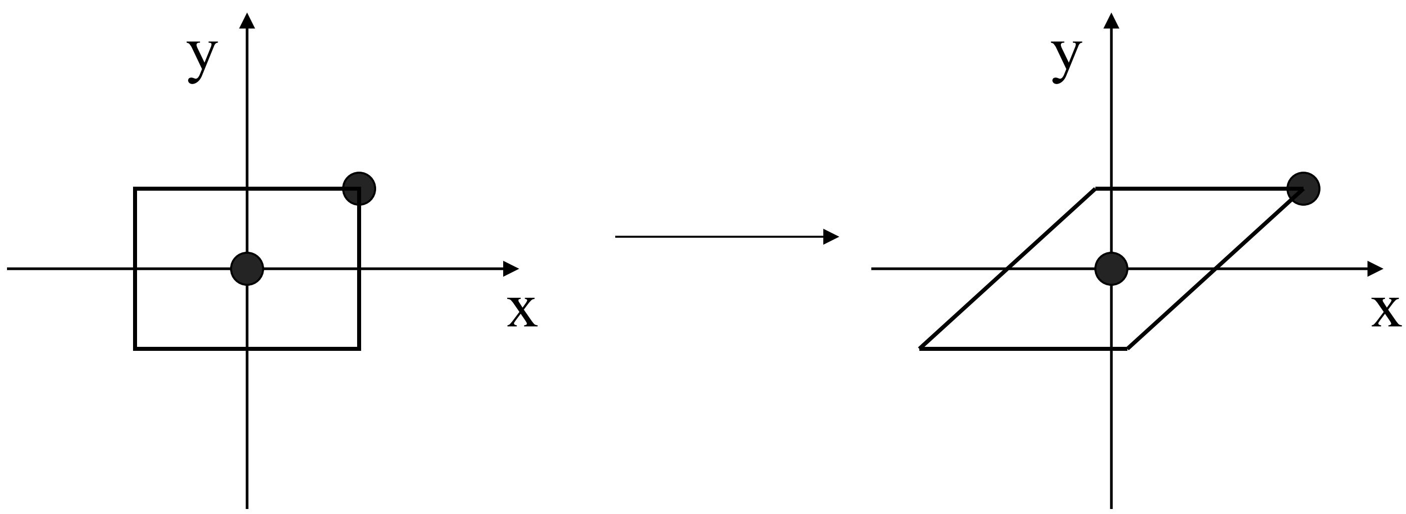
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix}$$



Shear

- shear along x axis
  - push points to right in proportion to height

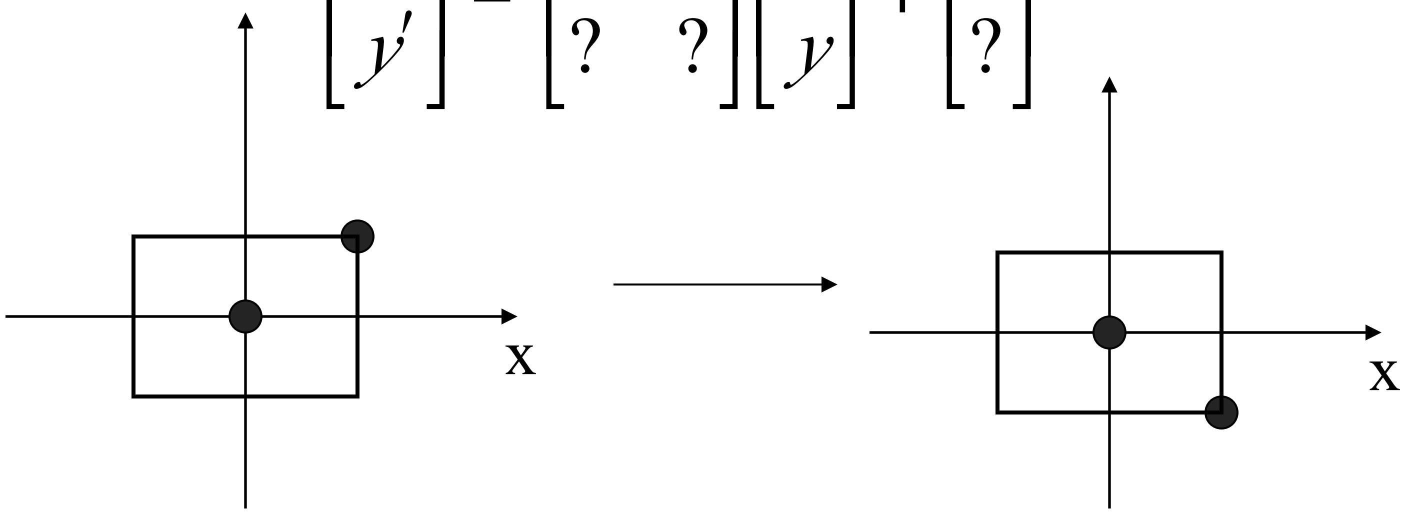
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Reflection

- reflect across x axis
  - mirror

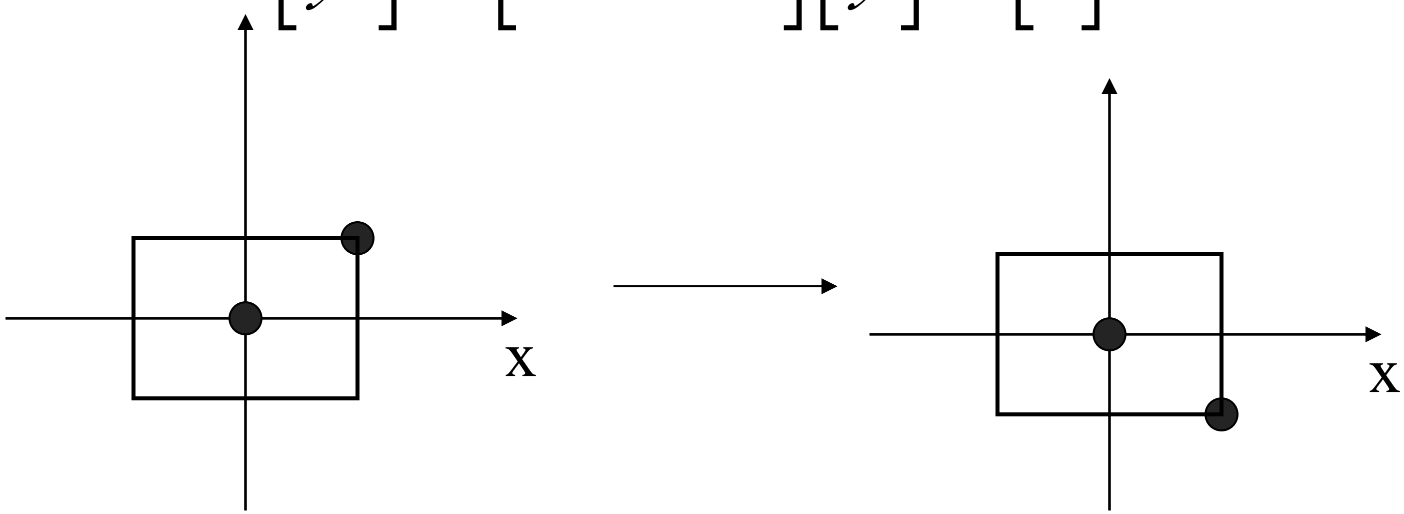
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix}$$



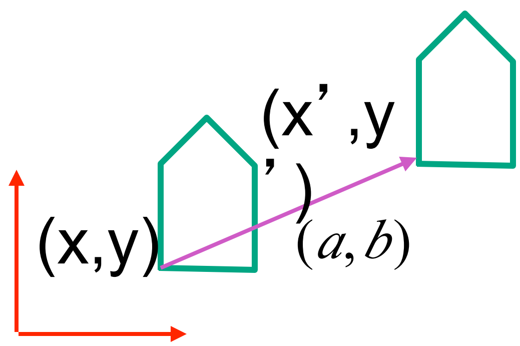
Reflection

- reflect across x axis
  - mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

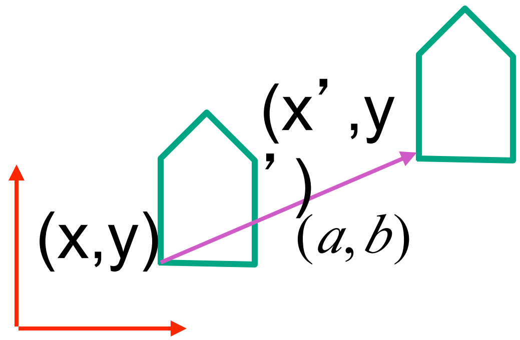


2D Translation



$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

2D Translation



$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

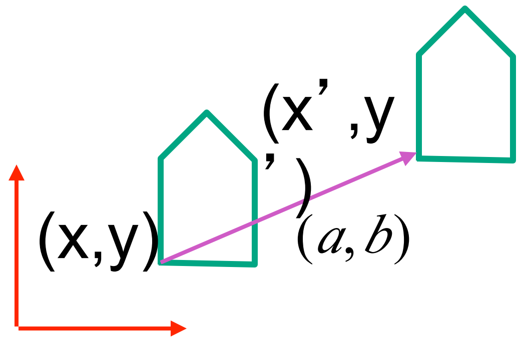
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

scaling matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\text{rotation matrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

rotation matrix

2D Translation



vector addition

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

matrix multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

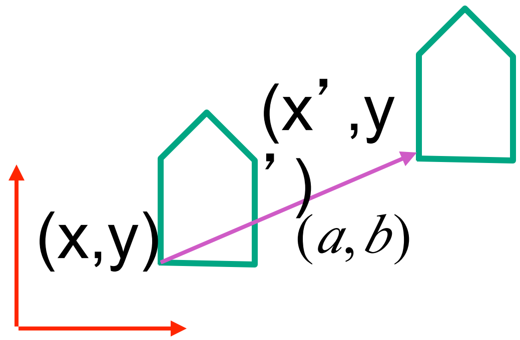
scaling matrix

matrix multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\text{rotation matrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

rotation matrix

2D Translation



vector addition

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

matrix multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix}$$

scaling matrix

matrix multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix}$$

rotation matrix

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

translation multiplication matrix??

Linear Transformations

- linear transformations are combinations of
  - shear
  - scale
  - rotate
  - reflect

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = ax + by$$
$$y' = cx + dy$$

- properties of linear transformations
  - satisifes T(s**x**+t**y**) = s T(**x**) + t T(**y**)
  - origin maps to origin
  - lines map to lines
  - parallel lines remain parallel
  - ratios are preserved
  - closed under composition

Challenge

- matrix multiplication
  - for everything except translation
  - how to do everything with multiplication?
    - then just do composition, no special cases
- homogeneous coordinates trick
  - represent 2D coordinates (x,y) with 3-vector (x,y,1)

Homogeneous Coordinates

- our 2D transformation matrices are now 3x3:

$$\mathbf{R}otation = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{S}cale = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

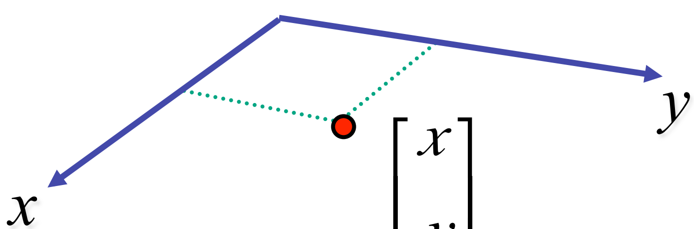
$$\mathbf{T}ranslation = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix}$$

- use rightmost column

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x*1 + a*1 \\ y*1 + b*1 \\ 1 \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \\ 1 \end{bmatrix}$$

Homogeneous Coordinates Geometrically

- point in 2D cartesian

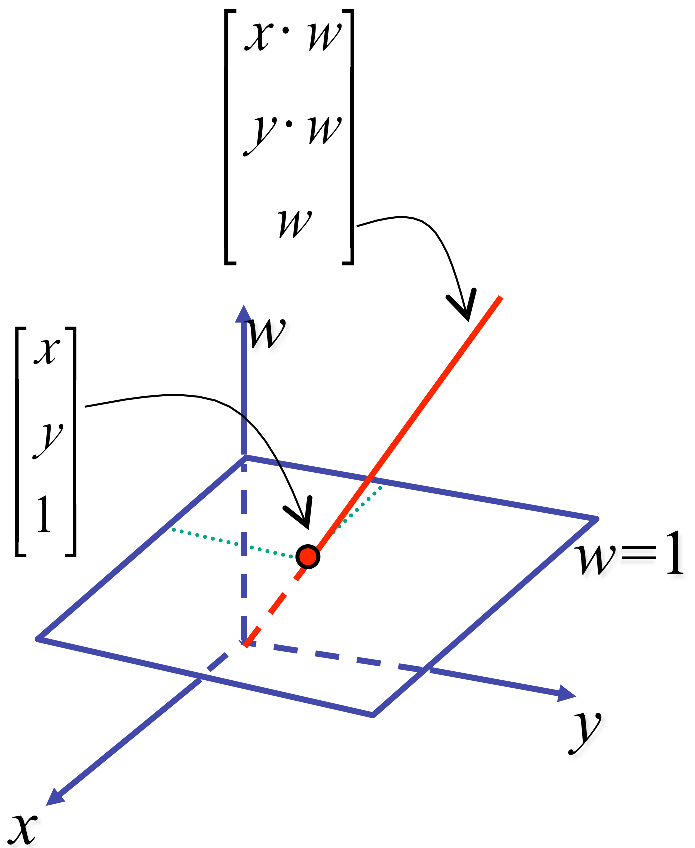


$$\begin{bmatrix} x \\ y \end{bmatrix}$$

Homogeneous Coordinates Geometrically

homogeneous | **cartesian**

$$(x, y, w) \xrightarrow{/\,w} \left(\frac{x}{w}, \frac{y}{w}\right)$$

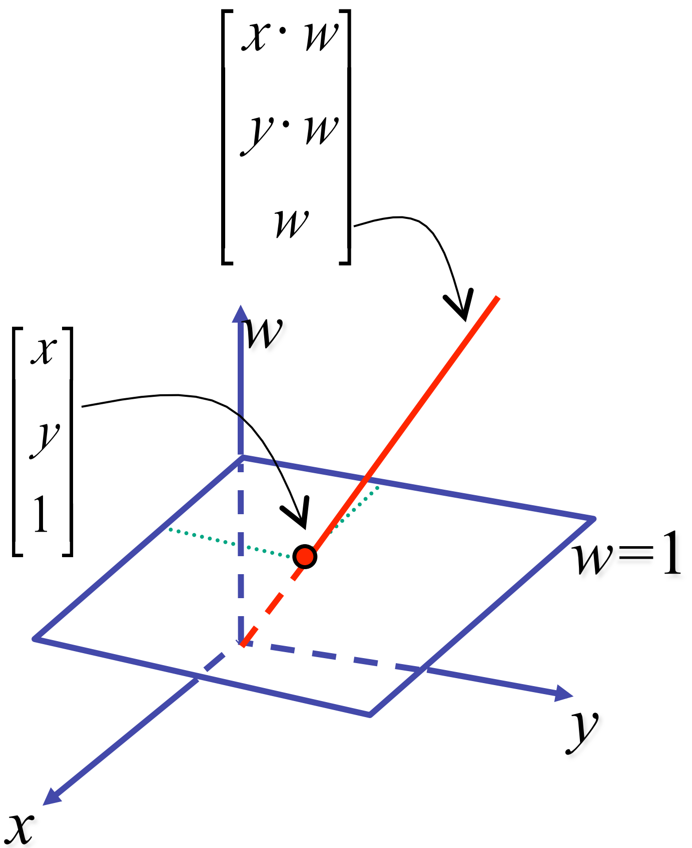


- point in 2D cartesian + weight w = point P in 3D homog. coords
- multiples of (x,y,w)
  - form a line L in 3D
  - all homogeneous points on L represent same 2D cartesian point
  - example: (2,2,1) = (4,4,2) = (1,1,0.5)

Homogeneous Coordinates Geometrically

homogeneous **cartesian**

$$(x, y, w) \xrightarrow{/\,w} \left(\frac{x}{w}, \frac{y}{w}\right)$$



- homogenize to convert homog. 3D point to cartesian 2D point:
  - divide by w to get (x/w, y/w, 1)
  - projects line to point onto w=1 plane
  - like normalizing, one dimension up
- when w=0, consider it as direction
  - points at infinity
  - these points cannot be homogenized
  - lies on x-y plane
- (0,0,0) is undefined

Affine Transformations

- affine transforms are combinations of
  - linear transformations
  - translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

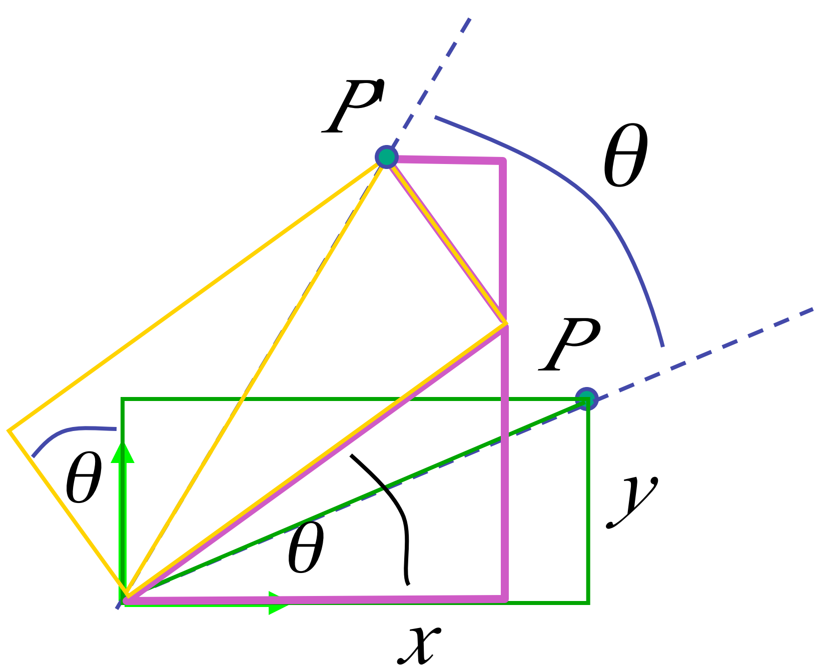
- properties of affine transformations
  - origin does not necessarily map to origin
  - lines map to lines
  - parallel lines remain parallel
  - ratios are preserved
  - closed under composition

Homogeneous Coordinates Summary

- may seem unintuitive, but they make graphics operations much easier
- allow all affine transformations to be expressed through matrix multiplication
  - we'll see even more later...
- use 3x3 matrices for 2D transformations
  - use 4x4 matrices for 3D transformations

3D Transformations

3D Rotation About Z Axis



$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- general OpenGL command

  glRotatef(angle,x,y,z);

- rotate in z

  glRotatef(angle,0,0,1);

3D Rotation in X, Y

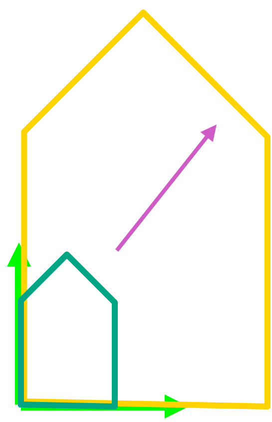
around x axis: **glRotatef(angle,1,0,0);**

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

around y axis: **glRotatef(angle,0,1,0);**

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

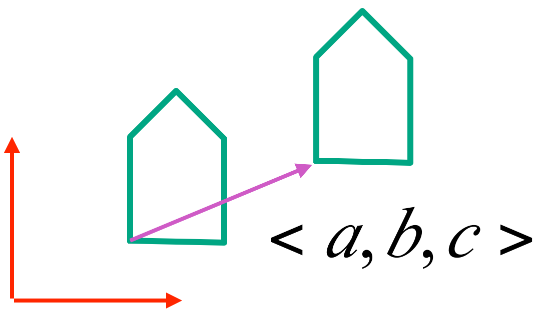
3D Scaling



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

glScalef(a,b,c);

3D Translation



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

glTranslatef(a,b,c);

3D Shear

- general shear

$$shear(hxy, hxz, hyx, hyz, hzx, hzy) = \begin{bmatrix} 1 & hyx & hzx & 0 \\ hxy & 1 & hzy & 0 \\ hxz & hyz & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- to avoid ambiguity, always say "shear along <axis> in direction of <axis>"

$$shearAlongXinDirectionOfY(h) = \begin{bmatrix} 1 & h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$shearAlongXinDirectionOfZ(h) = \begin{bmatrix} 1 & 0 & h & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$shearAlongYinDirectionOfX(h) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ h & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$shearAlongYinDirectionOfZ(h) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & h & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$shearAlongZinDirectionOfX(h) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ h & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$shearAlongZinDirectionOfY(h) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & h & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary: Transformations

translate(a,b,c)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & & a \\ & 1 & & b \\ & & 1 & c \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

scale(a,b,c)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$\text{Rotate}\,(x,\theta)$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$\text{Rotate}\,(y,\theta)$

$$\begin{bmatrix} \cos\theta & & \sin\theta & \\ & 1 & & \\ -\sin\theta & & \cos\theta & \\ & & & 1 \end{bmatrix}$$

$\text{Rotate}\,(z,\theta)$

$$\begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Undoing Transformations: Inverses

$$\mathbf{T}(x, y, z)^{-1} = \mathbf{T}(-x, -y, -z)$$

$$\mathbf{T}(x, y, z)\ \mathbf{T}(-x, -y, -z) = \mathbf{I}$$

$$\mathbf{R}(z, \theta)^{-1} = \mathbf{R}(z, -\theta) = \mathbf{R}^{\mathbf{T}}(z, \theta)$$ **(R is orthogonal)**

$$\mathbf{R}(z, \theta)\ \mathbf{R}(z, -\theta) = \mathbf{I}$$

$$\mathbf{S}(sx, sy, sz)^{-1} = \mathbf{S}\left(\frac{1}{sx}, \frac{1}{sy}, \frac{1}{sz}\right)$$

$$\mathbf{S}(sx, sy, sz)\,\mathbf{S}\left(\frac{1}{sx}, \frac{1}{sy}, \frac{1}{sz}\right) = \mathbf{I}$$

Composing Transformations

Composing Transformations

- translation

$$T1 = T(dx_1, dy_1) = \begin{bmatrix} 1 & & & dx_1 \\ & 1 & & dy_1 \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad T2 = T(dx_2, dy_2) = \begin{bmatrix} 1 & & & dx_2 \\ & 1 & & dy_2 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$P'' = T2 \bullet P' = T2 \bullet [T1 \bullet P] = [T2 \bullet T1] \bullet P, where$$

$$T2 \bullet T1 = \begin{bmatrix} 1 & & & dx_1 + dx_2 \\ & 1 & & dy_1 + dy_2 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

so translations add

Composing Transformations

- scaling

$$S2 \bullet S1 = \begin{bmatrix} sx_1 * dx_2 & & & \\ & sy_1 * sy_2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

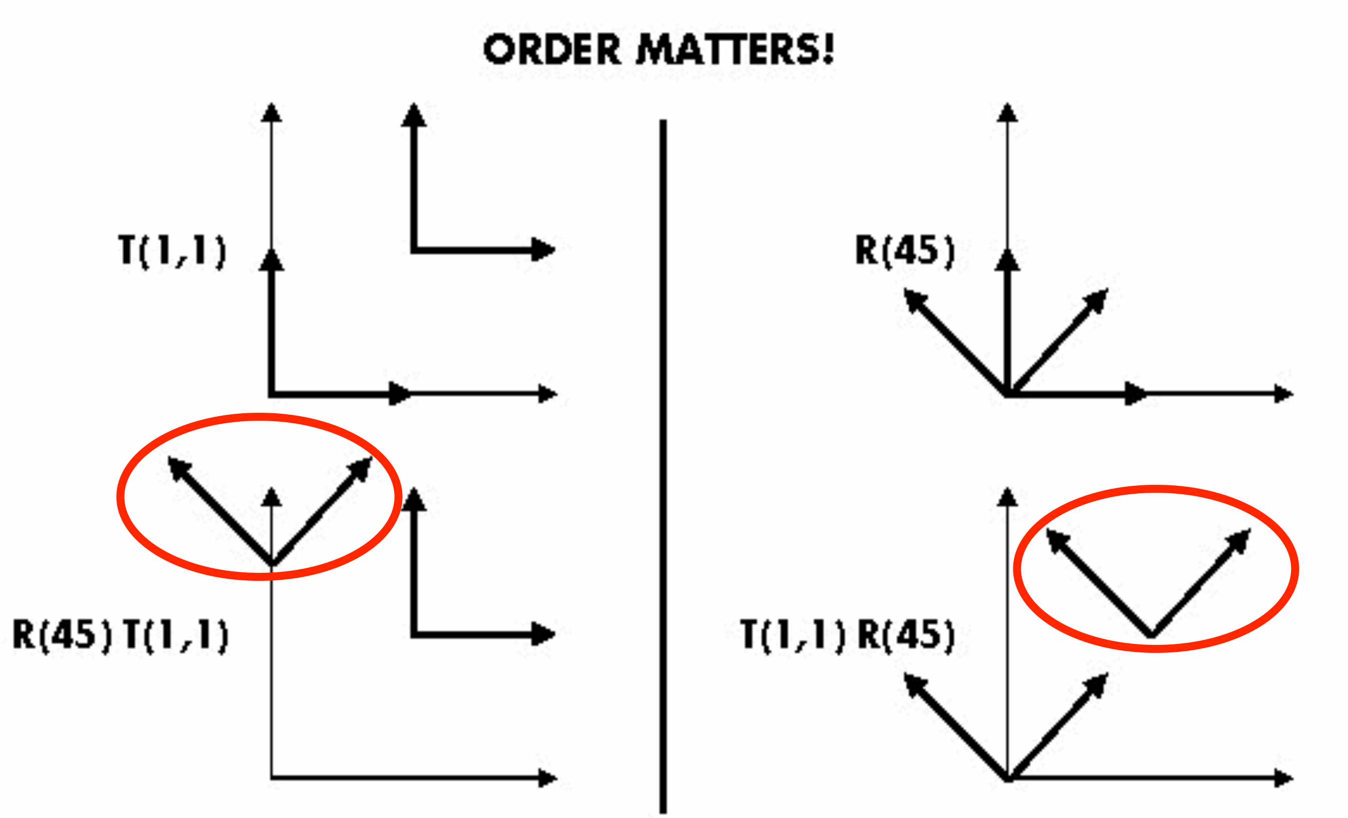
so scales multiply

- rotation

$$R2 \bullet R1 = \begin{bmatrix} \cos(\theta 1 + \theta 2) & -\sin(\theta 1 + \theta 2) & & \\ \sin(\theta 1 + \theta 2) & \cos(\theta 1 + \theta 2) & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

so rotations add

Composing Transformations

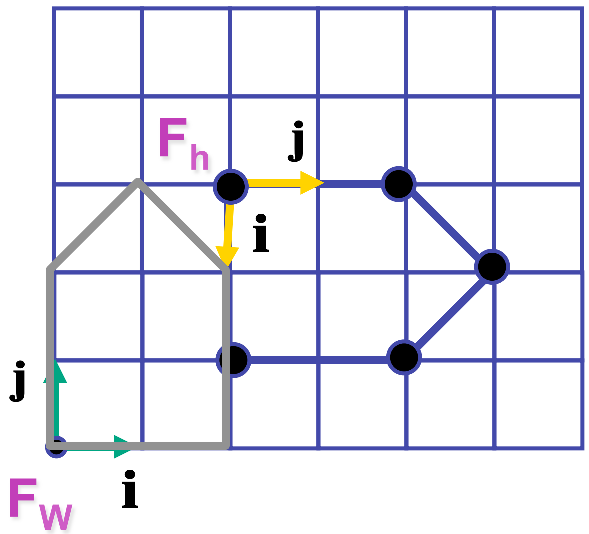


Ta Tb = Tb Ta, but Ra Rb != Rb Ra and Ta Rb != Rb Ta

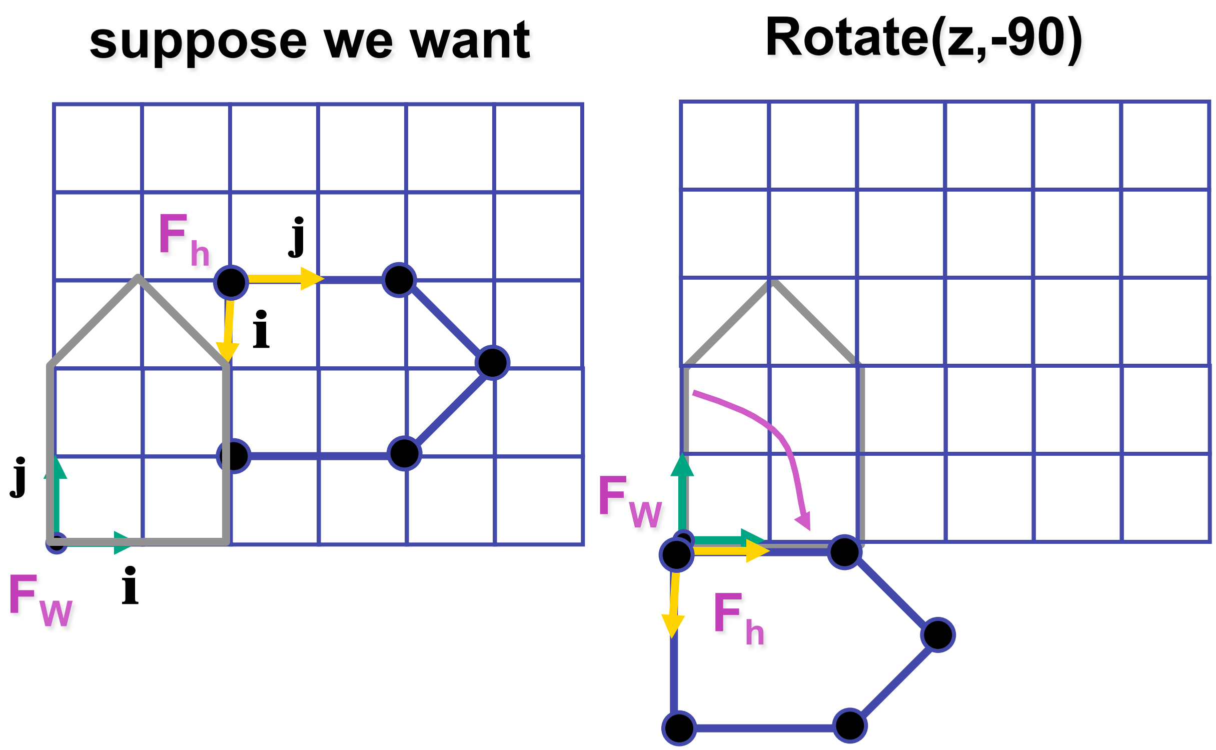
- translations commute
- rotations around same axis commute
- rotations around different axes do not commute
- rotations and translations do not commute

Composing Transformations

suppose we want

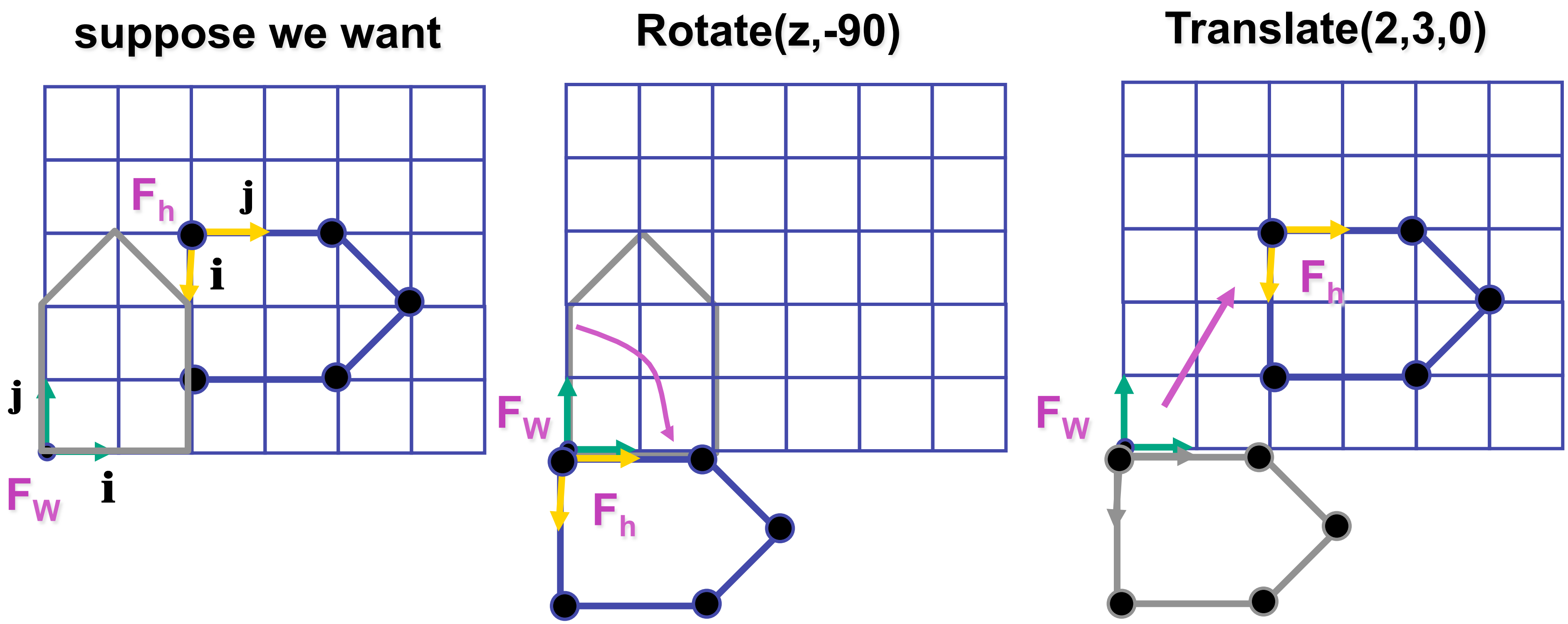


Composing Transformations



$$\mathbf{p'} = \mathbf{R}(z,-90)\,\mathbf{p}$$

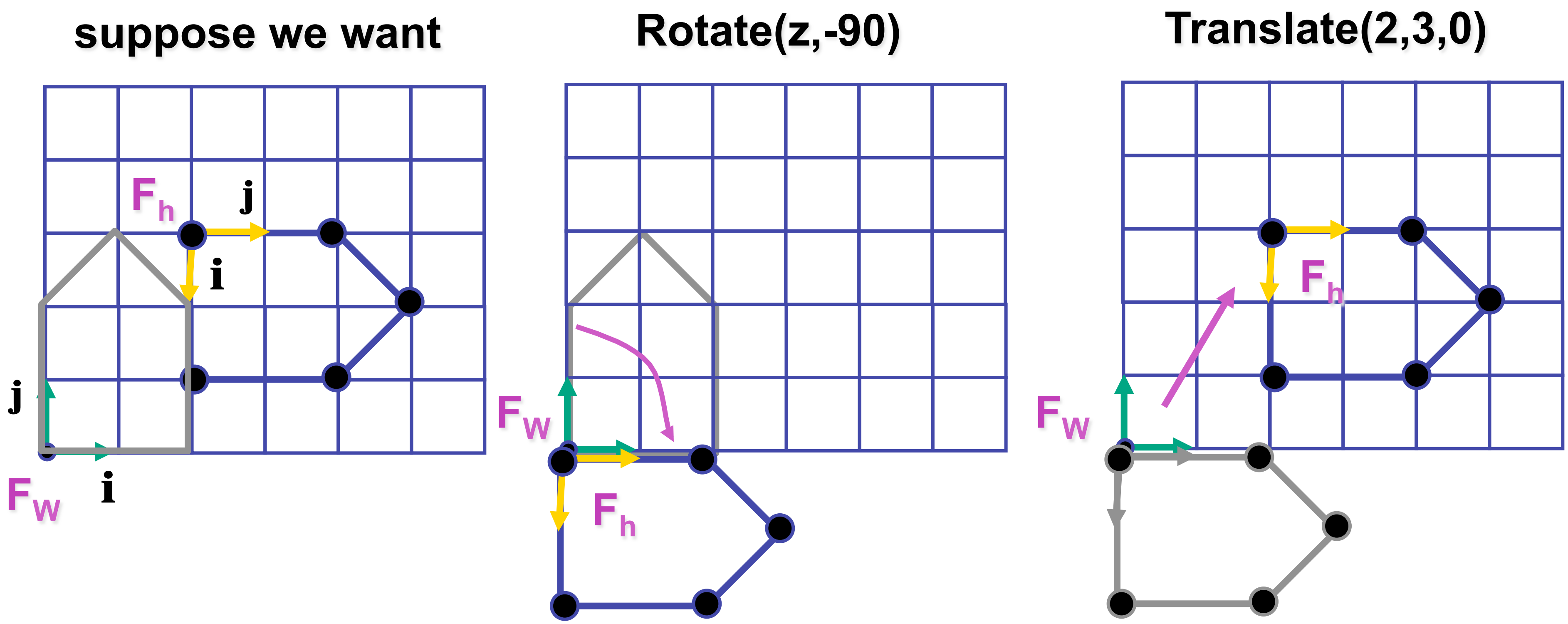
Composing Transformations



$$\mathbf{p}' = \mathbf{R}(z, -90)\,\mathbf{p}$$

$$\mathbf{p}'' = \mathbf{T}(2,3,0)\,\mathbf{p}'$$

Composing Transformations



$$\mathbf{p}' = \mathbf{R}(z, -90)\,\mathbf{p}$$

$$\mathbf{p}'' = \mathbf{T}(2,3,0)\,\mathbf{p}'$$

$$\mathbf{p}'' = \mathbf{T}(2,3,0)\,\mathbf{R}(z,-90)\,\mathbf{p} = \mathbf{TRp}$$

Composing Transformations

$$\mathbf{p}' = \mathbf{TRp}$$

- which direction to read?

Composing Transformations

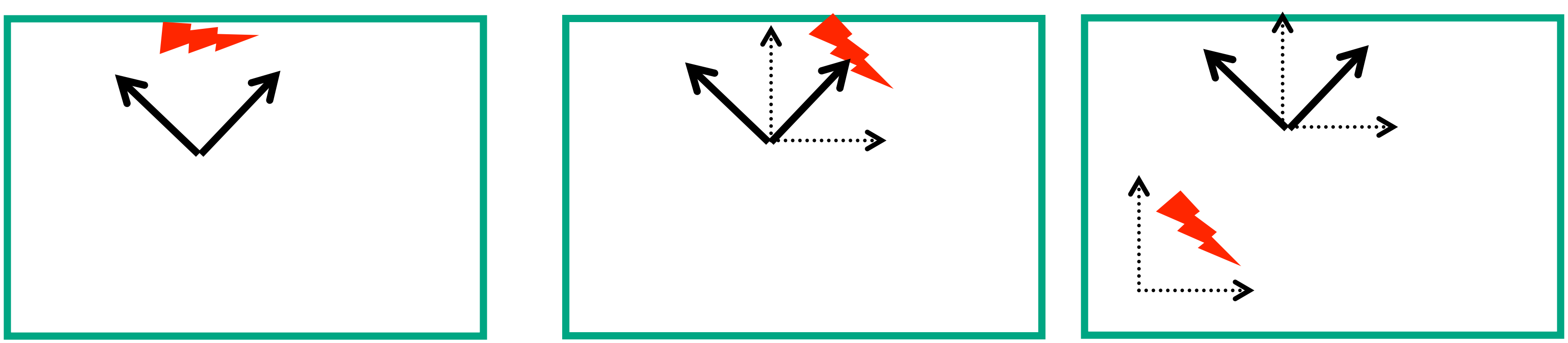
$$\mathbf{p}' = \mathbf{T}\mathbf{R}\mathbf{p}$$

- which direction to read?
  - right to left
    - interpret operations wrt fixed coordinates
    - moving object
  - left to right
    - interpret operations wrt local coordinates
    - changing coordinate system
    - in OpenGL, cannot move object once it is drawn!!
      - object specified as set of coordinates wrt specific coord sys

Composing Transformations

$$\mathbf{p'} = \mathbf{TRp}$$

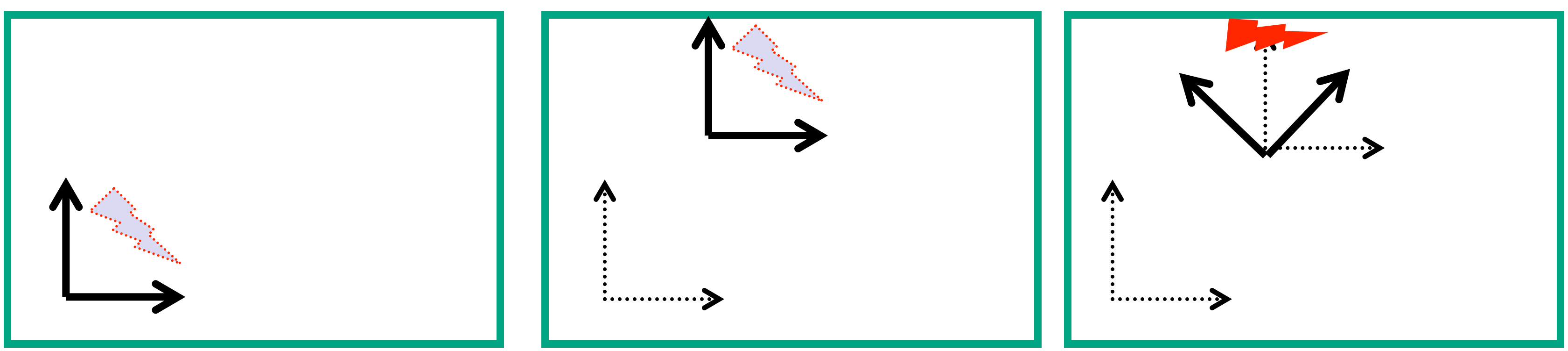
- which direction to read?
  - right to left 
    - interpret operations wrt fixed coordinates
    - moving object
      - draw thing
      - rotate thing by -45 degrees wrt origin
      - translate it (-2, -3) over



Composing Transformations

$$\mathbf{p}' = \mathbf{TRp}$$

- which direction to read?
  - left to right
    - interpret operations wrt local coordinates
    - changing coordinate system
      - translate coordinate system (2, 3) over
      - rotate coordinate system 45 degrees wrt origin
      - draw object in current coordinate system



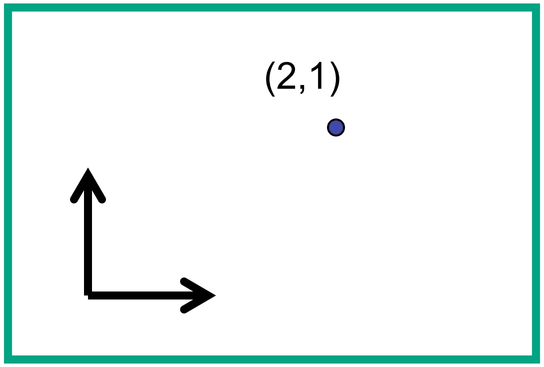
Composing Transformations

$$\mathbf{p}' = \mathbf{TRp}$$

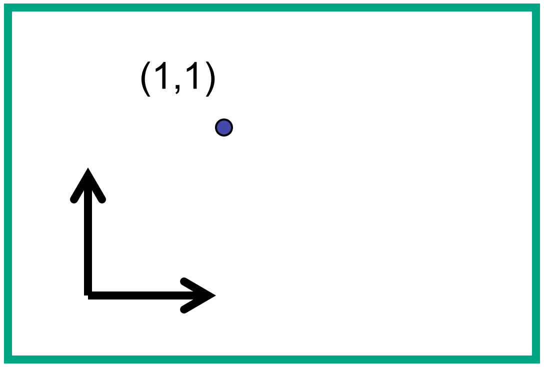
- which direction to read?
  - right to left
    - interpret operations wrt fixed coordinates
    - moving object
  - left to right **OpenGL pipeline ordering!**
    - interpret operations wrt local coordinates
    - changing coordinate system
    - OpenGL updates current matrix with postmultiply
      - glTranslatef(2,3,0);
      - glRotatef(-90,0,0,1);
      - glVertexf(1,1,1);
    - specify vector last, in final coordinate system
    - first matrix to affect it is specified second-to-last

Interpreting Transformations

translate by (-1,0)

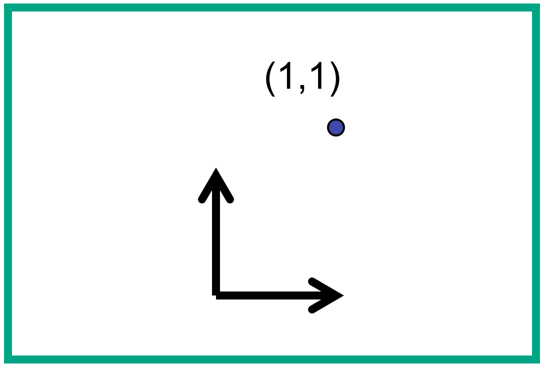


moving object



intuitive?

changing coordinate system



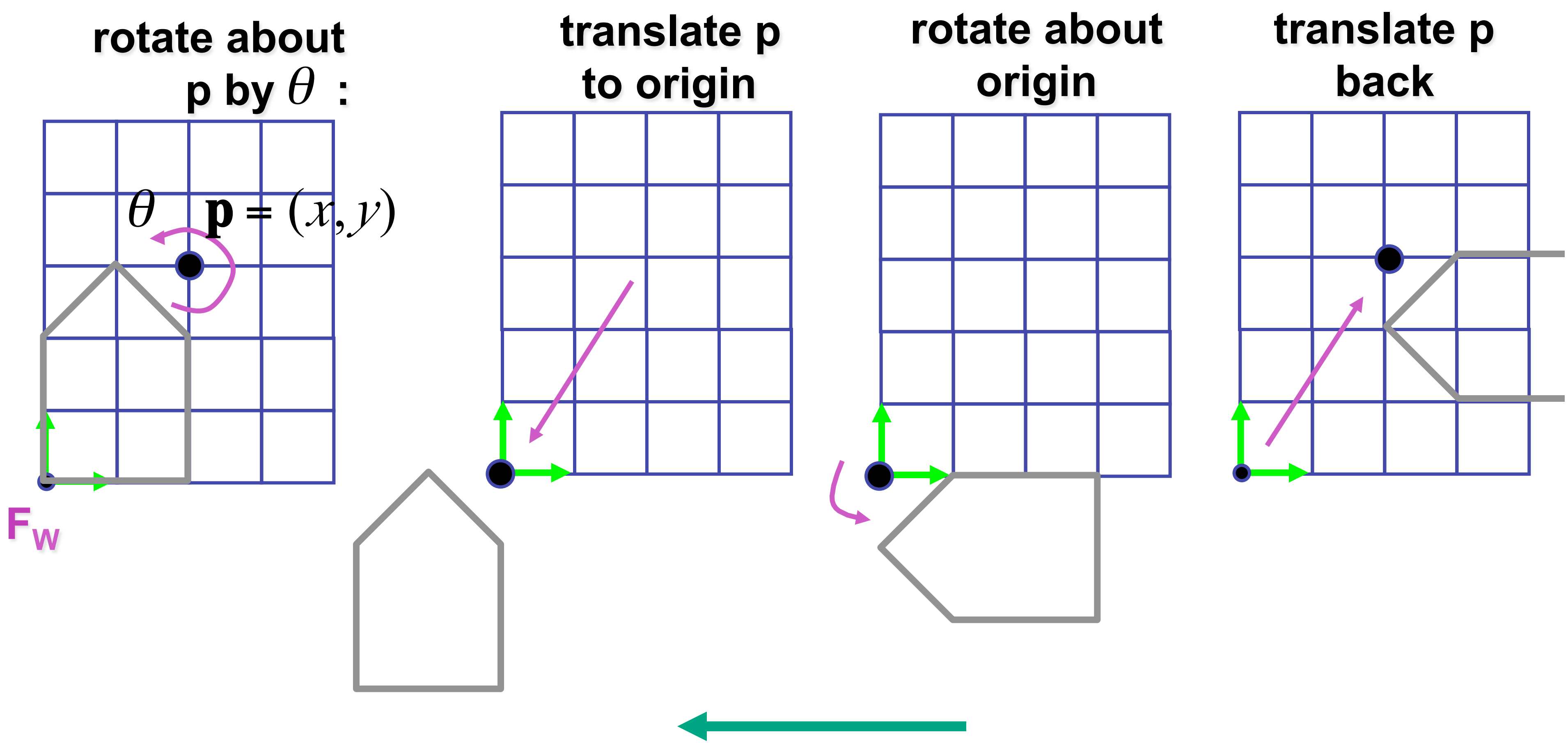
OpenGL

- same relative position between object and basis vectors

Matrix Composition

- matrices are convenient, efficient way to represent series of transformations
  - general purpose representation
  - hardware matrix multiply
  - matrix multiplication is associative
    - **p′** = (T*(R*(S***p**)))
    - **p′** = (T*R*S)***p**
- procedure
  - correctly order your matrices!
  - multiply matrices together
  - result is one matrix, multiply vertices by this matrix
  - all vertices easily transformed with one matrix multiply

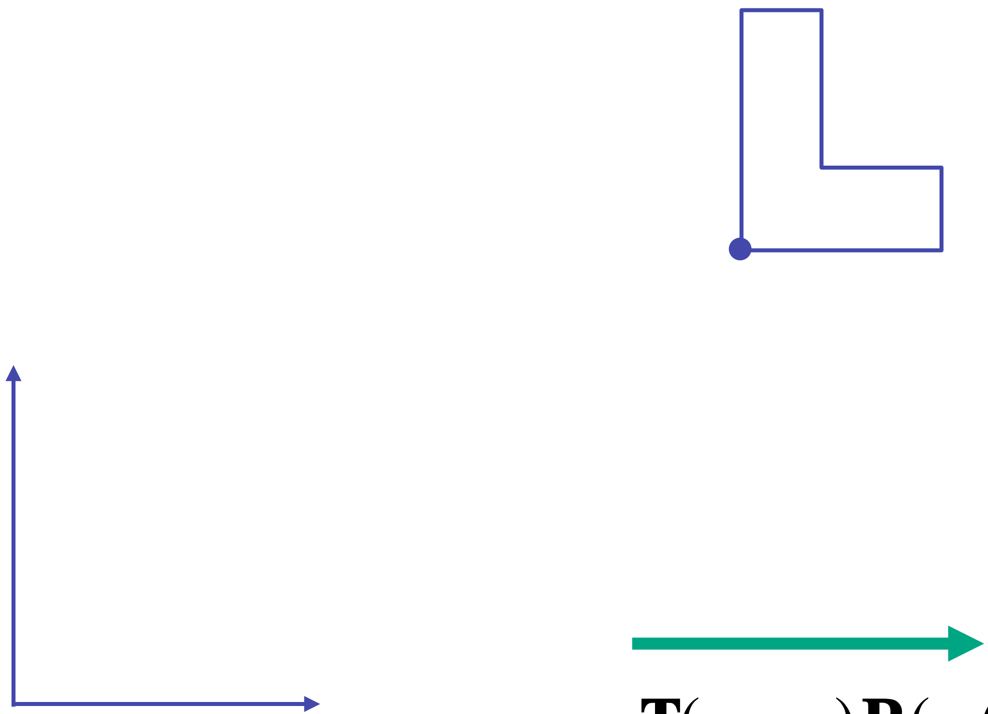
Rotation About a Point: Moving Object



$$\mathbf{T}(x, y, z)\mathbf{R}(z, \theta)\mathbf{T}(-x, -y, -z)$$

Rotation: Changing Coordinate Systems

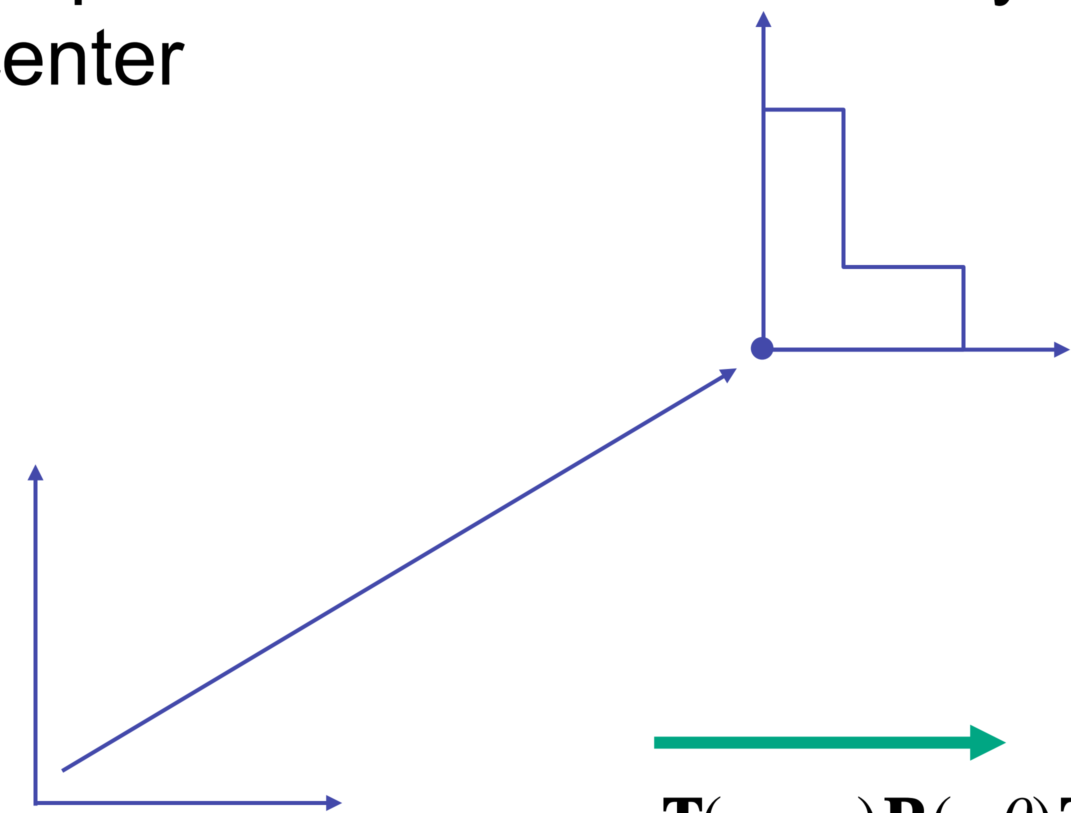
- same example: rotation around arbitrary center



$$\mathbf{T}(x, y, z)\,\mathbf{R}(z, \theta)\,\mathbf{T}(-x, -y, -z)$$

Rotation: Changing Coordinate Systems

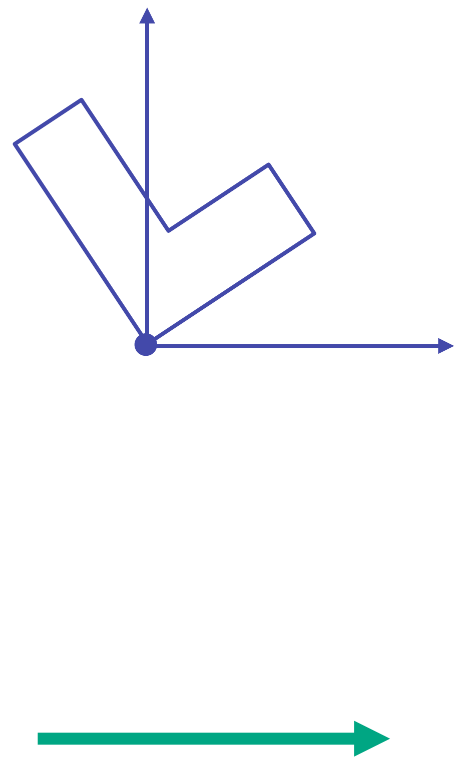
- rotation around arbitrary center
  - step 1: translate coordinate system to rotation center



$$\mathbf{T}(x,y,z)\,\mathbf{R}(z,\theta)\,\mathbf{T}(-x,-y,-z)$$

Rotation: Changing Coordinate Systems

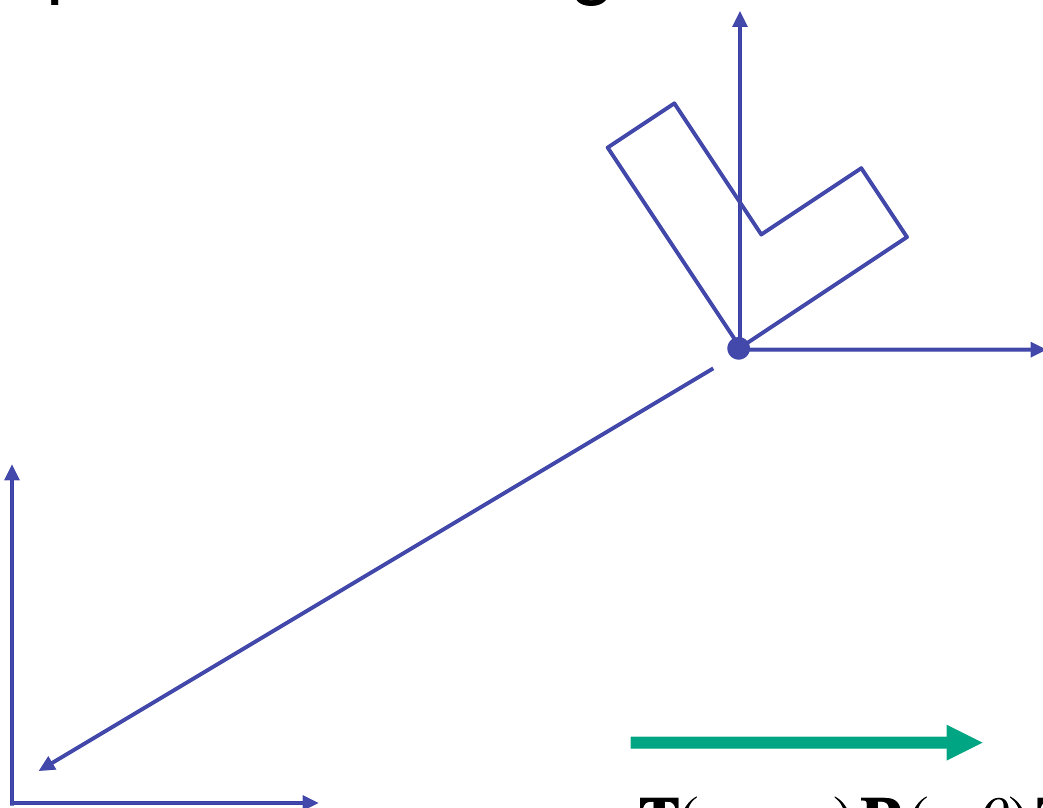
- rotation around arbitrary center
  - step 2: perform rotation



$$\mathbf{T}(x,y,z)\,\mathbf{R}(z,\theta)\,\mathbf{T}(-x,-y,-z)$$

Rotation: Changing Coordinate Systems

- rotation around arbitrary center
  - step 3: back to original coordinate system



$$\mathbf{T}(x,y,z)\,\mathbf{R}(z,\theta)\,\mathbf{T}(-x,-y,-z)$$

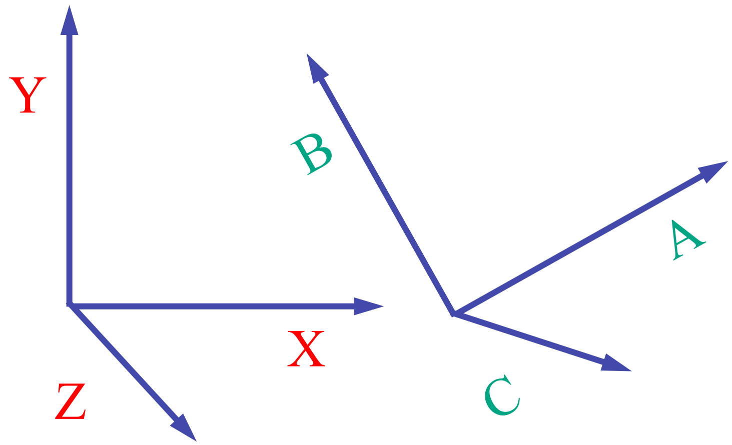
General Transform Composition

- transformation of geometry into coordinate system where operation becomes simpler
  - typically translate to origin
- perform operation
- transform geometry back to original coordinate system

Rotation About an Arbitrary Axis

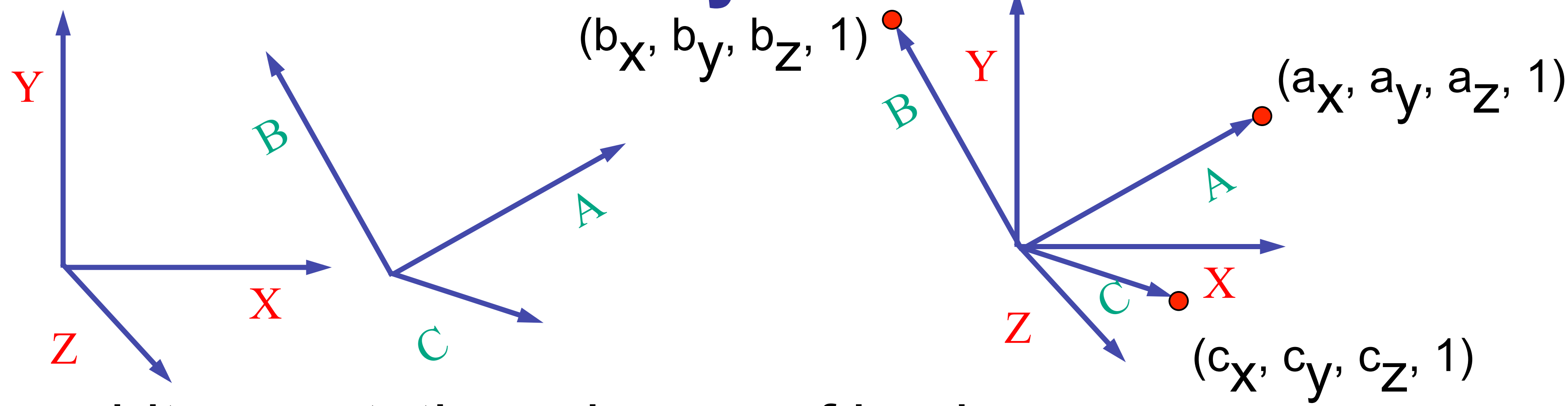
- axis defined by two points
- translate point to the origin
- rotate to align axis with z-axis (or x or y)
- perform rotation
- undo aligning rotations
- undo translation

Arbitrary Rotation



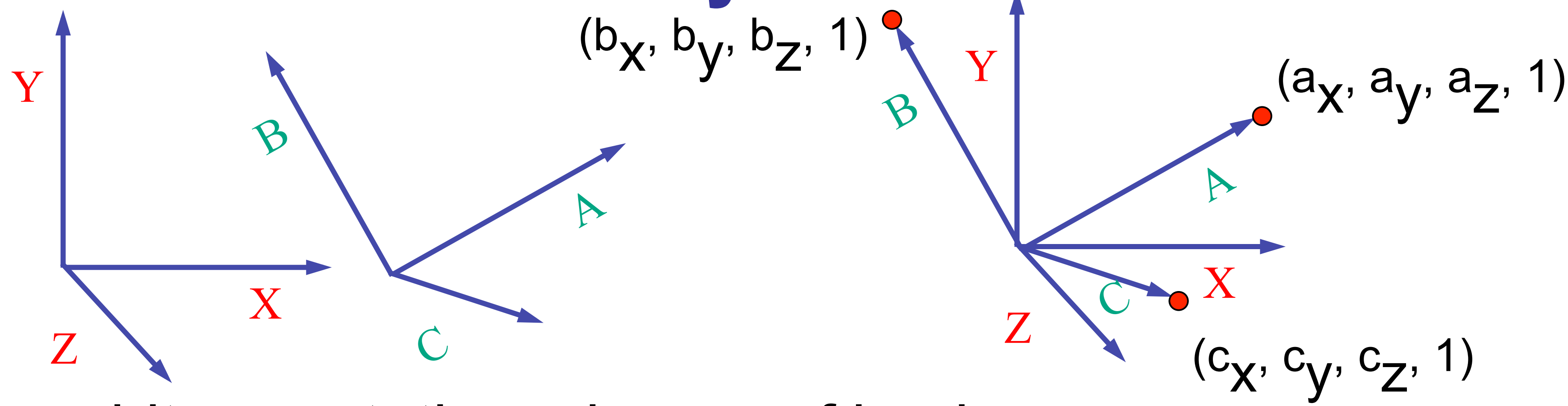
- arbitrary rotation: change of basis
  - given two orthonormal coordinate systems *XYZ* and *ABC*
    - *A*'s location in the XYZ coordinate system is $(a_x, a_y, a_z, 1)$, ...

Arbitrary Rotation



- arbitrary rotation: change of basis
  - given two orthonormal coordinate systems *XYZ* and *ABC*
    - *A*'s location in the XYZ coordinate system is $(a_x, a_y, a_z, 1)$, ...

Arbitrary Rotation



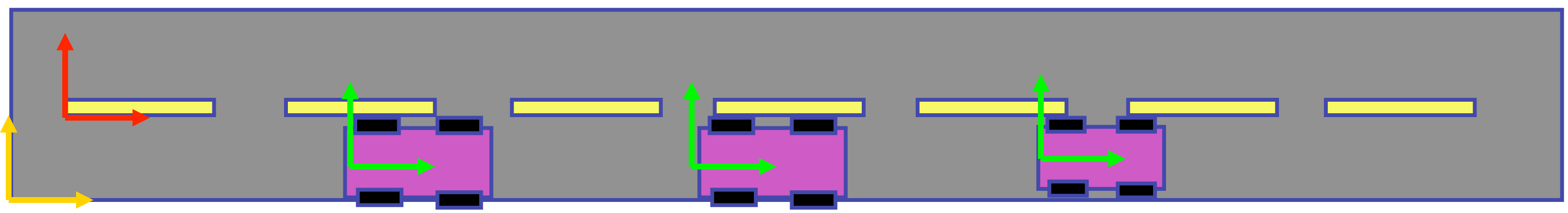
- arbitrary rotation: change of basis
  - given two orthonormal coordinate systems *XYZ* and *ABC*
    - *A*'s location in the XYZ coordinate system is $(a_X, a_Y, a_Z, 1)$, ...
- transformation from one to the other is matrix R whose columns are *A,B,C:*

$$R(X) = \begin{bmatrix} a_x & b_x & c_x & 0 \\ a_y & b_y & c_y & 0 \\ a_z & b_z & c_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = (a_x, a_y, a_z, 1) = A$$

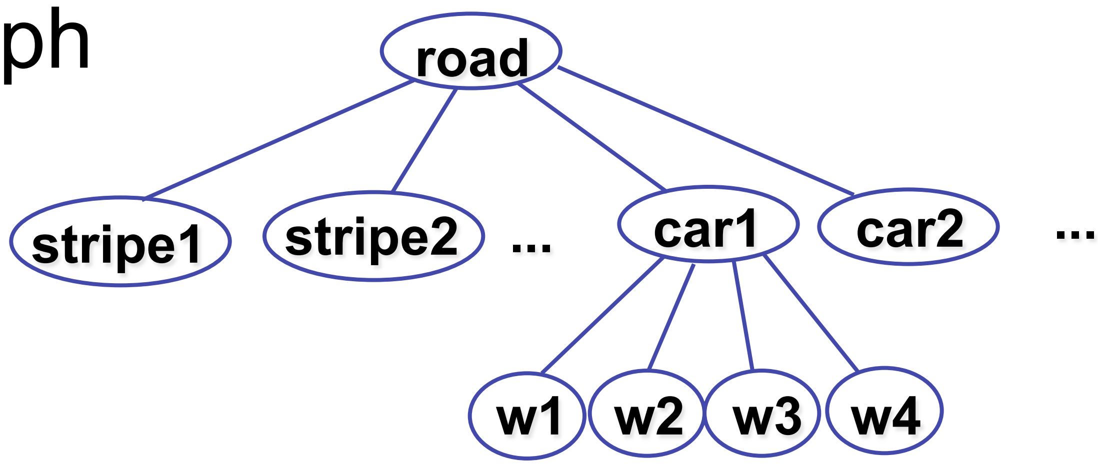
Transformation Hierarchies

Transformation Hierarchies

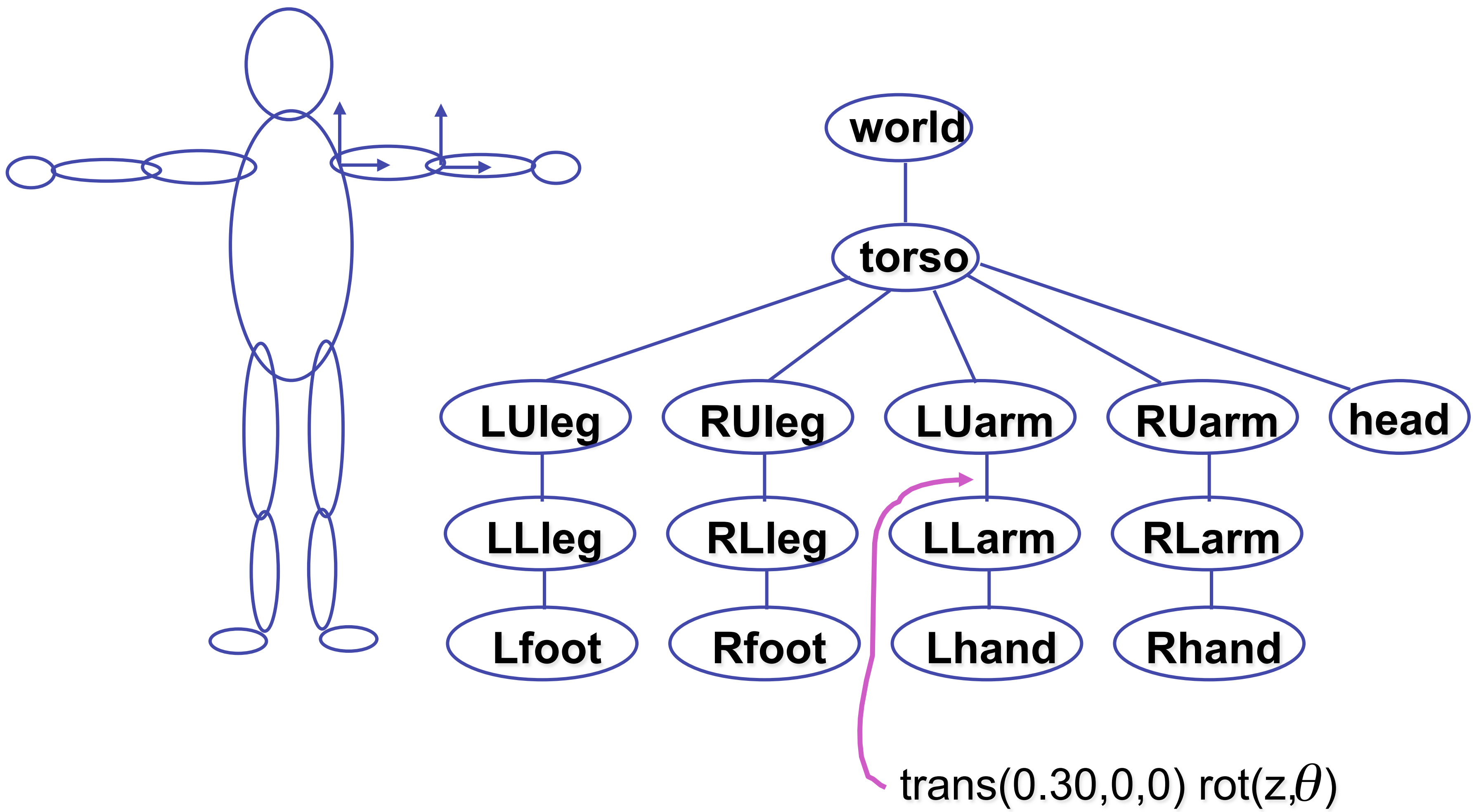
- scene may have a hierarchy of coordinate systems
  - stores matrix at each level with incremental transform from parent's coordinate system



- scene graph

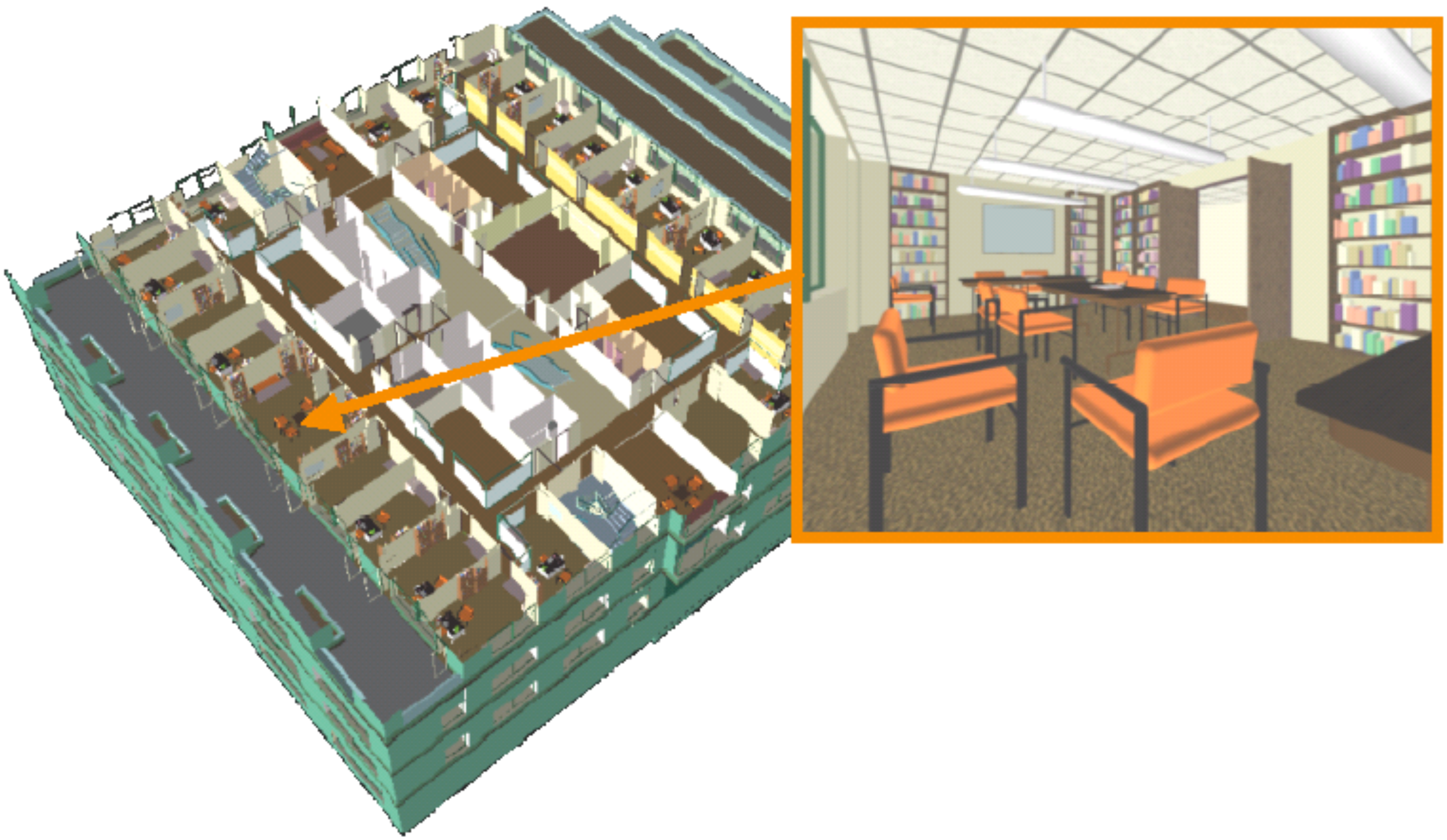


Transformation Hierarchy Example 1



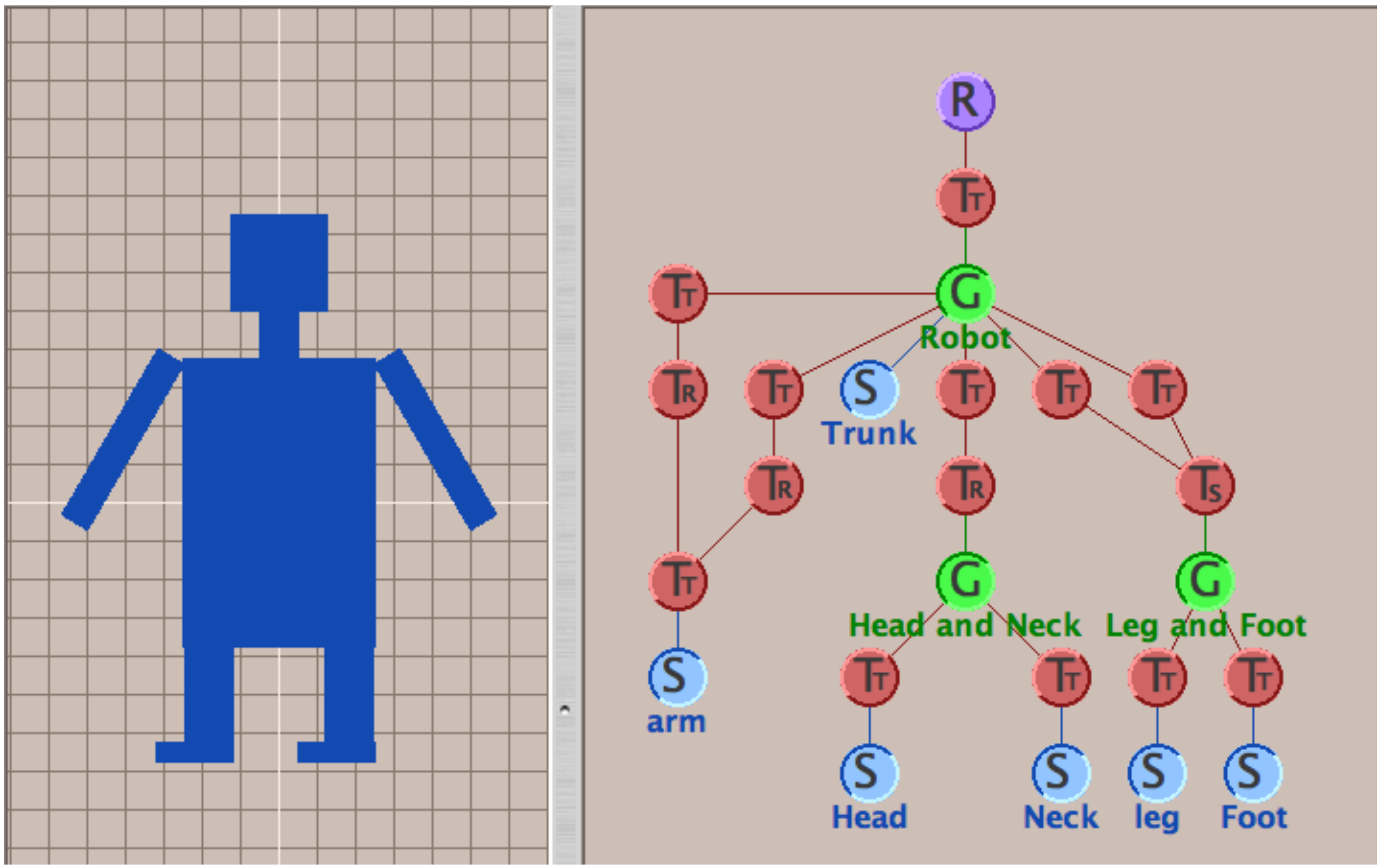
Transformation Hierarchy Example 2

- draw same 3D data with different transformations: instancing



Transformation Hierarchies Demo

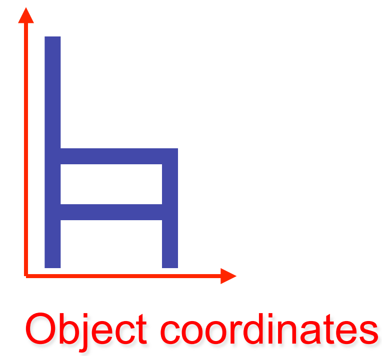
- transforms apply to graph nodes beneath

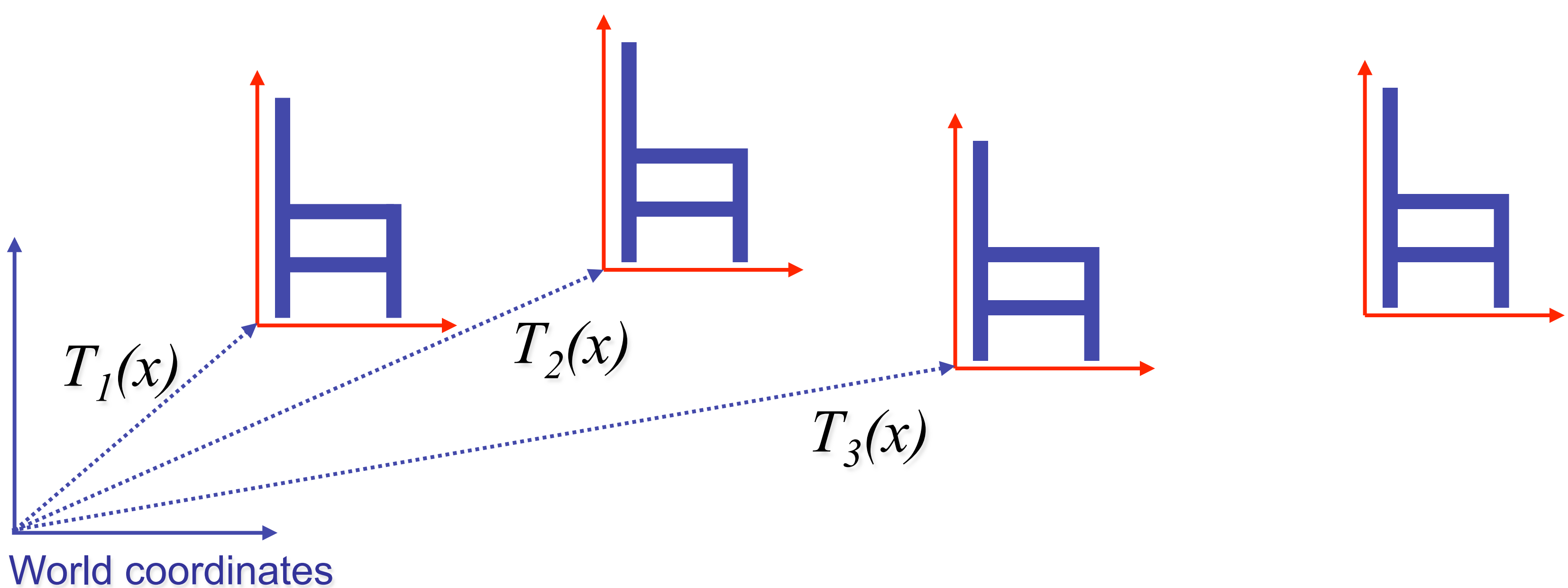


http://www.cs.brown.edu/exploratories/freeSoftware/catalogs/scenegraphs.html

Matrix Stacks

- challenge of avoiding unnecessary computation
  - using inverse to return to origin
  - computing incremental T_1 -> T_2

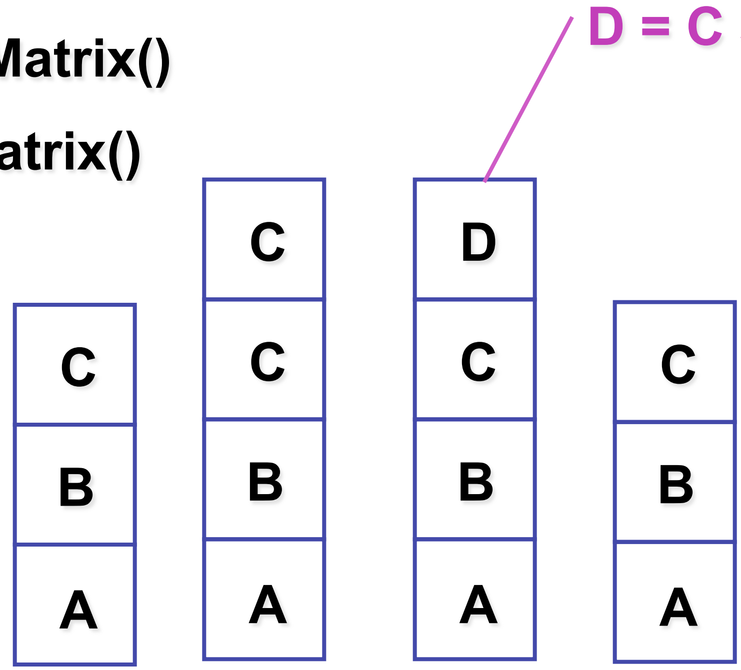




Matrix Stacks

glPushMatrix()

glPopMatrix()

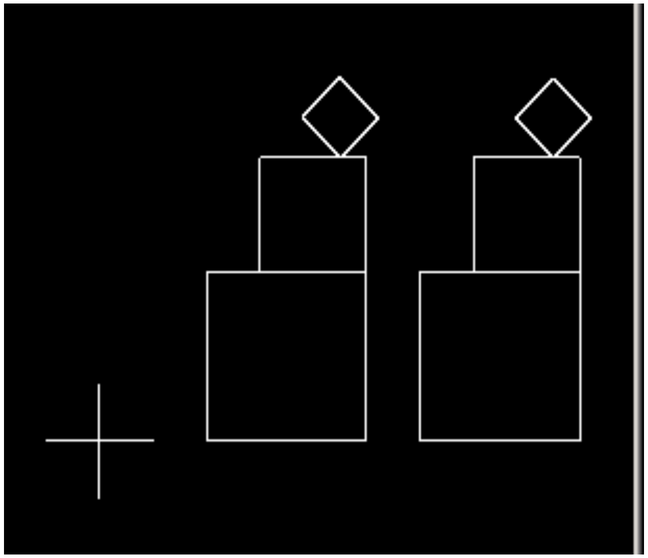


DrawSquare()
glPushMatrix()
glScale3f(2,2,2)
glTranslate3f(1,0,0)
DrawSquare()
glPopMatrix()

Modularization

- drawing a scaled square
  - push/pop ensures no coord system change

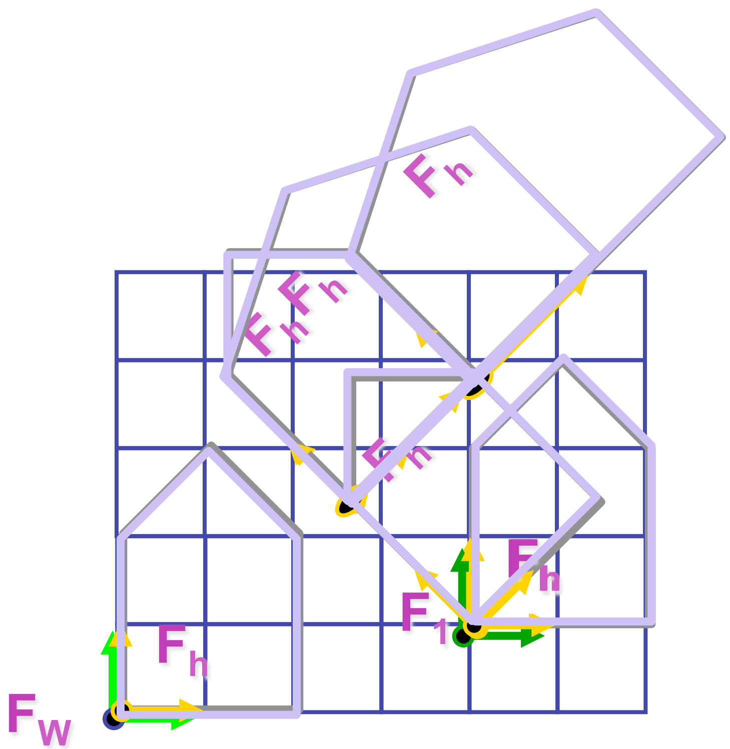
```
void drawBlock(float k) {
  glPushMatrix();

  glScalef(k,k,k);
  glBegin(GL_LINE_LOOP);
  glVertex3f(0,0,0);
  glVertex3f(1,0,0);
  glVertex3f(1,1,0);
  glVertex3f(0,1,0);
  glEnd();

  glPopMatrix();
}
```



Matrix Stacks

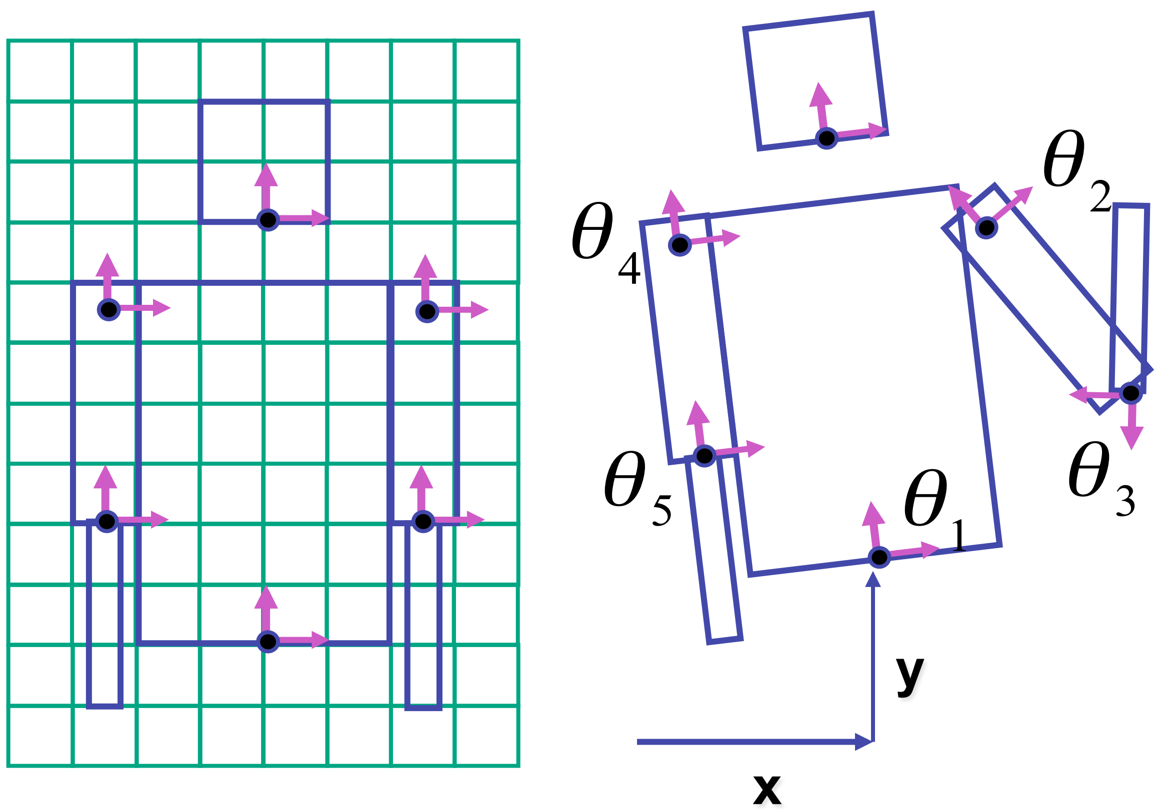
- advantages
  - no need to compute inverse matrices all the time
  - modularize changes to pipeline state
  - avoids incremental changes to coordinate systems
    - accumulation of numerical errors
- practical issues
  - in graphics hardware, depth of matrix stacks is limited
    - (typically 16 for model/view and about 4 for projective matrix)

Transformation Hierarchy Example 3




```
glLoadIdentity();
glTranslatef(4,1,0);
glPushMatrix();
glRotatef(45,0,0,1);
glTranslatef(0,2,0);
glScalef(2,1,1);
glTranslate(1,0,0);
glPopMatrix();
```

Transformation Hierarchy Example 4




```
glTranslate3f(x,y,0);
glRotatef(θ1,0,0,1);
DrawBody();
glPushMatrix();
  glTranslate3f(0,7,0);
  DrawHead();
glPopMatrix();
glPushMatrix();
  glTranslate(2.5,5.5,0);
  glRotatef(θ2,0,0,1);
  DrawUArm();
  glTranslate(0,-3.5,0);
  glRotatef(θ3,0,0,1);
  DrawLArm();
glPopMatrix();
... (draw other arm)
```

Hierarchical Modelling

- advantages
  - define object once, instantiate multiple copies
  - transformation parameters often good control knobs
  - maintain structural constraints if well-designed
- limitations
  - expressivity: not always the best controls
  - can’t do closed kinematic chains
    - keep hand on hip
  - can’t do other constraints
    - collision detection
      - self-intersection
      - walk through walls

Display Lists

Display Lists

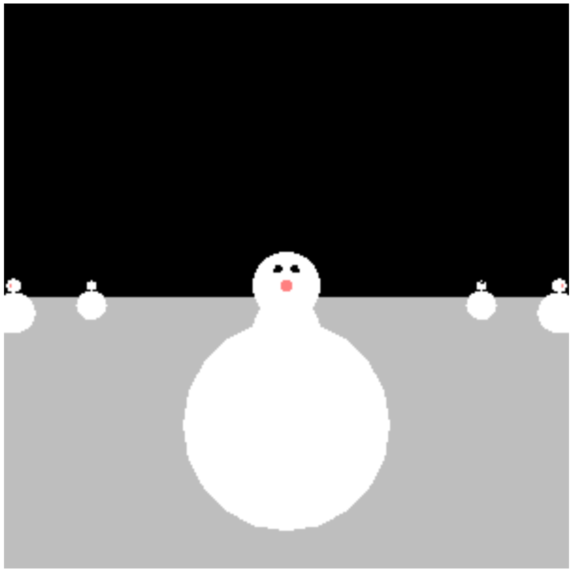
- precompile/cache block of OpenGL code for reuse
  - usually more efficient than immediate mode
    - exact optimizations depend on driver
  - good for multiple instances of same object
    - but cannot change contents, not parametrizable
  - good for static objects redrawn often
    - display lists persist across multiple frames
    - interactive graphics: objects redrawn every frame from new viewpoint from moving camera
  - can be nested hierarchically
- snowman example

  http://www.lighthouse3d.com/opengl/displaylists

One Snowman

```
void drawSnowMan() {

glColor3f(1.0f, 1.0f, 1.0f);

// Draw Body
glTranslatef(0.0f ,0.75f, 0.0f);
glutSolidSphere(0.75f,20,20);

// Draw Head
glTranslatef(0.0f, 1.0f, 0.0f);
glutSolidSphere(0.25f,20,20);

// Draw Eyes
glPushMatrix();
glColor3f(0.0f,0.0f,0.0f);
glTranslatef(0.05f, 0.10f, 0.18f);
glutSolidSphere(0.05f,10,10);
glTranslatef(-0.1f, 0.0f, 0.0f);
glutSolidSphere(0.05f,10,10);
glPopMatrix();

// Draw Nose
glColor3f(1.0f, 0.5f , 0.5f);
glRotatef(0.0f,1.0f, 0.0f, 0.0f);
glutSolidCone(0.08f,0.5f,10,2);
}
```

Instantiate Many Snowmen



```
// Draw 36 Snowmen
for(int i = -3; i < 3; i++)
  for(int j=-3; j < 3; j++) {
  glPushMatrix();
  glTranslatef(i*10.0, 0, j * 10.0);
  // Call the function to draw a snowman
  drawSnowMan();
  glPopMatrix();
}
```

36K polygons, 55 FPS

Making Display Lists

```
GLuint createDL() {
GLuint snowManDL;
// Create the id for the list
snowManDL = glGenLists(1);
glNewList(snowManDL,GL_COMPILE);
drawSnowMan();
glEndList();
return(snowManDL); }

snowmanDL = createDL();
for(int i = -3; i < 3; i++)
  for(int j=-3; j < 3; j++) {
    glPushMatrix();
    glTranslatef(i*10.0, 0, j * 10.0);
    glCallList(snowManDL);
    glPopMatrix(); }
```

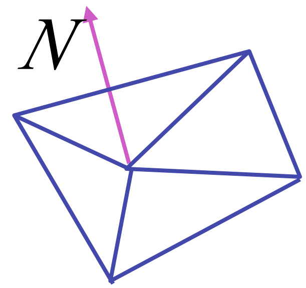
36K polygons, 153 FPS

Transforming Normals

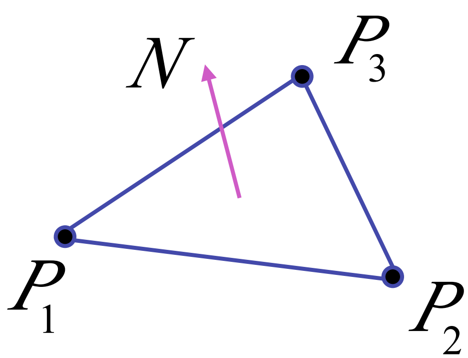
Transforming Geometric Objects

- lines, polygons made up of vertices
  - transform the vertices
  - interpolate between
- does this work for everything? no!
  - normals are trickier

Computing Normals



- normal
  - direction specifying orientation of polygon
    - w=0 means direction with homogeneous coords
    - vs. w=1 for points/vectors of object vertices
  - used for lighting
    - must be normalized to unit length
  - can compute if not supplied with object



$$N = (P_2 - P_1) \times (P_3 - P_1)$$

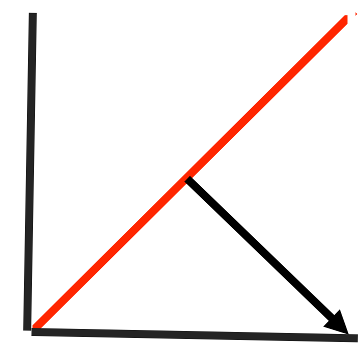
Transforming Normals

$$\begin{bmatrix} x' \\ y' \\ z' \\ 0 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & T_x \\ m_{21} & m_{22} & m_{23} & T_y \\ m_{31} & m_{32} & m_{33} & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

- so if points transformed by matrix **M**, can we just transform normal vector by **M** too?
  - translations OK: w=0 means unaffected
  - rotations OK
  - uniform scaling OK
- these all maintain direction

Transforming Normals

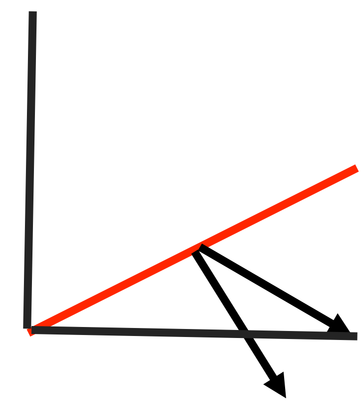
- nonuniform scaling does not work
- x-y=0 plane
  - line x=y
  - normal: [1,-1,0]
    - direction of line x=-y
    - (ignore normalization for now)



Transforming Normals

- apply nonuniform scale: stretch along x by 2
  - new plane x = 2y
- transformed normal: [2,-1,0]

$$\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$



  - normal is direction of line x = -2y or x+2y=0
  - not perpendicular to plane!
  - should be direction of 2x = -y

Planes and Normals

- plane is all points perpendicular to normal
  - $N \bullet P = 0$ (with dot product)
  - $N^T \bullet P = 0$ (matrix multiply requires transpose)

$$N = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, P = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

- explicit form: plane = $ax + by + cz + d$

Finding Correct Normal Transform

- transform a plane

$$P \longrightarrow P' = MP$$
$$N \longrightarrow N' = QN$$

given M,
what should Q be?

$$N'^T P' = 0$$ **stay perpendicular**

$$(QN)^T (MP) = 0$$ **substitute from above**

$$N^T Q^T M P = 0$$ $$(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}} \mathbf{A}^{\mathbf{T}}$$

$$Q^T M = I$$ $$\mathbf{N}^{\mathbf{T}} \mathbf{P} = 0 \text{ if } \mathbf{Q}^{\mathbf{T}} \mathbf{M} = \mathbf{I}$$

$$\mathbf{Q} = \left(\mathbf{M}^{-1}\right)^{\mathbf{T}}$$

thus the normal to any surface can be transformed by the inverse transpose of the modelling transformation

Reading for Next Topic: Viewing

- FCG Chapter 7 Viewing
- FCG Section 6.3.1 Windowing Transforms

- RB rest of Chap Viewing
- RB rest of App Homogeneous Coords