# Graphical Models

## parameter learning in undirected models

Siamak Ravanbakhsh

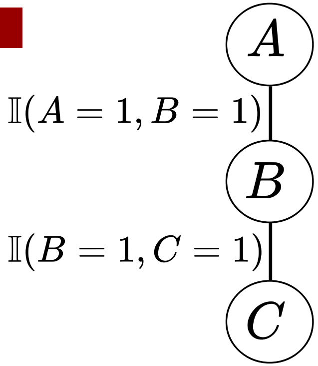
Winter 2018

# Learning objectives

- the form of likelihood for undirected models
  - why is it difficult to optimize?
- **conditional** likelihood in undirected models
- different approximations for parameter learning

# Likelihood in MRFs

example

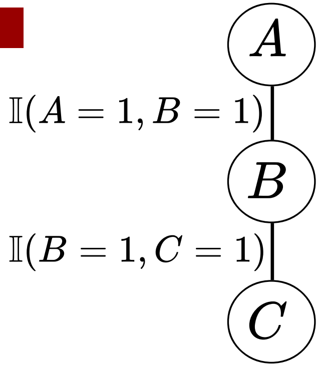


probability dist.

$$p(A, B, C; \theta) = \frac{1}{Z} \exp(\theta_1 \mathbb{I}(A = 1, B = 1) + \theta_2 \mathbb{I}(B = 1, C = 1))$$

# Likelihood in MRFs

example



probability dist.

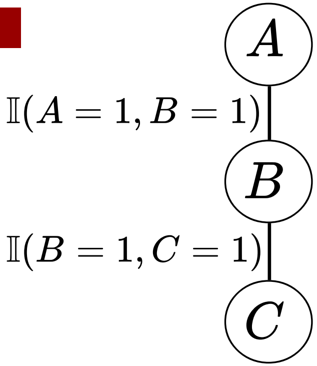
$$p(A, B, C; \theta) = \frac{1}{Z} \exp(\theta_1 \mathbb{I}(A = 1, B = 1) + \theta_2 \mathbb{I}(B = 1, C = 1))$$

observations $|\mathcal{D}| = 100$

- $\mathbb{E}_{\mathcal{D}}[\mathbb{I}(A = 1, B = 1)] = .4, \mathbb{E}_{\mathcal{D}}[\mathbb{I}(B = 1, C = 1)] = .4$

# Likelihood in MRFs

example



probability dist.

$$p(A, B, C; \theta) = \frac{1}{Z}\exp(\theta_1 \mathbb{I}(A=1, B=1) + \theta_2 \mathbb{I}(B=1, C=1))$$

observations $|\mathcal{D}| = 100$

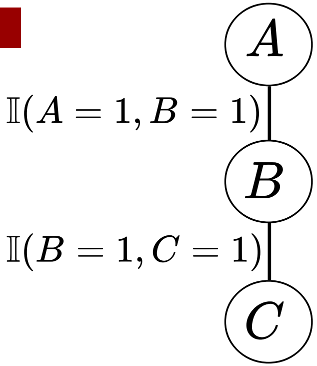
- $\mathbb{E}_{\mathcal{D}}[\mathbb{I}(A=1, B=1)] = .4, \mathbb{E}_{\mathcal{D}}[\mathbb{I}(B=1, C=1)] = .4$

log-likelihood: $\log p(\mathcal{D}; \theta) = \sum_{a,b,c \in \mathcal{D}} \theta_1 \mathbb{I}(a=1, b=1) + \theta_2 \mathbb{I}(b=1, c=1) - 100 \log Z(\theta)$

$$= 40\theta_1 + 40\theta_2 - 100 \log Z(\theta)$$

# Likelihood in MRFs

example



probability dist.

$$p(A,B,C;\theta) = \frac{1}{Z}\exp(\theta_1 \mathbb{I}(A=1,B=1) + \theta_2 \mathbb{I}(B=1,C=1))$$

observations $|\mathcal{D}| = 100$

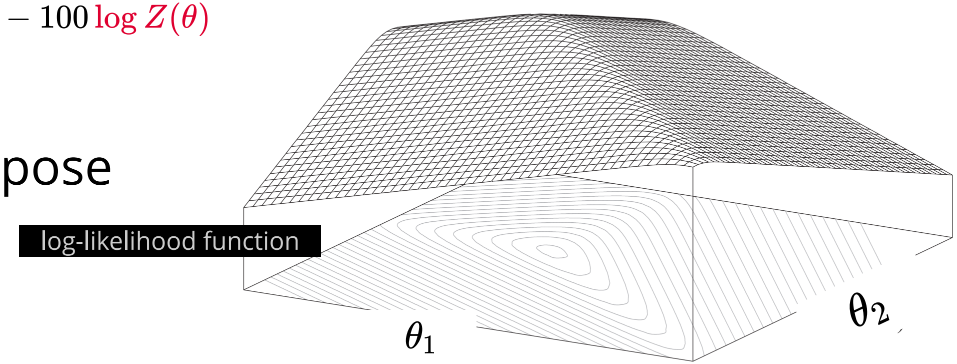
- $\mathbb{E}_{\mathcal{D}}[\mathbb{I}(A=1,B=1)] = .4, \mathbb{E}_{\mathcal{D}}[\mathbb{I}(B=1,C=1)] = .4$

log-likelihood: $\log p(\mathcal{D};\theta) = \sum_{a,b,c \in \mathcal{D}} \theta_1 \mathbb{I}(a=1,b=1) + \theta_2 \mathbb{I}(b=1,c=1) - 100 \log Z(\theta)$

$$= 40\theta_1 + 40\theta_2 - 100 \log Z(\theta)$$

because of the partition function

the likelihood does not decompose



# Likelihood in **linear exponential family** (log-linear models)

probability distribution $p(x;\theta) = \frac{1}{Z(\theta)} \exp(\langle \theta, \phi(x) \rangle)$

sufficient statistics

# Likelihood in linear exponential family (log-linear models)

probability distribution $p(x;\theta) = \frac{1}{Z(\theta)} \exp(\langle \theta, \phi(x) \rangle)$

sufficient statistics

log-likelihood of $\mathcal{D}$ $\ell(\mathcal{D}, \theta) = \log p(\mathcal{D}; \theta) = \sum_{x \in \mathcal{D}} \langle \theta, \phi(x) \rangle - |\mathcal{D}| \log Z(\theta)$

# Likelihood in **linear exponential family** (log-linear models)

probability distribution $p(x;\theta) = \frac{1}{Z(\theta)} \exp(\langle \theta, \phi(x) \rangle)$

sufficient statistics

log-likelihood of $\mathcal{D}$

$$\ell(\mathcal{D},\theta) = \log p(\mathcal{D};\theta) = \sum_{x \in \mathcal{D}} \langle \theta, \phi(x) \rangle - |\mathcal{D}| \log Z(\theta)$$

$$\ell(\mathcal{D},\theta) = |\mathcal{D}| \left( \langle \theta, \mathbb{E}_{\mathcal{D}}[\phi(x)] \rangle - \log Z(\theta) \right)$$

expected sufficient statistics $\mu_{\mathcal{D}}$

# Likelihood in linear exponential family (log-linear models)

probability distribution $p(x;\theta) = \frac{1}{Z(\theta)} \exp(\langle \theta, \phi(x) \rangle)$

sufficient statistics

log-likelihood of $\mathcal{D}$ $\ell(\mathcal{D}, \theta) = \log p(\mathcal{D}; \theta) = \sum_{x \in \mathcal{D}} \langle \theta, \phi(x) \rangle - |\mathcal{D}| \log Z(\theta)$

$$\ell(\mathcal{D}, \theta) = |\mathcal{D}| \left( \langle \theta, \mathbb{E}_{\mathcal{D}}[\phi(x)] \rangle - \log Z(\theta) \right)$$

expected sufficient statistics $\mu_{\mathcal{D}}$

**example**

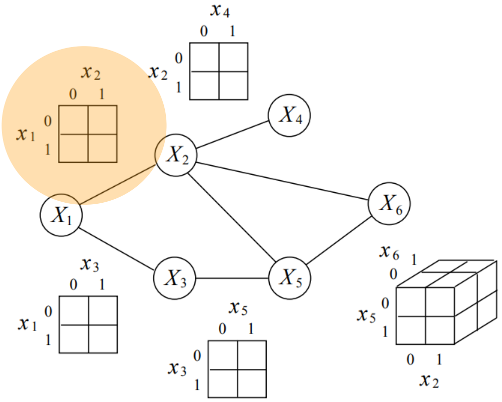


image: Michael Jordan's draft

| expected sufficient statistics | params. | sufficient statistics |
|---|---|---|
| $\mathbb{E}_{\mathcal{D}}[\mathbb{I}(X_1 = 0, X_2 = 0)] = P(X_1 = 0, X_2 = 0)$ | $\theta_{1,2,0,0}$ | $\mathbb{I}(X_1 = 0, X_2 = 0)$ |
| $\mathbb{E}_{\mathcal{D}}[\mathbb{I}(X_1 = 1, X_2 = 0)] = P(X_1 = 1, X_2 = 0)$ | $\theta_{1,2,1,0}$ | $\mathbb{I}(X_1 = 1, X_2 = 0)$ |
| $\mathbb{E}_{\mathcal{D}}[\mathbb{I}(X_1 = 0, X_2 = 1)] = P(X_1 = 0, X_2 = 1)$ | $\theta_{1,2,0,1}$ | $\mathbb{I}(X_1 = 0, X_2 = 1)$ |
| $\mathbb{E}_{\mathcal{D}}[\mathbb{I}(X_1 = 1, X_2 = 1)] = P(X_1 = 1, X_2 = 1)$ | $\theta_{1,2,1,1}$ | $\mathbb{I}(X_1 = 1, X_2 = 1)$ |

# Likelihood in **linear exponential family** (log-linear models)

probability distribution $p(x;\theta) = \frac{1}{Z(\theta)} \exp(\langle \theta, \phi(x) \rangle)$

sufficient statistics

log-likelihood of $\mathcal{D}$ $\ell(\mathcal{D}, \theta) = \log p(\mathcal{D}; \theta) = \sum_{x \in \mathcal{D}} \langle \theta, \phi(x) \rangle - |\mathcal{D}| \log Z(\theta)$

$$\ell(\mathcal{D}, \theta) = |\mathcal{D}| \left( \langle \theta, \mathbb{E}_{\mathcal{D}}[\phi(x)] \rangle - \log Z(\theta) \right)$$

expected sufficient statistics $\mu_{\mathcal{D}}$

$\log Z(\theta)$ has interesting properties

$$\frac{\partial}{\partial \theta_i} \log Z(\theta) = \frac{\frac{\partial}{\partial \theta_i} \sum_x \exp(\langle \theta, \phi(x) \rangle)}{Z(\theta)} = \frac{1}{Z(\theta)} \sum_x \phi_i(x) \exp(\langle \theta, \phi(x) \rangle) = \mathbb{E}_p[\phi_i(x)]$$

so $\nabla_\theta \log Z(\theta) = \mathbb{E}_\theta[\phi(x)]$

# Likelihood in linear exponential family (log-linear models)

probability distribution $p(x;\theta) = \frac{1}{Z(\theta)} \exp(\langle \theta, \phi(x) \rangle)$

sufficient statistics

log-likelihood of $\mathcal{D}$ $\ell(\mathcal{D},\theta) = \log p(\mathcal{D};\theta) = \sum_{x\in\mathcal{D}} \langle \theta, \phi(x) \rangle - |\mathcal{D}| \log Z(\theta)$

$$\ell(\mathcal{D},\theta) = |\mathcal{D}| \left( \langle \theta, \mathbb{E}_{\mathcal{D}}[\phi(x)] \rangle - \log Z(\theta) \right)$$

expected sufficient statistics $\mu_{\mathcal{D}}$

$\log Z(\theta)$ has interesting properties

$$\frac{\partial}{\partial\theta_i} \log Z(\theta) = \frac{\frac{\partial}{\partial\theta_i} \sum_x \exp(\langle\theta,\phi(x)\rangle)}{Z(\theta)} = \frac{1}{Z(\theta)} \sum_x \phi_i(x) \exp(\langle \theta, \phi(x) \rangle) = \mathbb{E}_p[\phi_i(x)]$$ so $\nabla_\theta \log Z(\theta) = \mathbb{E}_\theta[\phi(x)]$

$$\frac{\partial^2}{\partial\theta_i \partial\theta_j} \log Z(\theta) = \mathbb{E}[\phi_i(x)\phi_j(x)] - \mathbb{E}[\phi_i(x)]\mathbb{E}[\phi_j(x)] = Cov(\phi_i, \phi_j)$$

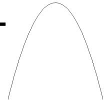
so the Hessian matrix is positive definite → $\log Z(\theta)$ is convex

# Likelihood in **linear exponential family** (log-linear models)

probability distribution $p(x;\theta) = \frac{1}{Z(\theta)} \exp(\langle \theta, \phi(x) \rangle)$

log-likelihood of $\mathcal{D}$

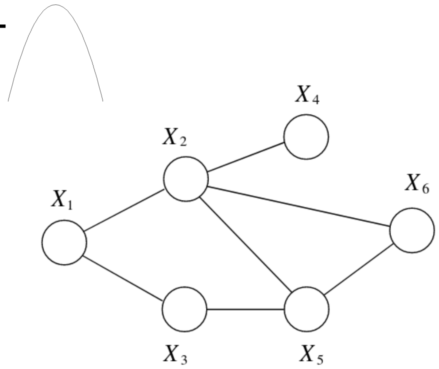
$$\ell(\mathcal{D}, \theta) = |\mathcal{D}| \underbrace{\underbrace{(\langle \theta, \mathbb{E}_{\mathcal{D}}[\phi(x)] \rangle}_{\text{linear in } \theta} - \underbrace{\log Z(\theta))}_{\text{convex}}}_{\text{concave}}$$



# Likelihood in **linear exponential family** (log-linear models)

probability distribution $p(x;\theta) = \frac{1}{Z(\theta)} \exp(\langle \theta, \phi(x) \rangle)$

log-likelihood of $\mathcal{D}$ $\ell(\mathcal{D},\theta) = |\mathcal{D}| \left( \underbrace{\underbrace{\langle \theta, \mathbb{E}_{\mathcal{D}}[\phi(x)] \rangle}_{\text{linear in } \theta} - \underbrace{\log Z(\theta)}_{\text{convex}}}_{\text{concave}} \right)$



should be easy to maximize (?) NO!

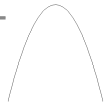
- estimating $Z(\theta)$ is a difficult inference problem
- how about just using the gradient info?
  - involves inference as well $\nabla_\theta \log Z(\theta) = \mathbb{E}_\theta[\phi(x)]$
    - any combination of inference-gradient based optimization for learning undirected models

# **Moment matching** for linear exponential family

probability distribution $p(x;\theta) = \frac{1}{Z(\theta)} \exp(\langle \theta, \phi(x) \rangle)$

log-likelihood of $\mathcal{D}$

$$\ell(\mathcal{D}, \theta) = |\mathcal{D}| \, \underbrace{(\underbrace{\langle \theta, \mathbb{E}_{\mathcal{D}}[\phi(x)] \rangle}_{\text{linear in } \theta} - \underbrace{\log Z(\theta))}_{\text{convex}}}_{\text{concave}}$$



set its derivative to zero $\nabla_\theta \ell(\theta, \mathcal{D}) = |\mathcal{D}|(\mathbb{E}_{\mathcal{D}}[\phi(x)] - \mathbb{E}_{p_\theta}[\phi(x)]) = 0$

$$\Rightarrow \mathbb{E}_{p_\theta}[\phi(x)] = \mathbb{E}_{\mathcal{D}}[\phi(x)]$$

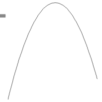
find the parameter $\theta$

that results in the same expected sufficient statistics as the data

# Moment matching for linear exponential family

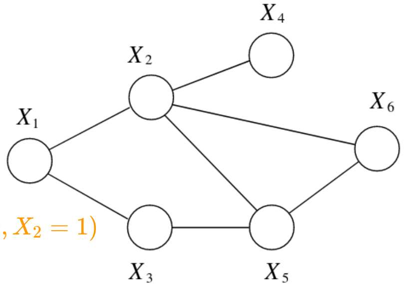
probability distribution $p(x;\theta) = \frac{1}{Z(\theta)} \exp(\langle \theta, \phi(x) \rangle)$

log-likelihood of $\mathcal{D}$ $\ell(\mathcal{D}, \theta) = \underbrace{\underbrace{|\mathcal{D}| \left( \langle \theta, \mathbb{E}_{\mathcal{D}}[\phi(x)] \rangle \right.}_{\text{linear in } \theta} - \underbrace{\left. \log Z(\theta) \right)}_{\text{convex}}}_{\text{concave}}$



set its derivative to zero $\nabla_\theta \ell(\theta, \mathcal{D}) = |\mathcal{D}| (\mathbb{E}_{\mathcal{D}}[\phi(x)] - \mathbb{E}_{p_\theta}[\phi(x)]) =$

$\Rightarrow \mathbb{E}_{p_\theta}[\phi(x)] = \mathbb{E}_{\mathcal{D}}[\phi(x)]$



$p(X_1 = 0, X_2 = 1; \theta) = p_{\mathcal{D}}(X_1 = 0, X_2 = 1)$

find the parameter $\theta$

that results in the same expected sufficient statistics as the data

# Learning needs inference in an inner loop

maximizing the likelihood: $\arg\max_\theta \log p(\mathcal{D}|\theta)$

- gradient $\propto \mathbb{E}_{\mathcal{D}}[\phi(x)] - \mathbb{E}_{p_\theta}[\phi(x)]$
- optimality condition

$$\underset{\substack{\downarrow\\ \text{easy to calculate}}}{\mathbb{E}_{\mathcal{D}}[\phi(x)]} = \underset{\substack{\downarrow\\ \text{inference in the graphical model}}}{\mathbb{E}_{p_\theta}[\phi(x)]}$$

example: in discrete pairwise MRF

$$\underset{\substack{\downarrow\\ \text{empirical marginals}}}{p_{\mathcal{D}}(x_i, x_j)} = \underset{\substack{\downarrow\\ \text{marginals in our current model}}}{p(x_i, x_j; \theta)} \quad \forall i, j \in \mathcal{E}$$

# Learning needs inference in an inner loop

maximizing the likelihood: $\arg\max_\theta \log p(\mathcal{D}|\theta)$

- gradient $\propto \mathbb{E}_{\mathcal{D}}[\phi(x)] - \mathbb{E}_{p_\theta}[\phi(x)]$
- optimality condition

$$\underset{\substack{\downarrow \\ \text{easy to calculate}}}{\mathbb{E}_{\mathcal{D}}[\phi(x)]} = \underset{\substack{\downarrow \\ \text{inference in the graphical model}}}{\mathbb{E}_{p_\theta}[\phi(x)]}$$

example: in discrete pairwise MRF

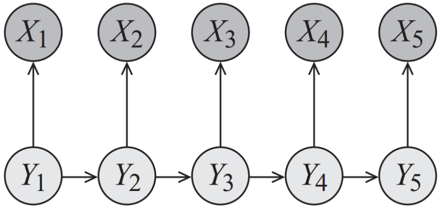
$$\underset{\substack{\downarrow \\ \text{empirical marginals}}}{p_{\mathcal{D}}(x_i, x_j)} = \underset{\substack{\downarrow \\ \text{marginals in our current model}}}{p(x_i, x_j; \theta)} \quad \forall i, j \in \mathcal{E}$$

what if exact inference is infeasible?

- learning with approx. inference often $\equiv$ exact optimization of approx. objective
  - use sampling, variational inference ...

# Conditional training

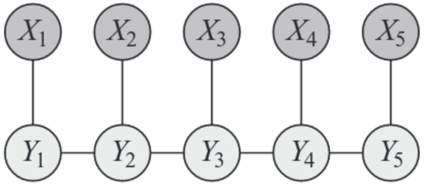
**Recall** generative vs. discriminative training



Hidden Markov Model (HMM) trained generatively

$$\ell(\mathcal{D}, \theta) = \sum_{(x,y)\in\mathcal{D}} \log p(x, y)$$

- easy to train the Bayes-net
- the likelihood decomposes



Conditional random fields (CRF)

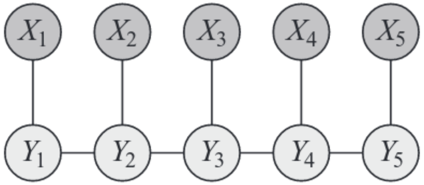
- trained discriminatively
- maximizing conditional log-likelihood

$$\ell_{Y|X}(\mathcal{D}, \theta) = \sum_{(x,y)\in\mathcal{D}} \log p(y|x)$$

- how to maximize this?

# Conditional training

objective: $\arg\max_\theta \ell_{Y|X}(\mathcal{D},\theta) = \arg\max_\theta \sum_{(x,y)\in\mathcal{D}} \log p(y|x)$



again consider the gradient

$$\nabla_\theta \ell_{Y|X}(\mathcal{D},\theta) = \sum_{(x',y')\in\mathcal{D}} \phi(x',y') - \mathbb{E}_{p(.|x;\theta)}[\phi(x',y)]$$

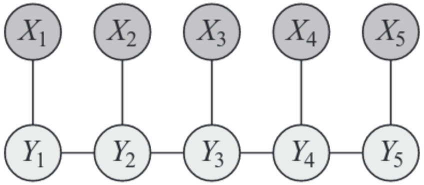
- conditional expectation of sufficient statistics
- it's conditioned on the observed x'

to obtain the gradient:

- for each instance $(x,y)\in\mathcal{D}$
  - run inference conditioned on x

# Conditional training

objective: $\arg\max_\theta \ell_{Y|X}(\mathcal{D},\theta) = \arg\max_\theta \sum_{(x,y)\in\mathcal{D}} \log p(y|x)$



again consider the gradient

$$\nabla_\theta \ell_{Y|X}(\mathcal{D},\theta) = \sum_{(x',y')\in\mathcal{D}} \phi(x',y') - \mathbb{E}_{p(.|x;\theta)}[\phi(x',y)]$$

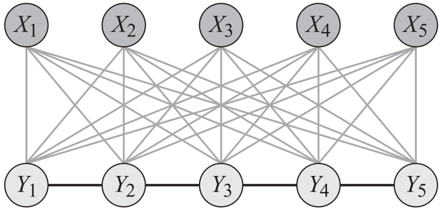
- conditional expectation of sufficient statistics
- it's conditioned on the observed x'

to obtain the gradient:

- for each instance $(x,y)\in\mathcal{D}$
  - run inference conditioned on x
- compared to generative training in undirected models

**pro:** conditioning could simplify inference

**con:** have to run inference for each datapoint



inference on the reduced MRF is easy in this case

# Local priors & regularization

max-likelihood can lead to over-fitting

Bayesian approach:

- in Bayes-nets: decomposed prior $p(\theta)$ → decomposed posterior $p(\theta \mid \mathcal{D})$
- in Markov nets: posterior does not decompose (because of the likelihood)

# Local priors & regularization

max-likelihood can lead to over-fitting

Bayesian approach:

- in Bayes-nets: decomposed prior $p(\theta)$ → decomposed posterior $p(\theta \mid \mathcal{D})$
- in Markov nets: posterior does not decompose (because of the likelihood)

**alternative** to a full-Bayesian approach

MAP inference: find the maximum of the posterior $\arg\max_\theta \log p(\mathcal{D}|\theta) + \underline{\log p(\theta)}$

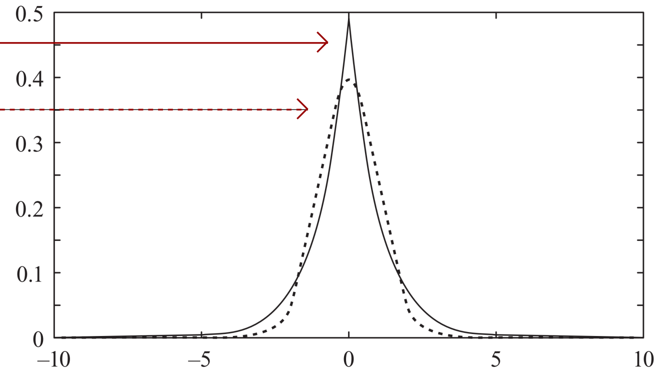
- serves as a regularization
- does not have to be conjugate

- does not model uncertainty
- sensitive to parametrization

# Gaussian & Laplacian priors

MAP inference: find the maximum of the posterior $\arg\max_\theta \log p(\mathcal{D}|\theta) + \log p(\theta)$

$p(\theta)$
- the product of univariate Laplacian (L1 reg.)
- the product of univariate Gaussian (L2 reg.)

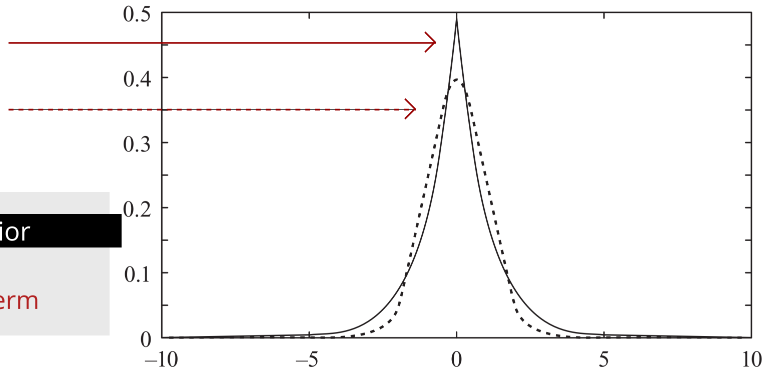


# Gaussian & Laplacian priors

MAP inference: find the maximum of the posterior $\arg\max_\theta \log p(\mathcal{D}|\theta) + \log p(\theta)$

$p(\theta)$

- the product of univariate Laplacian (L1 reg.)
- the product of univariate Gaussian (L2 reg.)



$$p(\theta \mid \sigma) \propto \prod_i \exp(-\frac{\theta_i^2}{2\sigma^2}) \Rightarrow$$

**Gaussian prior**

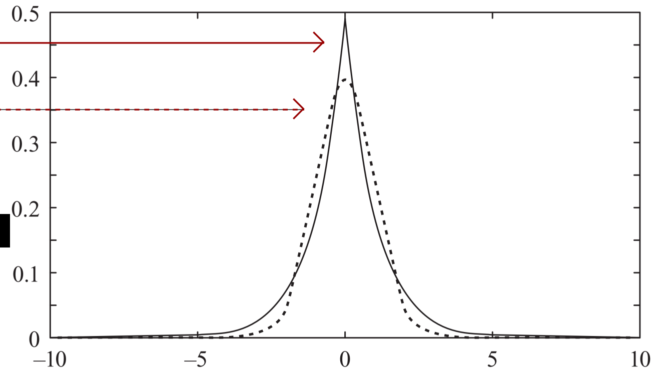
$$\log p(\theta \mid \sigma) = -\frac{1}{2\sigma^2}\sum_i \theta_i^2 + c$$

L2 regularization penalty term

# Gaussian & Laplacian priors

MAP inference: find the maximum of the posterior $\arg\max_\theta \log p(\mathcal{D}|\theta) + \log p(\theta)$

$p(\theta)$
- the product of univariate Laplacian (L1 reg.)
- the product of univariate Gaussian (L2 reg.)



$$p(\theta \mid \sigma) \propto \prod_i \exp(-\frac{\theta_i^2}{2\sigma^2}) \Rightarrow$$

Gaussian prior

$$\log p(\theta \mid \sigma) = -\frac{1}{2\sigma^2} \sum_i \theta_i^2 + c$$

L2 regularization penalty term

$$p(\theta \mid \beta) = \prod_i \frac{1}{2\beta} \exp(-\frac{|\theta_i|}{\beta}) \Rightarrow$$

Laplacian prior

sparsity-inducing

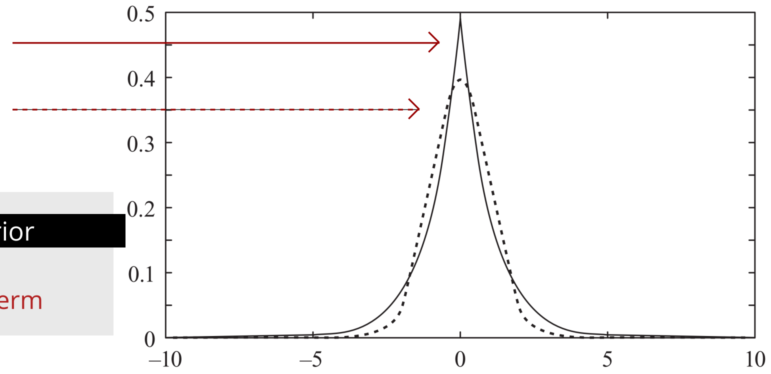
$$\log p(\theta \mid \beta) = -\frac{1}{\beta} \sum_i |\theta_i|$$

L1 regularization penalty term

# Gaussian & Laplacian priors

MAP inference: find the maximum of the posterior $\arg\max_\theta \log p(\mathcal{D}|\theta) + \log p(\theta)$

$p(\theta)$
- the product of univariate Laplacian (L1 reg.)
- the product of univariate Gaussian (L2 reg.)



$$p(\theta \mid \sigma) \propto \prod_i \exp(-\frac{\theta_i^2}{2\sigma^2}) \Rightarrow$$

**Gaussian prior**

$$\log p(\theta \mid \sigma) = -\frac{1}{2\sigma^2} \sum_i \theta_i^2 + c$$

L2 regularization penalty term

$$p(\theta \mid \beta) = \prod_i \frac{1}{2\beta} \exp(-\frac{|\theta_i|}{\beta}) \Rightarrow$$

**Laplacian prior** sparsity-inducing

$$\log p(\theta \mid \beta) = -\frac{1}{\beta} \sum_i |\theta_i|$$

L1 regularization penalty term

- both of these bias the posterior towards smaller parameters
  - why is this a good idea?

# Pseudo-moment matching

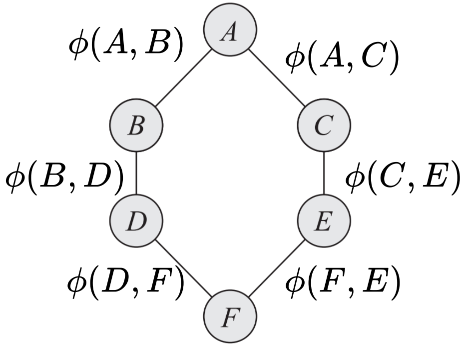
we want to set the parameters $\theta$ such that

if/when loopy BP converges:

$$p_{\mathcal{D}}(A, B) = \hat{p}(A, B; \theta), p_{\mathcal{D}}(B, D) = \hat{p}(B, D; \theta) \dots$$

empirical marginals

marginals using BP



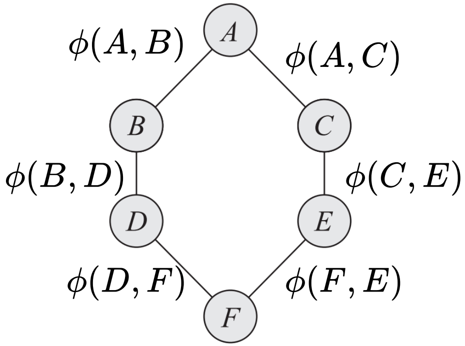
# Pseudo-moment matching

we want to set the parameters $\theta$ such that

if/when loopy BP converges:

$$p_{\mathcal{D}}(A, B) = \hat{p}(A, B; \theta), p_{\mathcal{D}}(B, D) = \hat{p}(B, D; \theta) \ldots$$

empirical marginals marginals using BP



**idea:** use the reparametrization in BP

$$p(A, B, C, D, E, F) \propto \frac{\hat{p}(A,B)\ldots\hat{p}(C,A)}{\hat{p}(A)\ldots\hat{p}(F)}$$

→ *product of clique marginals*

→ *cancel the double-counts*

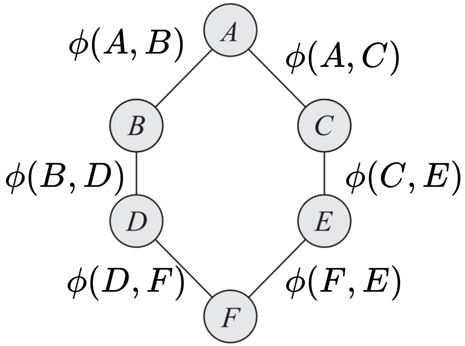
# Pseudo-moment matching

we want to set the parameters $\theta$ such that

if/when loopy BP converges:

$$p_{\mathcal{D}}(A,B) = \hat{p}(A,B;\theta), p_{\mathcal{D}}(B,D) = \hat{p}(B,D;\theta)\dots$$

empirical marginals marginals using BP



**idea:** use the reparametrization in BP

☀ $p(A,B,C,D,E,F) \propto \frac{\hat{p}(A,B)\dots\hat{p}(C,A)}{\hat{p}(A)\dots\hat{p}(F)}$

*product of clique marginals*
*cancel the double-counts*

set the factors using empirical marginals

- e.g., $\phi(A,B) \leftarrow p_{\mathcal{D}}(A,B)/p_{\mathcal{D}}(A)$
- each term in the numerator & denominator of ☀ should be used exactly once
- if we run BP on the resulting model we will have $p_{\mathcal{D}}(A,B) = \hat{p}(A,B;\theta), p_{\mathcal{D}}(B,D) = \hat{p}(B,D;\theta)\dots$

# Pseudo-likelihood

log-likelihood: $\log p(\mathcal{D};\theta) = \sum_{x\in\mathcal{D}} \sum_i \log p(x_i | x_1, \ldots, x_{i-1}; \theta)$ using the chain rule

# Pseudo-likelihood

log-likelihood: $\log p(\mathcal{D};\theta) = \sum_{x\in\mathcal{D}} \sum_i \log p(x_i|x_1,\dots,x_{i-1};\theta)$ using the chain rule

pseudo log-likelihood is an approximation

$[x_1,\dots,x_{i-1},x_{i+1},\dots,x_n]$

$$\log p(\mathcal{D};\theta) \approx \sum_{x\in\mathcal{D}} \sum_i \log p(x_i|x_{-i};\theta)$$

# Pseudo-likelihood

log-likelihood: $\log p(\mathcal{D};\theta) = \sum_{x\in\mathcal{D}} \sum_i \log p(x_i|x_1,\ldots,x_{i-1};\theta)$ using the chain rule

pseudo log-likelihood is an approximation

$$\log p(\mathcal{D};\theta) \approx \sum_{x\in\mathcal{D}} \sum_i \log \underbrace{p(x_i|\overset{[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]}{x_{-i}};\theta)}_{\frac{p(x;\theta)}{\sum_{x_i} p(x;\theta)} = \frac{\tilde{p}(x;\theta)}{\sum_{x_i} \tilde{p}(x;\theta)}}$$

eliminated the normalization constant

# Pseudo-likelihood

log-likelihood: $\log p(\mathcal{D};\theta) = \sum_{x\in\mathcal{D}} \sum_i \log p(x_i|x_1,\ldots,x_{i-1};\theta)$ using the chain rule

pseudo log-likelihood is an approximation

$$\log p(\mathcal{D};\theta) \approx \sum_{x\in\mathcal{D}} \sum_i \log \underline{p(x_i|\overset{[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]}{x_{-i}};\theta)}_{\frac{p(x;\theta)}{\sum_{x_i} p(x;\theta)} = \frac{\tilde{p}(x;\theta)}{\sum_{x_i} \tilde{p}(x;\theta)}}$$

eliminated the normalization constant

this assumption simplifies the gradient:

- instead of calculating $\sum_{x\in\mathcal{D}} \phi_k(x) \; - \; |\mathcal{D}| \, \mathbb{E}_{p_\theta}[\phi_k(x)]$ expensive!
- use $\sum_{x\in\mathcal{D}} \phi_k(x) - \sum_i \mathbb{E}_{p(.|x_{-i})}[\phi_k(x'_i, x_{-i})]$ can be further simplified using Markov blanket for each node...
- **upshot:** only conditional expectations are used (tractable!)

# Pseudo-likelihood

log-likelihood: $\log p(\mathcal{D};\theta) = \sum_{x\in\mathcal{D}} \sum_i \log p(x_i|x_1,\ldots,x_{i-1};\theta)$ using the chain rule

pseudo log-likelihood is an approximation

$$\log p(\mathcal{D};\theta) \approx \sum_{x\in\mathcal{D}} \sum_i \log \underbrace{p(x_i|\overset{[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]}{x_{-i}};\theta)}_{\frac{p(x;\theta)}{\sum_{x_i} p(x;\theta)} = \frac{\tilde{p}(x;\theta)}{\sum_{x_i} \tilde{p}(x;\theta)}}$$

eliminated the normalization constant

this assumption simplifies the gradient:

- instead of calculating $\sum_{x\in\mathcal{D}} \phi_k(x) - |\mathcal{D}|\,\mathbb{E}_{p_\theta}[\phi_k(x)]$ expensive!
- use $\sum_{x\in\mathcal{D}} \phi_k(x) - \sum_i \mathbb{E}_{p(.|x_{-i})}[\phi_k(x'_i, x_{-i})]$ can be further simplified using Markov blanket for each node...
- **upshot:** only conditional expectations are used (tractable!)

at the limit of large data, this is exact!

# Pseudo-likelihood

log-likelihood: $\log p(\mathcal{D};\theta) = \sum_{x\in\mathcal{D}} \sum_i \log p(x_i|x_1,\ldots,x_{i-1};\theta)$ using the chain rule

pseudo log-likelihood is an approximation

$$\log p(\mathcal{D};\theta) \approx \sum_{x\in\mathcal{D}} \sum_i \log p(x_i|x_{-i};\theta)$$

this assumption simplifies the gradient

at the limit of large data, this is exact!

a combination of

- pseudo likelihood

+

- Laplacian prior

one of the most efficient methods for structure learning

# Contrastive methods

log-likelihood: $\log p(\mathcal{D};\theta) = \sum_{x\in\mathcal{D}} \log \tilde{p}(x;\theta) - \log Z(\theta)$

increase the unnormalize prob. of the data

- it's easy to evaluate: e.g, $\langle \theta, \phi(x) \rangle$

keep the total sum of unnormalized probabilities small $\log \sum_x \tilde{p}(x;\theta)$

- sum over exponentially many terms

# Contrastive methods

log-likelihood: $\log p(\mathcal{D};\theta) = \sum_{x\in\mathcal{D}} \log \tilde{p}(x;\theta) - \log Z(\theta)$

increase the unnormalize prob. of the data

- it's easy to evaluate: e.g, $\langle\theta, \phi(x)\rangle$

keep the total sum of unnormalized probabilities small $\log \sum_x \tilde{p}(x;\theta)$

- sum over exponentially many terms

contrastive methods: replace $\log Z(\theta)$ with a tractable alternative

- **contrastive divergence minimization:** only look at a small neighborhood of the data
- **margin-based training:** consider $\log \max_{x'\neq x} \tilde{p}(x';\theta)$
  - only for conditional training

# Summary

- parameter learning in MRFs is difficult
    - normalization constant ties the parameters together
        - likelihood does not decompose
        - Bayesian inference is also difficult

# Summary

- parameter learning in MRFs is difficult
  - normalization constant ties the parameters together
    - likelihood does not decompose
    - Bayesian inference is also difficult
- (conditional) log-likelihood is convex
  - gradient steps: need inference on the current model
  - global optima satisfies moment-matching condition
  - combine inference methods + gradient descent for learning

# Summary

- parameter learning in MRFs is difficult
    - normalization constant ties the parameters together
        - likelihood does not decompose
        - Bayesian inference is also difficult
- (conditional) log-likelihood is convex
    - gradient steps: need inference on the current model
    - global optima satisfies moment-matching condition
    - combine inference methods + gradient descent for learning
- alternative approaches:
    - *pseudo moment matching, pseudo likelihood, contrastive divergence, margin-based training*