Graphical Models

Clique trees & Belief Propagation

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Winter 2018

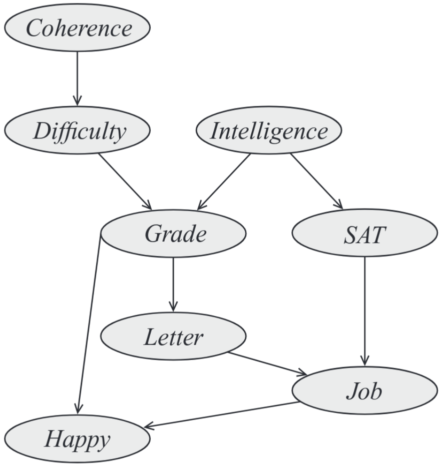
Learning objectives

- message passing on clique trees
- its relation to variable elimination
- two different forms of belief propagation

Recap: variable elimination (VE)

- marginalize over a subset - e.g., $P(J)$
- expensive to calculate (why?) $\sum_{C,D,I,G,S,L,H} P(C,D,I,G,S,L,J,H)$
- use the factorized form of P

$$\sum_{C,D,I,G,S,L} P(D|C)P(G|D,I)P(S|I)P(L|G)P(J|L,S)P(H|G,J)$$



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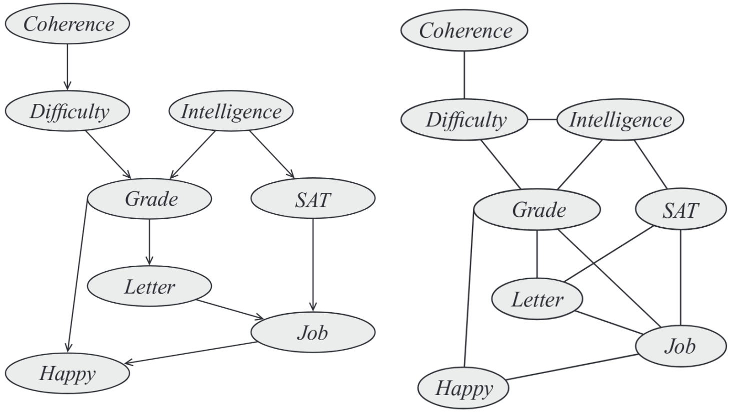
$$\downarrow$$

$$\phi_2(H, G, J)$$



note that they do not encode the same CIs

think of this as a factor/potential
same treatment of

- Bayes-nets
- Markov nets

for inference

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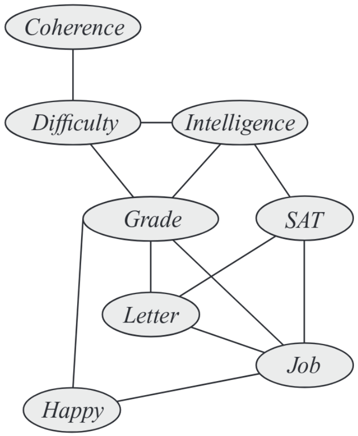
- use the factorized form of P

$$\sum_{C,D,I,G,S,L} \phi_1(D,C)\phi_2(G,D,I)\phi_3(S,I)\phi_4(L,G)\phi_5(J,L,S)\phi_6(H,G,J)$$



$$= .... \sum_I \phi_3(S|I) \sum_D \phi_2(G,D,I) \sum_C \phi_1(D,C)$$

$\psi'_1(D)$ $\psi_1(D,C)$

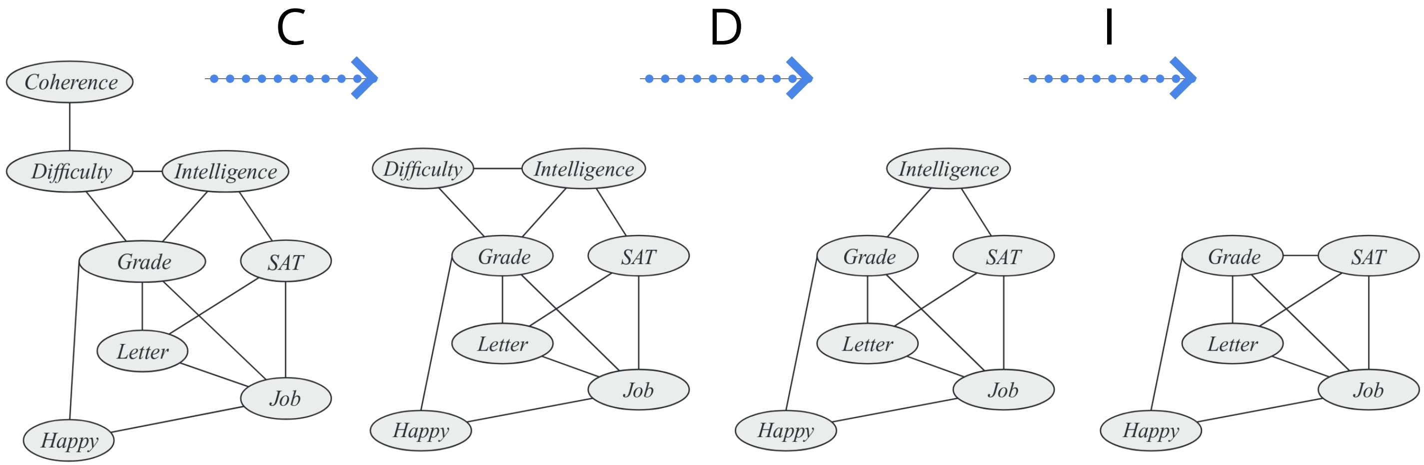
- repeat this

$$= .... \sum_I \phi_3(S,I) \sum_D \phi_2(G,D,I)\psi'_1(D)$$

$\psi'_2(G,I)$ $\psi_2(G,I,D)$

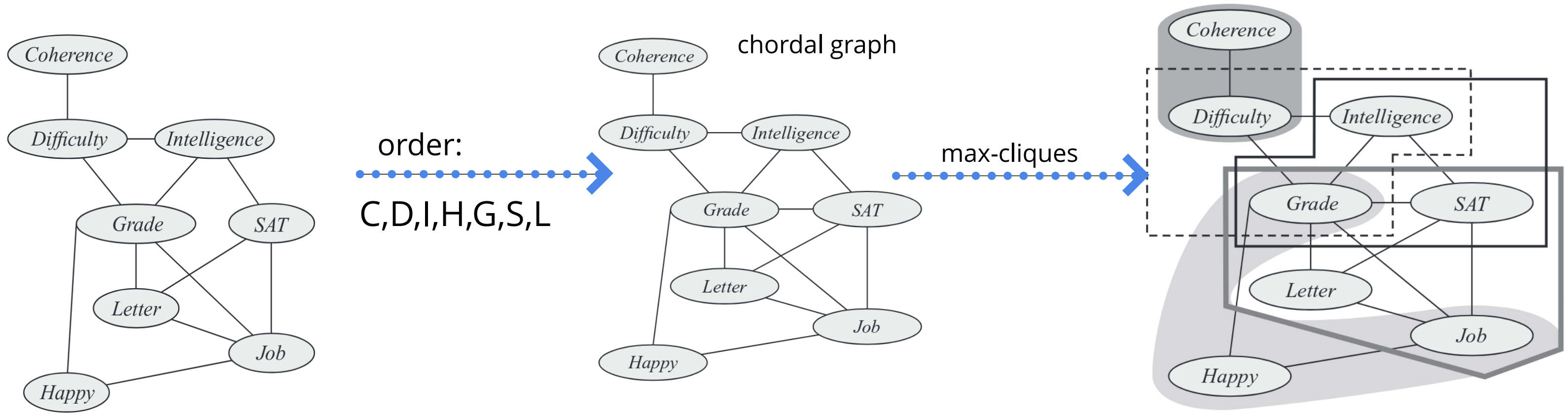
Recap: variable elimination (VE)

- marginalize over a subset - e.g., $P(J)$
- expensive to calculate (why?) $\sum_{C,D,I,G,S,L,H} P(C,D,I,G,S,L,J,H)$
- eliminate variables in some order



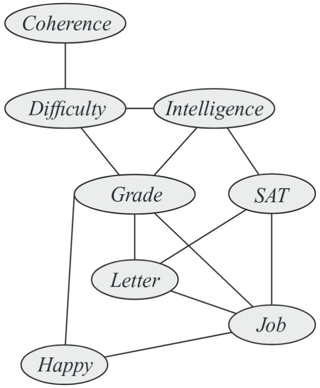
Recap: variable elimination (VE)

- eliminate variables in some order
- creates a chordal graph
- maximal cliques are the factors created during VE (ψ_t)

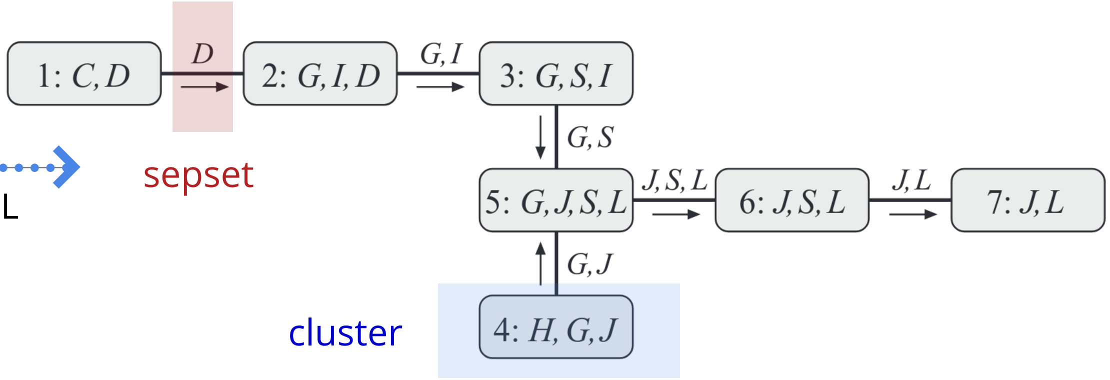


Clique-tree

- summarize the *VE computation* using a clique-tree



order:
C,D,I,H,G,S,L



sepset

cluster

- clusters are maximal cliques (factors that are marginalized)

$$C_i = Scope[\psi_i]$$

$$P(J) = ... \sum_I P(S|I) \sum_D P(G|D,I) \sum_C P(D|C)$$

$\psi'_1(D)$ $\psi_1(D,C)$

Clique-tree

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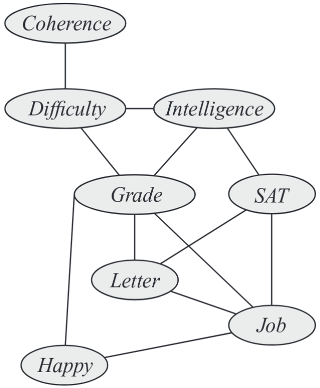

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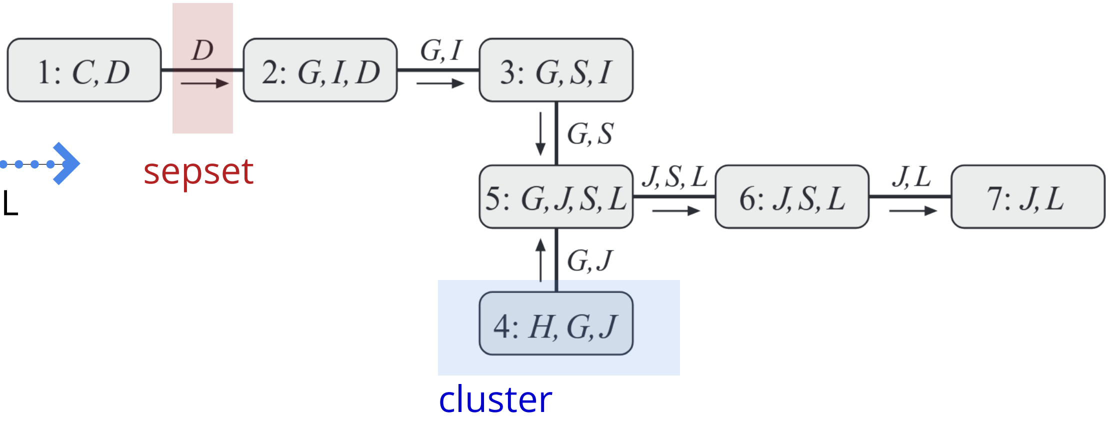


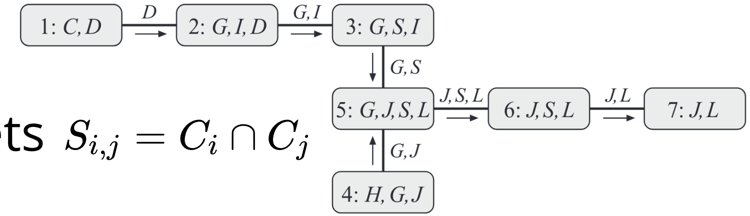
- clusters are maximal cliques (factors that are marginalized)
- sepsets are the result of marginalization over cliques

$$C_i = Scope[\psi_i]$$

$$S_{i,j} = Scope[\psi_i']$$

$$S_{i,j} = C_i \cap C_j$$

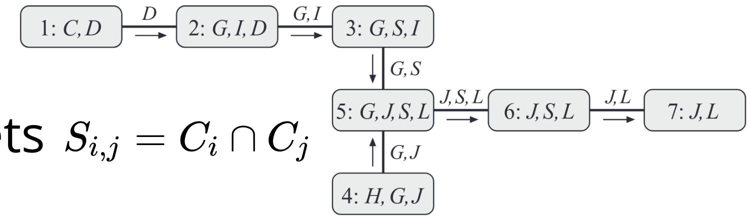
Clique-tree: properties



- a tree $\mathcal{T}$ from clusters C_i and sepsets $S_{i,j} = C_i \cap C_j$

- **family-preserving property:** $\alpha(\phi) = j$
  - each factor ϕ is associated with a cluster C_j s.t. $Scope[\phi] \subseteq C_j$

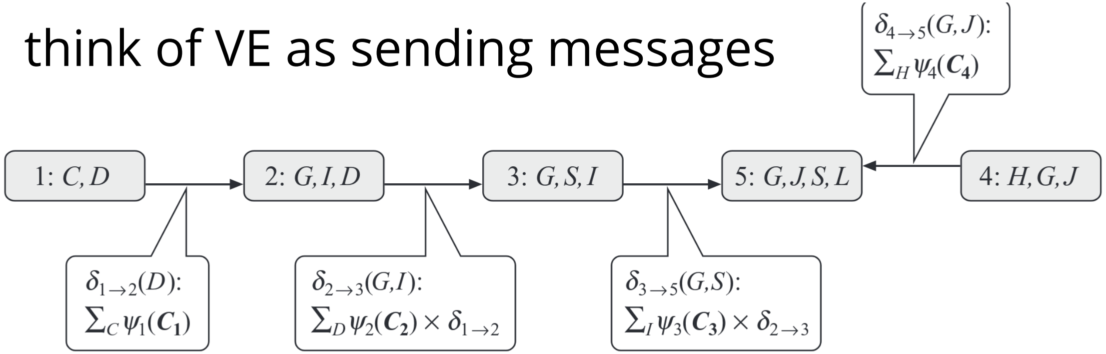
Clique-tree: properties

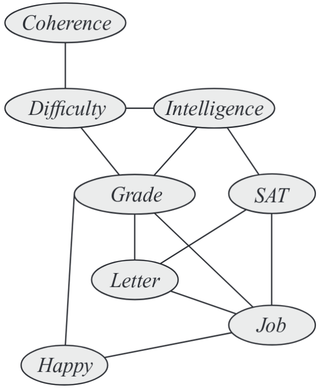


- a tree $\mathcal{T}$ from clusters C_i and sepsets $S_{i,j} = C_i \cap C_j$
- **family-preserving property:** $\alpha(\phi) = j$
  - each factor ϕ is associated with a cluster C_j s.t. $Scope[\phi] \subseteq C_j$
- **running intersection property:**
  - if $X \in C_i, C_j$ then $X \in C_k$ for C_k in the path $C_i \to \ldots \to C_j$

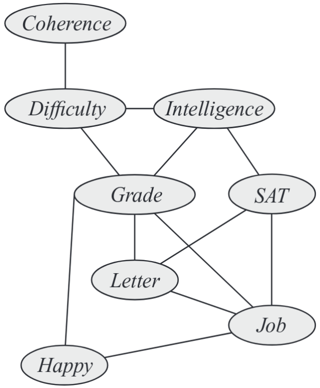
VE as message passing

- think of VE as sending messages





VE as message passing



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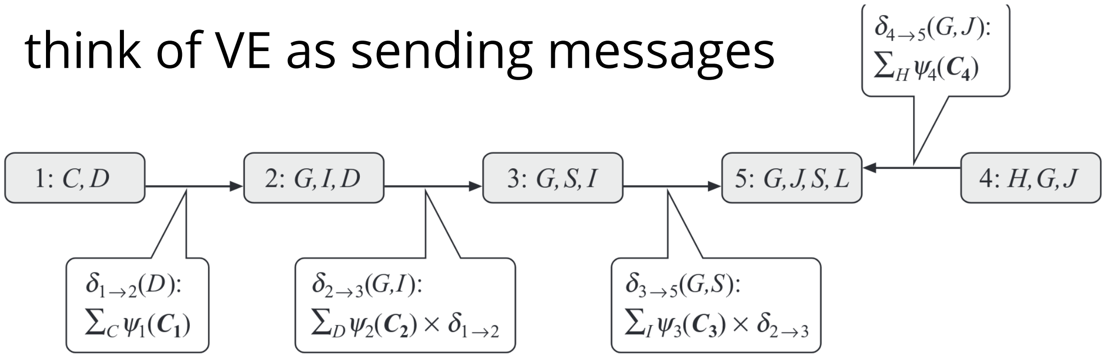


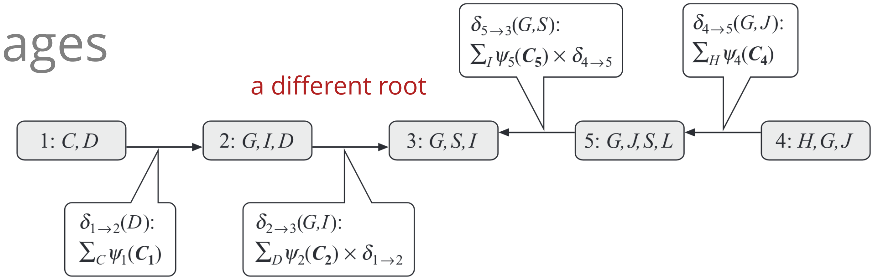
calculate the product of factors in each clique $\psi_i(C_i) \triangleq \prod_{\phi:\alpha(\phi)=i} \phi$

send messages from the leaves towards a root:

$$\delta_{i\to j}(S_{i,j}) = \sum_{C_i - S_{i,j}} \psi_i(C_i) \prod_{k \in Nb_i - j} \delta_{k\to i}(S_{i,k})$$

message passing

- think of VE as sending messages



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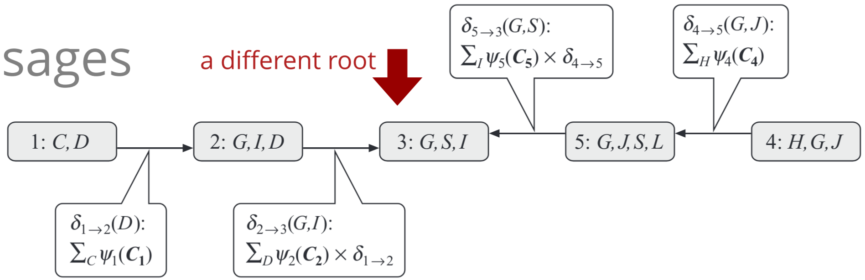
$$= \sum_{\mathcal{V}_{\prec(i\to j)}} \prod_{\phi\in\mathcal{F}_{\prec(i\to j)}} \phi$$

- the message is the marginal from one side of the tree

message passing

- think of VE as sending messages



- send messages from the leaves towards a root:

$$\delta_{i\to j}(S_{i,j}) \triangleq \sum_{C_i - S_{i,j}} \psi_i(C_i) \prod_{k\in Nb_i - j} \delta_{k\to i}(S_{i,k}) \;=\; \sum_{\mathcal{V}_{\prec(i\to j)}} \prod_{\phi\in\mathcal{F}_{\prec(i\to j)}} \phi$$

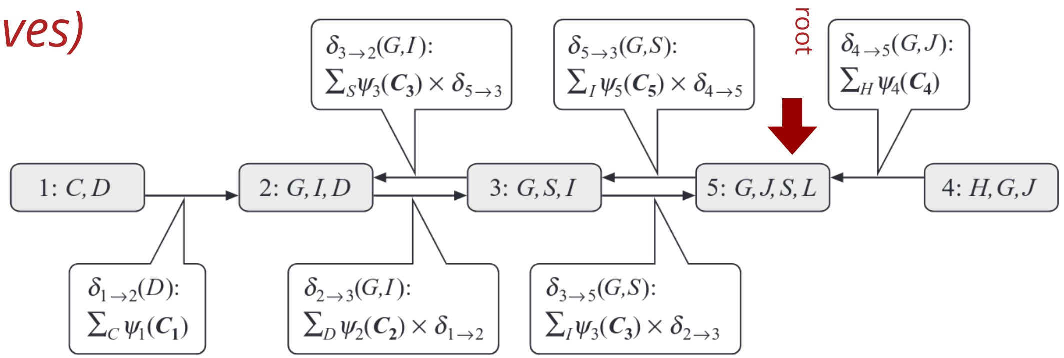
- the belief at the root clique is $\beta_r(C_r) \triangleq \psi_r(C_r) \prod_{k\in Nb_r} \delta_{k\to r}(S_{r,k})$

  proportional to the marginal $\beta_r(C_r) \propto \sum_{\mathbf{X}-C_i} P(\mathbf{X})$

message passing: downward pass

- what if we continue sending messages?

(from the root to leaves)



- clique **i** sends a message to clique **j** when received messages from all the **other** neighbors **k**

message passing: downward pass

- what if we continue sending messages?

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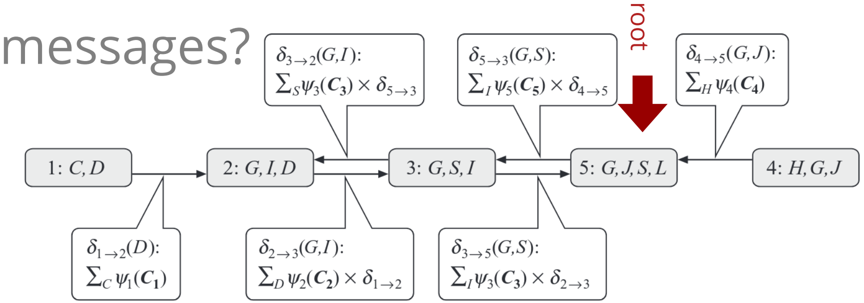

- sum-product **belief propagation** (BP)

$$\delta_{i\to j}(S_{i,j}) = \sum_{C_i - S_{i,j}} \psi_i(C_i) \prod_{k\in Nb_i - j} \delta_{k\to i}(S_{i,k})$$

async. message update

$$\mu_{i,j}(S_{i,j}) \triangleq \delta_{i\to j}(S_{i,j})\delta_{j\to i}(S_{i,j})$$

$$\beta_i(C_i) \triangleq \psi_i(C_i) \prod_{k\in Nb_i} \delta_{k\to i}(S_{i,k})$$

marginals

for any clique (not only root)

Clique-tree & queries

What **type of queries** can we answer?

- marginals over subset of cliques $P(A) \quad A \subseteq C_i$
- updating the beliefs after new evidence $P(A \mid E^{(t)} = e^{(t)}) \quad A \subseteq C_i, E \subseteq C_j$
  - multiply the *(previously calibrated)* beliefs $\beta(C_i)\mathbb{I}(E^{(t)} = e^{(t)})$
  - propagate to recalibrate

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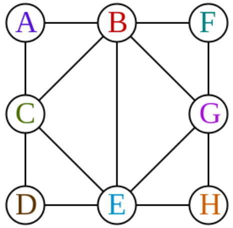
  - multiply the *(previously calibrated)* beliefs $\beta(C_i)\mathbb{I}(E^{(t)} = e^{(t)})$
  - propagate to recalibrate
- marginals outside cliques: $P(A, B) \quad A \subseteq C_i, B \subseteq C_j$
  - define a super-clique that has both A,B
  - a more efficient alternative?
- partition function Z

Chordal graph and clique-tree

any chordal graph gives a clique-tree



how to get a chordal graph? triangulation

- use the chordal graph from VE
  - min-neighbor, min-fill ...
- or find the optimal chordal graph
  - smallest tree-width
  - also smallest max-clique

NP-hard

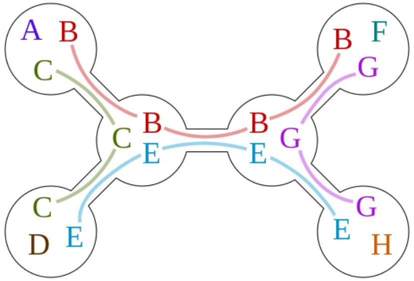
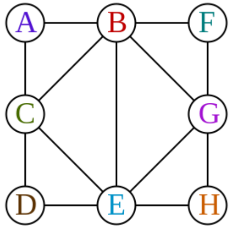


image: wikipedia

Chordal graph and clique-tree

Chordal graph = Markov $\cap$ Bayesian networks



from MRF to Bayes-net:

- triangulate
- build a clique-tree
- **within cliques:** fully connected directed edges
- **between cliques:** from a root to leaves

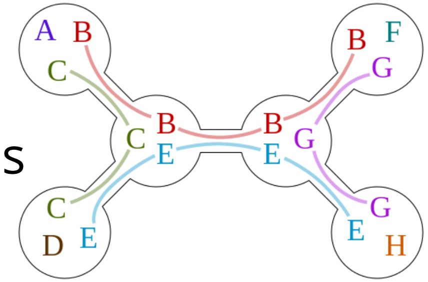
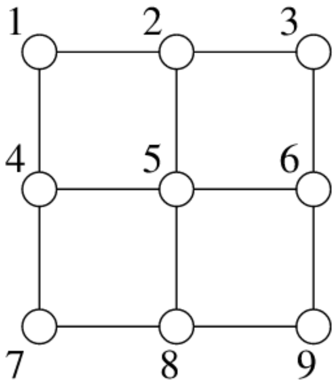


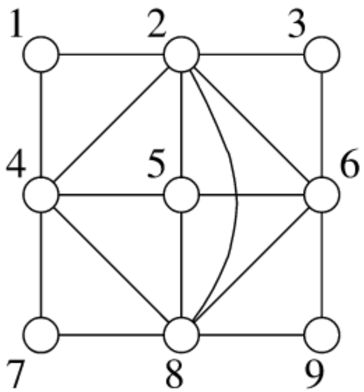
image: wikipedia

Building a clique-tree: example

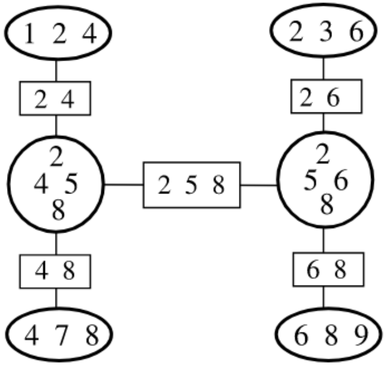
input



triangulated



clique-tree



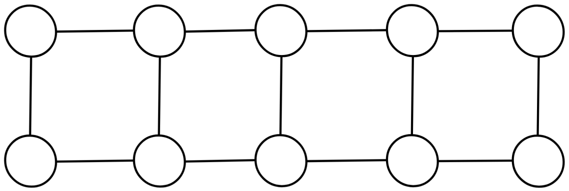
from: wainwright & jordan

clique-tree **quiz**

what clique-tree to use here?

what are the sepsets?

cost of exact inference?



Summary

- VE as message passing in a clique-tree
- clique-tree: running intersection & family preserving
- belief propagation updates:
  - message update
  - belief update
- types of queries
- how to build a clique-tree for exact inference

bonus slides

Clique-tree: calibration

represent P using marginals: $\frac{\prod_i \beta_i}{\prod_{i,j \in \mathcal{E}} \mu_{i,j}} = \frac{\prod_i \psi_i \prod_{k \to i} \delta_{k \to i}}{\prod_{i,j \in \mathcal{E}} \delta_{i \to j} \delta_{j \to i}} = \prod_i \psi_i = \tilde{P}$

how about about arbitrary assignments? $\beta_i, \mu_{i,j} \quad \forall i,j \in \mathcal{E}$

can they represent P as above?

an assignment is calibrated iff

BP produces calibrated beliefs

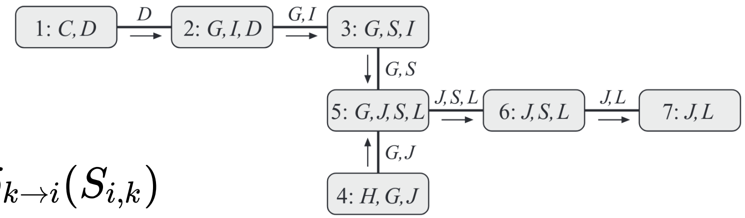
$$\mu_{i,j}(S_{i,j}) = \sum_{C_i - S_{i,j}} \beta_i(C_i) = \sum_{C_j - S_{i,j}} \beta_j(C_j)$$

for calibrated beliefs these "arbitrary assignments" have to be marginals

$$\tilde{P}(\mathbf{X}) \propto \frac{\prod \beta_i(C_i)}{\prod_{i,j \in \mathcal{E}} \mu_{i,j}(S_{i,j})} \quad \Leftrightarrow \quad \beta_i(C_i) \propto P(C_i)$$

BP: an alternative update

message update



$$\delta_{i\to j}(S_{i,j}) = \sum_{C_i - S_{i,j}} \psi_i(C_i) \prod_{k\in Nb_i - j} \delta_{k\to i}(S_{i,k})$$

calculate the beliefs **in the end** $\beta_i(C_i) = \psi_i(C_i) \prod_{k\in Nb_i} \delta_{k\to i}(S_{i,k})$

belief update

since $\delta_{i\to j}(S_{i,j}) = \frac{\sum_{C_i - S_{i,j}} \beta_i(C_i)}{\delta_{j\to i}(S_{i,j})}$ we can **update the beliefs**

instead of messages

BP: an alternative update

belief update

```
initialize
```
$\beta_i \leftarrow \psi_i = \prod_{\phi:\alpha(\phi)=i} \phi, \quad \mu_{i,j} \leftarrow 1$

```
until convergence:
        pick some
```
$(i,j) \in \mathcal{E}$

$\hat{\mu}_{i,j} \leftarrow \sum_{C_i - S_{i,j}} \beta_i$ // $\hat{\mu}_{i,j} = \delta_{i\to j}^{new} \delta_{j\to i}$

$\beta_j \leftarrow \beta_j \frac{\hat{\mu}_{i,j}}{\mu_{i,j}}$ // $\frac{\hat{\mu}_{i,j}}{\mu_{i,j}} = \frac{\delta_{i\to j}^{new} \delta_{j\to i}}{\delta_{i\to j}^{old} \delta_{j\to i}} = \frac{\delta_{i\to j}^{new}}{\delta_{i\to j}^{old}}$

$\mu_{i,j} \leftarrow \hat{\mu}_{i,j}$

at convergence, beliefs are calibrated $\sum_{C_i - S_{i,j}} \beta_i(C_i) = \sum_{C_j - S_{i,j}} \beta_j(C_j)$

and so they are $\propto$ marginals