Why Line-Search When You Can Plane-Search?

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Getting more without spending (much) more

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We could be using better step sizes

For many common machine learning (ML) problems, these step size strategies all have the same asymptotic cost

- fixed step size,
- line search,
- plane search

Plane search PHB

Polyak's Heavy Ball Method (PHB)

$$\mathbf{x_{t+1}} = \mathbf{x_t} - \alpha_t \nabla f(\mathbf{x_t}) + \beta_t(\mathbf{x_t} - \mathbf{x_{t-1}})$$

- Usually $\alpha_t = \alpha$ and $\beta_t = \beta$

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- Usually $\alpha_t = \alpha$ and $\beta_t = \beta$
- But you can do a plane search for the same cost

$$\alpha_t, \beta_t = \arg\min_{\alpha,\beta} f(\mathbf{x_t} - \alpha\nabla f(\mathbf{x_t}) + \beta(\mathbf{x_t} - \mathbf{x_{t-1}}))$$

Plane search PHB

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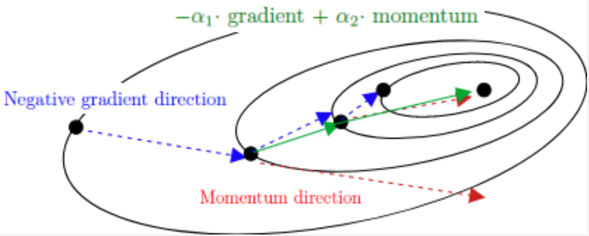
- Potentially more progress at every iteration
- No hyperparameter tuning

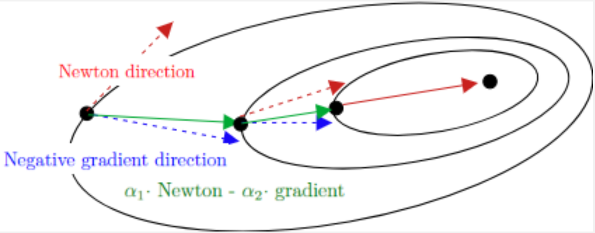
Plane search PHB

Adaptable gradient and momentum learning rates



Another use case: combination methods

Hybrid first- and second- order methods



Plane search for k directions

Subproblem to solve at every iteration t

Step sizes and search directions are

$$\boldsymbol{\alpha}_t = \begin{bmatrix} \alpha_{t,1} \\ \vdots \\ \alpha_{t,k} \end{bmatrix}, P_t = \begin{bmatrix} | & & | \\ p_{t,1} & \dots & p_{t,k} \\ | & & | \end{bmatrix}$$

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The update becomes

$$\mathbf{x_{t+1}} = \mathbf{x_t} + P_t \boldsymbol{\alpha}_t$$

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The update becomes

$$\mathbf{x_{t+1}} = \mathbf{x_t} + P_t \boldsymbol{\alpha}_t$$

Plane search solves this problem

$$\boldsymbol{\alpha}_t \in \arg\min_{\alpha} f\left(\mathbf{x_t} + P_t \boldsymbol{\alpha}_t\right)$$

When is plane search efficient?

Matrix-vector multiplications are bottlenecks

Linear composition problems (LCPs)

$$f(\mathbf{x}) = g(A\mathbf{x})$$

Examples:

- least squares: $g(A\mathbf{x}) = \frac{1}{2}\|A\mathbf{x} - \mathbf{y}\|^2$
- logistic regression: $g(A\mathbf{x}) = \log\left(1 + \exp(-\mathbf{y} \circ A\mathbf{x})\right)$

*Narkiss and Zibulevsky (2005) SESOP

Linear Composition Problem (LCP)

Cost of calculating function value

$$f(\mathbf{x_{t+1}}) = g(A(\mathbf{x_t} + P_t\boldsymbol{\alpha}_t)) = g(\mathbf{v_t} + \tilde{P}_t\boldsymbol{\alpha}_t)$$

where

$$\mathbf{v_t} = A\mathbf{x_t} \in \mathbb{R}^m \text{ and } \tilde{P}_t = AP_t \in \mathbb{R}^{m\times k}$$

Matrix-vector multiplications:

1. $\mathbf{v_t} = A\mathbf{x_t}$
2. $\tilde{P}_{t,1} = A^\intercal \mathbf{p_{t,1}}$
3. ...
4. $\tilde{P}_{t,k} = A^\intercal \mathbf{p_{t,k}}$

k direction plane search on LCP

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Subproblem at iteration t

Store $\mathbf{v_t}$ and $\tilde{P}_t$. Plane search is

$$\boldsymbol{\alpha}_t \in \arg\min_{\alpha} f(\mathbf{x_{t+1}}) = g(\mathbf{v_t} - \tilde{P}_t \boldsymbol{\alpha}) \tag{1}$$

- evaluating (1) requires no matrix-vector multiplication
- does require k vector-scalar multiplications
- significant gain if A is big and k is small
- buy 1 α get all the other αs for free

Plane search PHB on LCP

Subproblem at iteration t

PHB is a special case of k-direction plane search

$$\alpha_t = \begin{bmatrix} \alpha_{t,1} \\ \alpha_{t,2} \end{bmatrix}, \; P_t = \begin{bmatrix} | & | \\ -\nabla f(x_t) & (x_t - x_{t-1}) \\ | & | \end{bmatrix}$$

$$\boldsymbol{\alpha}_t \in \arg\min_{\boldsymbol{\alpha}} f(\mathbf{x_{t+1}}) = g(\mathbf{v_t} + \alpha_{t,1}\tilde{P}_{t,1} + \alpha_{t,2}(\mathbf{v_t} - \mathbf{v_{t-1}})) \quad (2)$$

Plane search PHB on LCP

Matrix-vector multiplications

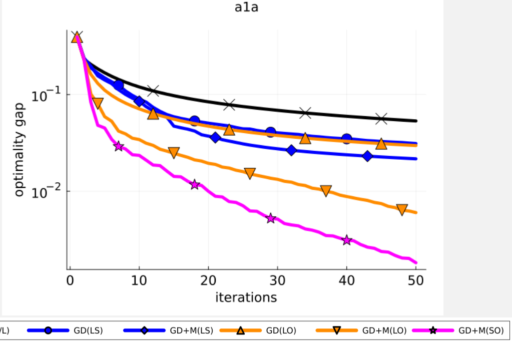
Matrix-vector multiplications

1. $\tilde{P}_{t,1} = -A^{\intercal}\nabla f(\mathbf{x_t})$.
2. $\mathbf{v_t} = A\mathbf{x_t}$. (Use $\mathbf{v_{t-1}}$ from previous iteration.)

Same as gradient descent with fixed step size and no momentum!

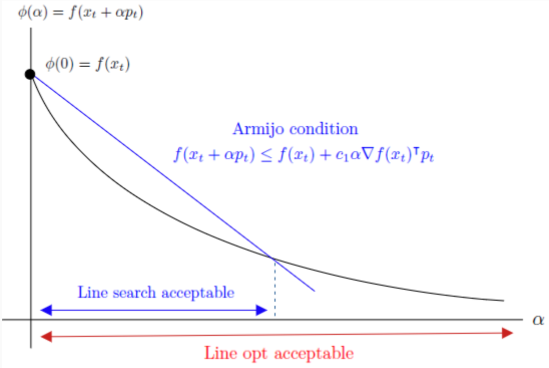
Experiments: GD+M, logistic regression

Fixed versus varying momentum weight



Line search blind spots

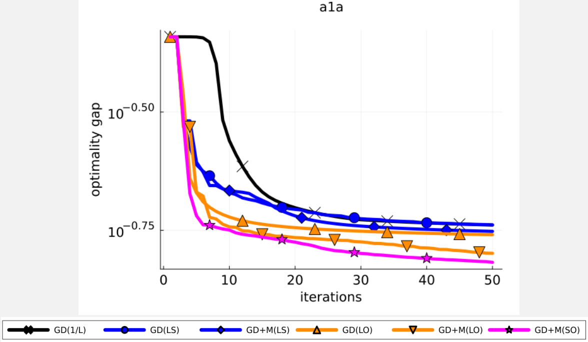
Armijo line search may rule out good step sizes



This affects Wolfe conditions too.

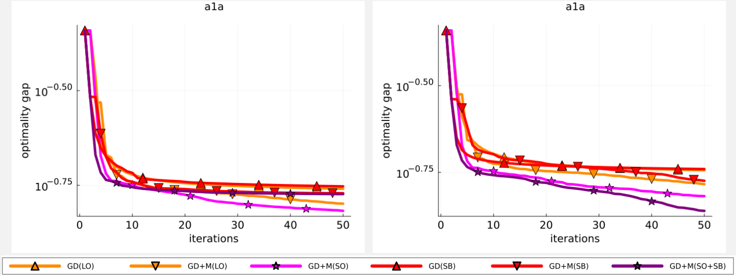
Experiments: GD+M, 2-layer NNs

No per-layer step sizes



Experiments: GD+M, 2-layer NNs

Per-layer step sizes without and with regularization



Experiments: observations

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Miscellaneous things we saw from the experiments

- large k (e.g. $k > 2$) does not appear to help much
- 3-term plane search implementation of Nesterov's acceleration
- can improve on Adam, quasi-Newton methods, etc.

Future directions

Software development: optimizers with built in plane search

- **Alyssa Zhang**,

  NSERC Undergraduate Student Research Award

Questions

- Does this work in the stochastic case?
- Does this work for modern architectures?

Thank you for listening

(or many) free step sizes

Experiments: GD+M, logistic regression

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Accelerated gradient methods