SVAN 2016 Mini-Course

Stochastic Convex Optimization Methods in Machine Learning

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Motivation for Parallel and Distributed

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  - We aren't making large gains in serial computation speed.
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- Two recent trends:
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  - Datasets no longer fit on a single machine.
- Result: we must use parallel and distributed computation.
- Two major issues:
  - Synchronization: we can't wait for the slowest machine.
  - Communication: we can't transfer all information.

Embarassing Parallelism in Machine Learning

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- These allow optimal linear speedups.
  - You should always consider this first!

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- You need to decrease step-size in proportion to asynchrony.
- Convergence rate decays elegantly with delay m.[Niu et al., 2011]

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- Only needs to communicate single coordinates.
- Again need to decrease step-size for convergence.
- Speedup is based on density of graph.[Richtarik & Takac, 2013]

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  - Each processor has its own parameter vector x_c.
  - Each processor only communicates with a limited number of neighbours $\text{nei}(c)$.

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$$x_c = \frac{1}{|\text{nei}(c)|} \sum_{c' \in \text{nei}(c)} x_c - \frac{\alpha_c}{M} \sum_{i=1}^{M} \nabla f_i(x_c).$$

- Gradient descent is special case where all neighbours communicate.
- With modified update, rate decays gracefully as graph becomes sparse.[Shi et al., 2014]
- Can also consider communication failures.[Agarwal & Duchi, 2011]

(pause)

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  - E.g., search over a sucessively-refined grid.
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  - Optimal rate for Lipschitz functions is $O(1/\epsilon^{1/D})$.
  - Can only solve low-dimensional problems.
- We'll go over recent local, global, and hybrid results..

PL Inequalty: Expanding the Second Phase

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- But you can also show linear convergence under many weaker assumptions:
  - Essential strong-convexity, weak strong-convexity, restricted secant inequality, restrictied secant inequality, quadratic growth property, optimal strong-convexity, error bounds.
- In fact, for our proof to work we only required

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which we call the Polyak-Łojasiewicz inequality:

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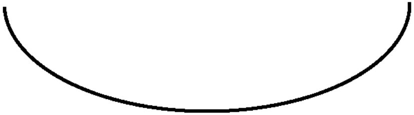
  - Does not imply convexity.
  - Also works for coordinate descent, can be generalized to proximal-gradient.

Global Linear Convergence with the PL Inequalty



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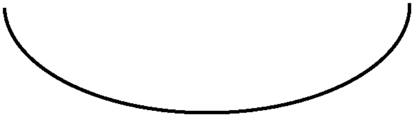

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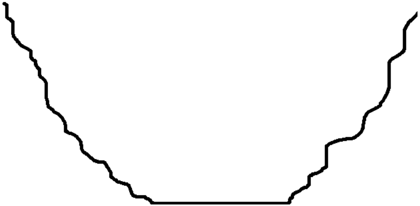

Function satisfying the PL inequality:

- Linear convergence rate for this non-convex function.
- Second phase of local solvers is larger than we thought.

General Global Non-Convex Rates?

- For strongly-convex smooth functions, we have

$$\|\nabla f(x^t)\|^2 = O(\rho^t), \quad f(x^t) - f(x^*) = O(\rho^t), \quad \|x_t - x_*\| = O(\rho^t).$$

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- You can get this rate for a random iteration of stochastic gradient.
  [Ghadimi & Lan, 2013].

Escaping Saddle Points

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  - Classical: trust-region methods allow negative eigenvalues.
  - Modify eigenvalues in Newton's method [Dauphin et al., 2014].
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  - Add random noise to stochastic gradient [Ge et al., 2015].
  - Cubic regularization of Newton [Nesterov & Polyak, 2006],

$$x^{k+1} = \min_d \left\{ f(x^k) + \langle \nabla f(x^k), d \rangle + \frac{1}{2} d^T \nabla^2 f(x^k) d + \frac{L}{6} \|d\|^3 \right\},$$

  if within ball of saddle point then next step:
    - Moves outside of ball.
    - Has lower objective than saddle-point.

Globally-Optimal Methods for Matrix Problems

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- Under certain assumptions, can solve UV^T dictionary learning and phase retrieval problems [Agarwal et al., 2014, Candes et al., 2015].
- Certain latent variable problems like training HMMs can be solved via SVD and tensor-decomposition methods [Hsu et al., 2012, Anandkumar et al, 2014].

Convex Relaxations/Representations

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- Exact convex re-formulations of non-convex problems:
  - Laserre [2001].
  - But the size may be enormous.

General Non-Convex Rates

Grid-search is optimal, but can be beaten:

- Convergence rate of Bayesian optimization [Bull, 2011]:
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- Regret bounds for Bayesian optimization:
  - Exponential scaling with dimensionality [Srinivas et al., 2010].
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- Other known faster-than-grid-search rates:
  - Simulated annealing under complicated non-singular assumption [Tikhomirov, 2010].
  - Particle filtering can improve under certain conditions [Crisan & Doucet, 2002].
  - Graduated Non-Convexity for σ-nice functions [Hazan et al., 2014].

Summary

- Parallel and distributed methods will be required in the future.
  - Need asynchronous methods with low communication and fault tolerance.
- We are starting to be able to understand non-convex problems, but there is a lot of work to do.

- Thank you for the invitation and I hope you learned some new things!