SVAN 2016 Mini-Course

Stochastic Convex Optimization Methods in Machine Learning

Mark Schmidt

University of British Columbia, May 2016

www.cs.ubc.ca/~schmidtm/SVAN16

Big-N Problems

- We can write our standard regularized optimization problem as

$$\min_{x \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} f_i(x) \quad + \quad r(x)$$

$$\text{data fitting term} \quad + \quad \text{regularizer}$$

Big-N Problems

- We can write our standard regularized optimization problem as

$$\min_{x \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} f_i(x) \quad + \quad r(x)$$

$$\text{data fitting term} \quad + \quad \text{regularizer}$$

- Gradient methods are effective when d is very large.
- What if number of training examples n is very large?
  - E.g., ImageNet has more than 14 million annotated images.

Stochastic vs. Deterministic Gradient Methods

- We consider minimizing $f(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x)$.

Stochastic vs. Deterministic Gradient Methods

- We consider minimizing $f(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x)$.
- Deterministic gradient method [Cauchy, 1847]:

$$x^{t+1} = x^t - \alpha_t \nabla f(x^t) = x^t - \frac{\alpha_t}{n}\sum_{i=1}^{n} \nabla f_i(x^t).$$

  - Iteration cost is linear in n.
  - Convergence with constant α_t or line-search.

Stochastic vs. Deterministic Gradient Methods

- We consider minimizing $f(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x)$.
- Deterministic gradient method [Cauchy, 1847]:

$$x^{t+1} = x^t - \alpha_t \nabla f(x^t) = x^t - \frac{\alpha_t}{n}\sum_{i=1}^{n} \nabla f_i(x^t).$$

  - Iteration cost is linear in n.
  - Convergence with constant α_t or line-search.
- Stochastic gradient method [Robbins & Monro, 1951]:
  - Random selection of i_t from $\{1, 2, \ldots, n\}$.

$$x^{t+1} = x^t - \alpha_t \nabla f_{i_t}(x^t).$$

Stochastic vs. Deterministic Gradient Methods

- We consider minimizing $f(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x)$.
- Deterministic gradient method [Cauchy, 1847]:

$$x^{t+1} = x^t - \alpha_t \nabla f(x^t) = x^t - \frac{\alpha_t}{n}\sum_{i=1}^{n} \nabla f_i(x^t).$$

  - Iteration cost is linear in n.
  - Convergence with constant α_t or line-search.
- Stochastic gradient method [Robbins & Monro, 1951]:
  - Random selection of i_t from $\{1, 2, \ldots, n\}$.

$$x^{t+1} = x^t - \alpha_t \nabla f_{i_t}(x^t).$$

  - Direction is an unbiased estimate of true gradient,

$$\mathbb{E}[f'_{i_t}(x)] = \frac{1}{n}\sum_{i=1}^{n} \nabla f_i(x) = \nabla f(x).$$

  - Iteration cost is independent of n.

Stochastic vs. Deterministic Gradient Methods

- We consider minimizing $f(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x)$.
- Deterministic gradient method [Cauchy, 1847]:

$$x^{t+1} = x^t - \alpha_t \nabla f(x^t) = x^t - \frac{\alpha_t}{n}\sum_{i=1}^{n} \nabla f_i(x^t).$$

  - Iteration cost is linear in n.
  - Convergence with constant α_t or line-search.
- Stochastic gradient method [Robbins & Monro, 1951]:
  - Random selection of i_t from $\{1, 2, \ldots, n\}$.

$$x^{t+1} = x^t - \alpha_t \nabla f_{i_t}(x^t).$$

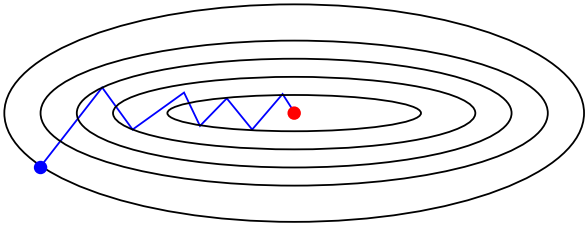
  - Direction is an unbiased estimate of true gradient,

$$\mathbb{E}[f'_{i_t}(x)] = \frac{1}{n}\sum_{i=1}^{n} \nabla f_i(x) = \nabla f(x).$$

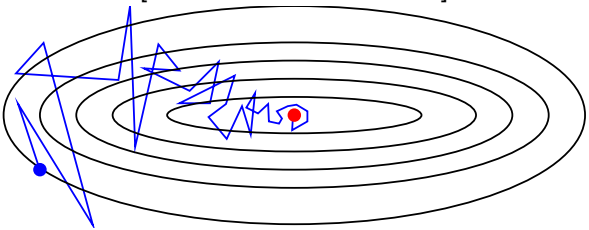
  - Iteration cost is independent of n.
  - Convergence requires $\alpha_t \to 0$.

Stochastic vs. Deterministic Gradient Methods

- We consider minimizing $f(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x)$.
- Deterministic gradient method [Cauchy, 1847]:



- Stochastic gradient method [Robbins & Monro, 1951]:



Stochastic vs. Deterministic Gradient Methods

Stochastic iterations are n times faster, but how many iterations?

Stochastic vs. Deterministic Gradient Methods

Stochastic iterations are n times faster, but how many iterations?

Assumption	Deterministic	Stochastic
Convex	$O(1/\sqrt{\epsilon})$	$O(1/\epsilon^2)$
Strongly	$O(\log(1/\epsilon))$	$O(1/\epsilon)$

Stochastic vs. Deterministic Gradient Methods

Stochastic iterations are n times faster, but how many iterations?

Assumption	Deterministic	Stochastic
Convex	$O(1/\sqrt{\epsilon})$	$O(1/\epsilon^2)$
Strongly	$O(\log(1/\epsilon))$	$O(1/\epsilon)$

- Stochastic has low iteration cost but slow convergence rate.
  - Sublinear rate even in strongly-convex case.
  - Bounds are unimprovable if only unbiased gradient available.

Stochastic vs. Deterministic Gradient Methods

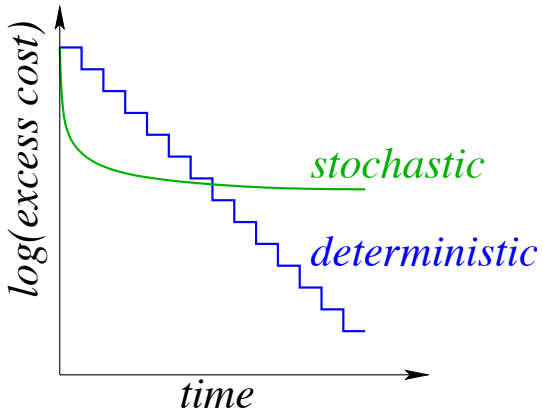
Stochastic iterations are n times faster, but how many iterations?

Assumption	Deterministic	Stochastic
Convex	$O(1/\sqrt{\epsilon})$	$O(1/\epsilon^2)$
Strongly	$O(\log(1/\epsilon))$	$O(1/\epsilon)$

- Stochastic has low iteration cost but slow convergence rate.
  - Sublinear rate even in strongly-convex case.
  - Bounds are unimprovable if only unbiased gradient available.
- Nesterov acceleration and momentum do not improve rate

Stochastic vs. Deterministic Convergence Rates

Plot of convergence rates in strongly-convex case:



Stochastic will be superior for low-accuracy/time situations.

Stochastic vs. Deterministic for Non-Smooth

- The story changes for non-smooth problems.
- Consider the binary support vector machine objective:

$$f(w) = \sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x_i)\} + \frac{\lambda}{2}\|w\|^2.$$

Stochastic vs. Deterministic for Non-Smooth

- The story changes for non-smooth problems.
- Consider the binary support vector machine objective:

$$f(w) = \sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x_i)\} + \frac{\lambda}{2}\|w\|^2.$$

- Rates for subgradient methods for non-smooth objectives:

Assumption	Deterministic	Stochastic
Convex	$O(1/\epsilon^2)$	$O(1/\epsilon^2)$
Strongly	$O(1/\epsilon)$	$O(1/\epsilon)$

- Other black-box methods (cutting plane) are not faster.

Stochastic vs. Deterministic for Non-Smooth

- The story changes for non-smooth problems.
- Consider the binary support vector machine objective:

$$f(w) = \sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x_i)\} + \frac{\lambda}{2}\|w\|^2.$$

- Rates for subgradient methods for non-smooth objectives:

Assumption	Deterministic	Stochastic
Convex	$O(1/\epsilon^2)$	$O(1/\epsilon^2)$
Strongly	$O(1/\epsilon)$	$O(1/\epsilon)$

- Other black-box methods (cutting plane) are not faster.
- For non-smooth problems:
  - Deterministic methods are not faster than stochastic method.
  - So use stochastic (iterations are n times faster).

Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

$$f(y) \geq f(x) + d^T (y - x), \forall y.$$

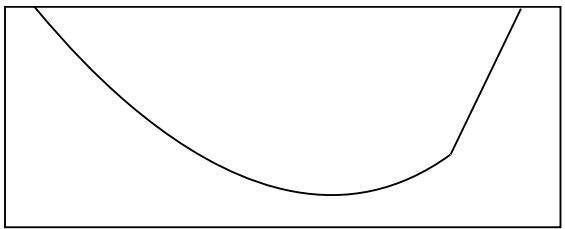
Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

$$f(y) \geq f(x) + d^T (y - x), \forall y.$$



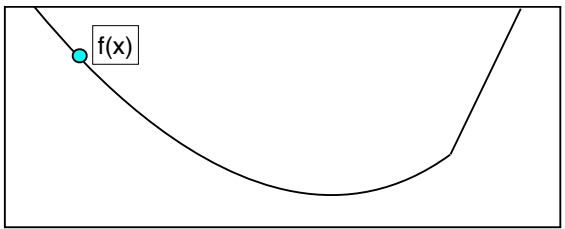
Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

$$f(y) \geq f(x) + d^T (y - x), \forall y.$$



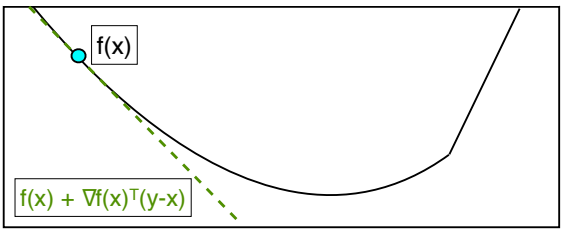
Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

$$f(y) \geq f(x) + d^T (y - x), \forall y.$$



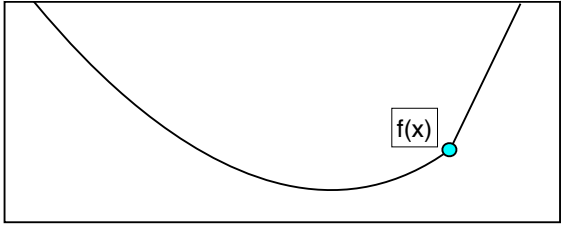
Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

$$f(y) \geq f(x) + d^T (y - x), \forall y.$$



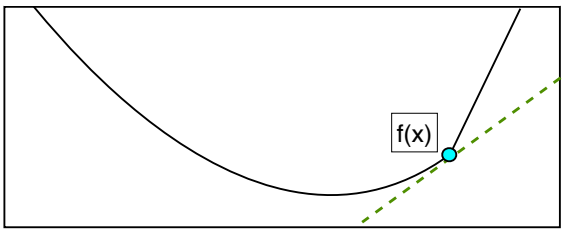
Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

$$f(y) \geq f(x) + d^T (y - x), \forall y.$$



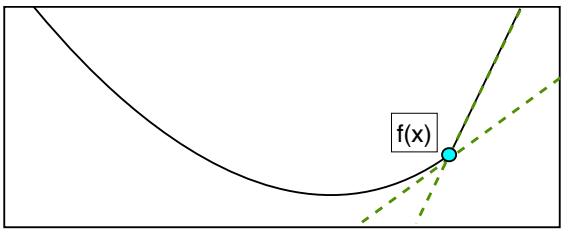
Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

$$f(y) \geq f(x) + d^T (y - x), \forall y.$$



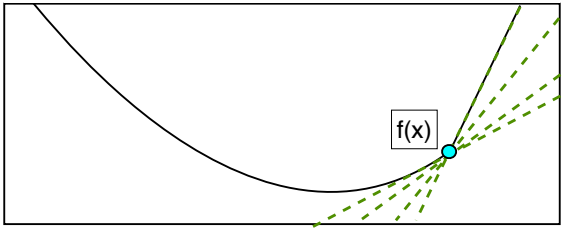
Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

$$f(y) \geq f(x) + d^T (y - x), \forall y.$$



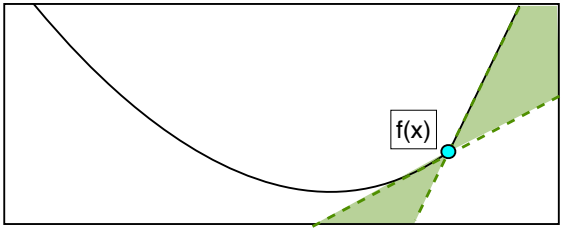
Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

$$f(y) \geq f(x) + d^T (y - x), \forall y.$$



Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T(y - x), \forall x, y.$$

A vector d is a subgradient of a convex function f at x if

$$f(y) \geq f(x) + d^T(y - x), \forall y.$$

- At differentiable x:
  - Only subgradient is $\nabla f(x)$.
- At non-differentiable x:
  - We have a set of subgradients.
  - Called the sub-differential, $\partial f(x)$.
  - Sub-differential is always non-empty for (almost) all convex functions.
- Note that $0 \in \partial f(x)$ iff x is a global minimum (generalizes $\nabla f(x) = 0$).

Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

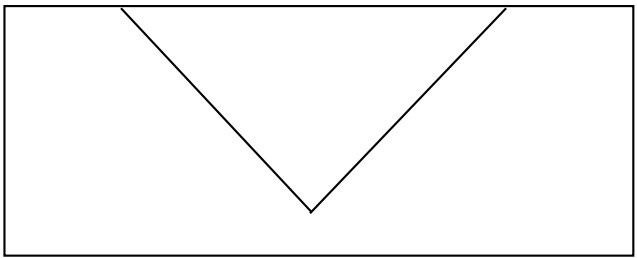
(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

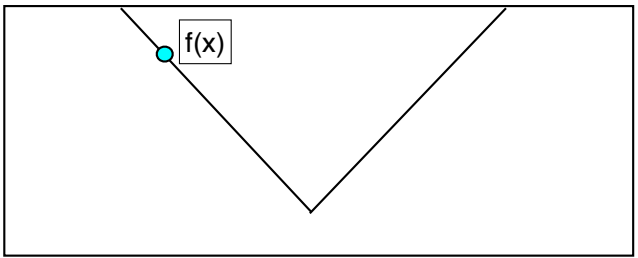


Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

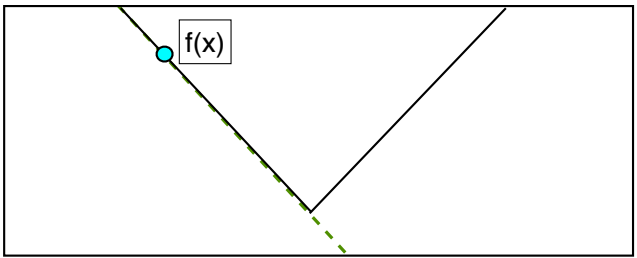


Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

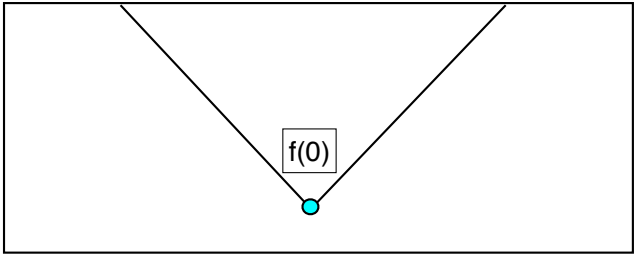


Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

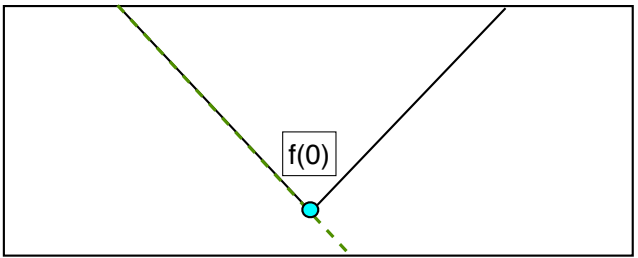


Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

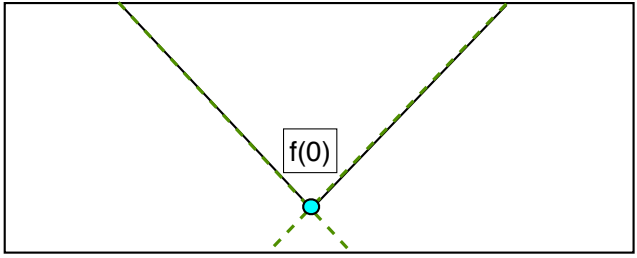


Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

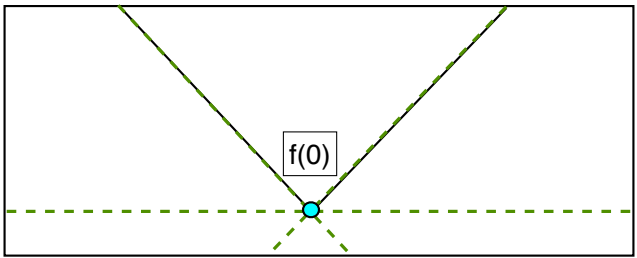


Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

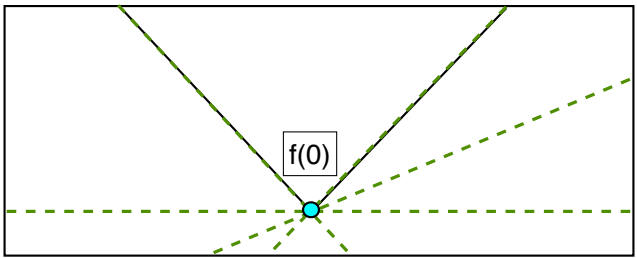


Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

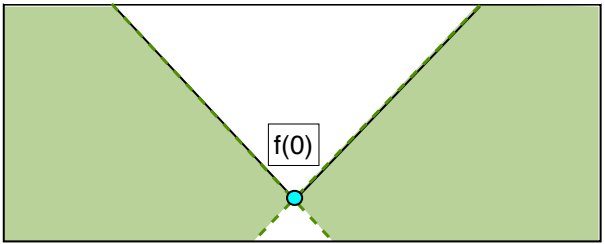


Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)



Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

- Sub-differential of sum of convex f_1 and f_2:

$$\partial(f_1(x) + f_2(x)) = \partial f_1(x) + \partial f_2(x).$$

Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

- Sub-differential of sum of convex f_1 and f_2:

$$\partial(f_1(x) + f_2(x)) = \partial f_1(x) + \partial f_2(x).$$

- Sub-differential of max of convex f_1 and f_2:

$$\partial \max\{f_1(x), f_2(x)\} =$$

Sub-Differential of Absolute Value and Max Functions

- Sub-differential of absolute value function:

$$\partial|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)

- Sub-differential of sum of convex f_1 and f_2:

$$\partial(f_1(x) + f_2(x)) = \partial f_1(x) + \partial f_2(x).$$

- Sub-differential of max of convex f_1 and f_2:

$$\partial \max\{f_1(x), f_2(x)\} = \begin{cases} \nabla f_1(x) & f_1(x) > f_2(x) \\ \nabla f_2(x) & f_2(x) > f_1(x) \\ \theta \nabla f_1(x) + (1 - \theta)\nabla f_2(x) & f_1(x) = f_2(x) \end{cases}$$

(any "convex combination" of the gradients of the argmax)

Subgradient Method

- The basic subgradient method:

$$x^{t+1} = x^t - \alpha_t g_t,$$

for some $g_t \in \partial f(x^t)$.

Subgradient Method

- The basic subgradient method:

$$x^{t+1} = x^t - \alpha_t g_t,$$

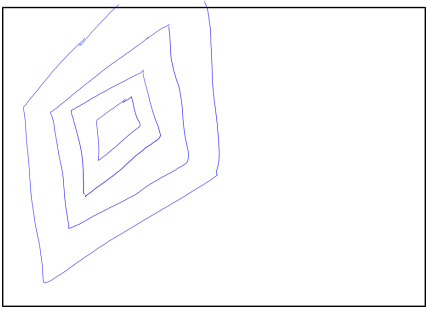
  for some $g_t \in \partial f(x^t)$.
- Unfortunately, may increase the objective even for small α_t.
- But, distance to solution decreases:
  - $\|x^{t+1} - x^*\| < \|x^t - x^*\|$ for small enough α.

Subgradient Method

- The basic subgradient method:

$$x^{t+1} = x^t - \alpha_t g_t,$$

  for some $g_t \in \partial f(x^t)$.
- Unfortunately, may increase the objective even for small α_t.
- But, distance to solution decreases:
  - $\|x^{t+1} - x^*\| < \|x^t - x^*\|$ for small enough α.

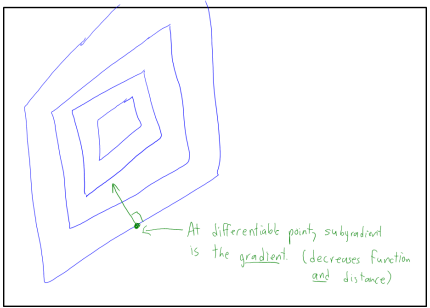


Subgradient Method

- The basic subgradient method:

$$x^{t+1} = x^t - \alpha_t g_t,$$

  for some $g_t \in \partial f(x^t)$.
- Unfortunately, may increase the objective even for small α_t.
- But, distance to solution decreases:
  - $\|x^{t+1} - x^*\| < \|x^t - x^*\|$ for small enough α.

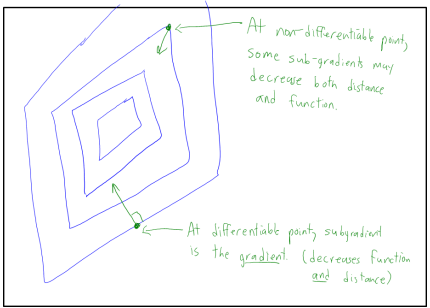


Subgradient Method

- The basic subgradient method:

$$x^{t+1} = x^t - \alpha_t g_t,$$

  for some $g_t \in \partial f(x^t)$.
- Unfortunately, may increase the objective even for small α_t.
- But, distance to solution decreases:
  - $\|x^{t+1} - x^*\| < \|x^t - x^*\|$ for small enough α.

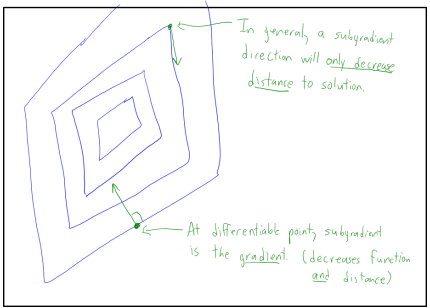


Subgradient Method

- The basic subgradient method:

$$x^{t+1} = x^t - \alpha_t g_t,$$

  for some $g_t \in \partial f(x^t)$.
- Unfortunately, may increase the objective even for small α_t.
- But, distance to solution decreases:
  - $\|x^{t+1} - x^*\| < \|x^t - x^*\|$ for small enough α.



Strong-Convexity Inequalities for Non-Differentiable f

- A "first-oder" relationship between subgradient and strong-convexity:
  - If f is μ-strongly convex then for all x and y we have

$$f(y) \geq f(x) + f'(y)^T(y - x) + \frac{\mu}{2}\|y - x\|^2,$$

    for $f'(y) \in \partial f(x)$.
  - The first-order definition of strong-convexity, but with subgradient replacing gradient.

Strong-Convexity Inequalities for Non-Differentiable f

- A "first-oder" relationship between subgradient and strong-convexity:
  - If f is μ-strongly convex then for all x and y we have

$$f(y) \geq f(x) + f'(y)^T(y-x) + \frac{\mu}{2}\|y-x\|^2,$$

    for $f'(y) \in \partial f(x)$.
  - The first-order definition of strong-convexity, but with subgradient replacing gradient.
  - Reversing y and x we can write

$$f(x) \geq f(y) + f'(x)^T(x-y) + \frac{\mu}{2}\|x-y\|^2,$$

    for $f'(x) \in \partial f(x)$.
  - Adding the above together gives

$$(f'(y) - f'(x)^T(y-x)) \geq \mu\|y-x\|^2.$$

Stochastic Subgradient Method

- The basic stochastic subgradient method:

$$x^{t+1} = x^t - \alpha g_{i_t},$$

for some $g_{i_t} \in \partial f_{i_t}(x^t)$ for some random $i_t \in \{1, 2, \ldots, n\}$.

Stochastic Subgradient Method

- The basic stochastic subgradient method:

$$x^{t+1} = x^t - \alpha g_{i_t},$$

  for some $g_{i_t} \in \partial f_{i_t}(x^t)$ for some random $i_t \in \{1, 2, \dots, n\}$.
- Stochastic subgradient is n times faster with similar convergence properties.
- We'll conisder it under the standard assumptions that
  - f is μ-strongly-convex:
  - $\mathbb{E}[\|g_t\|^2] \leq B^2$ (finite variance and bounded subgradients).

Convergence Rate of Stochastic Subgradient

- Since function value may not decrease, we analyze distance to x^*:

$$\begin{aligned}\|x^t - x^*\|^2 &= \|(x^{t-1} - \alpha_t g_{i_t}) - x^*\|^2 \\ &= \|(x^{t-1} - x^*) - \alpha_t g_{i_t}\|^2 \\ &= \|x^{t-1} - x^*\|^2 - 2\alpha_t g_{i_t}^T(x^{t-1} - x^*) + \alpha_t^2\|g_{i_t}\|^2.\end{aligned}$$

Convergence Rate of Stochastic Subgradient

- Since function value may not decrease, we analyze distance to x^*:

$$\begin{aligned}\|x^t - x^*\|^2 &= \|(x^{t-1} - \alpha_t g_{i_t}) - x^*\|^2 \\ &= \|(x^{t-1} - x^*) - \alpha_t g_{i_t}\|^2 \\ &= \|x^{t-1} - x^*\|^2 - 2\alpha_t g_{i_t}^T(x^{t-1} - x^*) + \alpha_t^2\|g_{i_t}\|^2.\end{aligned}$$

- Many analyses of distance to x^* start this way.
- First term is we what we want, we need to bound the second/third terms.

Convergence Rate of Stochastic Subgradient

- Expansion of distance:

$$\|x^t - x^*\|^2 = \|x^{t-1} - x^*\| - 2\alpha_t g_{i_t}^T (x^{t-1} - x^*) + \alpha_t^2 \|g_{i_t}\|^2.$$

Convergence Rate of Stochastic Subgradient

- Expansion of distance:

$$\|x^t - x^*\|^2 = \|x^{t-1} - x^*\| - 2\alpha_t g_{i_t}^T (x^{t-1} - x^*) + \alpha_t^2 \|g_{i_t}\|^2.$$

- Take expectation with respect to i_t:

$$\begin{aligned}\mathbb{E}[\|x^t - x^*\|^2] &= \mathbb{E}[\|x^{t-1} - x^*\|] - 2\alpha_t \mathbb{E}[g_{i_t}^T (x^{t-1} - x^*)] + \alpha_t^2 \mathbb{E}[\|g_{i_t}\|^2] \\ &\|x^{t-1} - x^*\|^2 - 2\alpha_t \mathbb{E}[g_{i_t}^T](x^{t-1} - x^*) + \alpha_t^2 \mathbb{E}[\|g_{i_t}\|^2] \\ &\leq \|x^{t-1} - x^*\|^2 - 2\alpha_t g_t^T (x^{t-1} - x^*) + \alpha_t^2 B^2.\end{aligned}$$

Convergence Rate of Stochastic Subgradient

- Expansion of distance:

$$\|x^t - x^*\|^2 = \|x^{t-1} - x^*\| - 2\alpha_t g_{i_t}^T(x^{t-1} - x^*) + \alpha_t^2 \|g_{i_t}\|^2.$$

- Take expectation with respect to i_t:

$$\begin{aligned}\mathbb{E}[\|x^t - x^*\|^2] &= \mathbb{E}[\|x^{t-1} - x^*\|] - 2\alpha_t \mathbb{E}[g_{i_t}^T(x^{t-1} - x^*)] + \alpha_t^2 \mathbb{E}[\|g_{i_t}\|^2] \\ &\|x^{t-1} - x^*\|^2 - 2\alpha_t \mathbb{E}[g_{i_t}^T](x^{t-1} - x^*) + \alpha_t^2 \mathbb{E}[\|g_{i_t}\|^2] \\ &\leq \|x^{t-1} - x^*\|^2 - 2\alpha_t g_t^T(x^{t-1} - x^*) + \alpha_t^2 B^2.\end{aligned}$$

- Using strong-convexity inequality,

$$(g_t - 0)^T(x^{t-1} - x^*) \geq \mu \|y - x\|^2,$$

gives

$$\begin{aligned}\mathbb{E}[\|x^t - x^*\|^2] &\leq \|x^{t-1} - x^*\|^2 - 2\alpha_t \mu \|x^{t-1} - x^*\|^2 + \alpha_t^2 B^2 \\ &= (1 - 2\alpha_t \mu)\|x^{t-1} - x^*\|^2 + \alpha_t^2 B^2.\end{aligned}$$

Stochastic Gradient with Constant Step Size

- Our bound on expected distance:

$$\mathbb{E}[\|x^t - x^*\|^2] \leq (1 - 2\alpha_t\mu)\|x^{t-1} - x^*\|^2 + \alpha_t^2 B^2.$$

- If α_t is *small* enough, shows distance to solution decreases.

Stochastic Gradient with Constant Step Size

- Our bound on expected distance:

$$\mathbb{E}[\|x^t - x^*\|^2] \leq (1 - 2\alpha_t\mu)\|x^{t-1} - x^*\|^2 + \alpha_t^2 B^2.$$

- If α_t is *small* enough, shows distance to solution decreases.
- Taking full expectation and applying recursively with constant $\alpha_t = \alpha$ gives:

$$\mathbb{E}[\|x^t - x^*\|^2] \leq (1 - 2\alpha\mu)^t\|x^0 - x^*\|^2 + \frac{\alpha B^2}{2\mu},$$

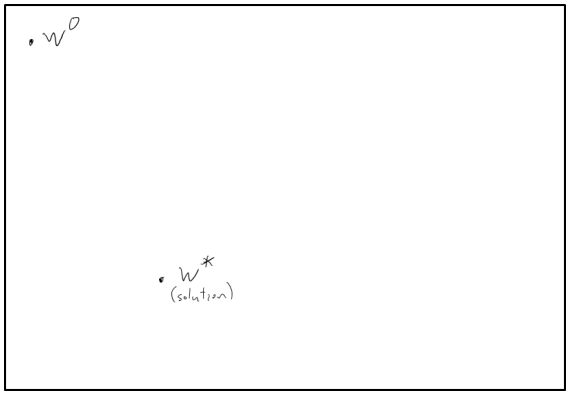
  after some of math (last term comes from bounding a geometric series).
- First term looks like linear convergence, but second term does not go to zero.

Stochastic Gradient with Constant Step Size

- Our bound on expected distance:

$$\mathbb{E}[\|x^t - x^*\|^2] \leq (1 - 2\alpha\mu)^t \|x^0 - x^*\|^2 + \frac{\alpha B^2}{2\mu}.$$

- First term looks like linear convergence, but second term does not go to zero.

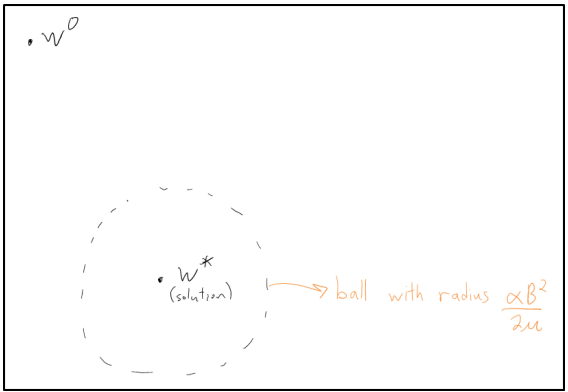


Stochastic Gradient with Constant Step Size

- Our bound on expected distance:

$$\mathbb{E}[\|x^t - x^*\|^2] \leq (1 - 2\alpha\mu)^t \|x^0 - x^*\|^2 + \frac{\alpha B^2}{2\mu}.$$

- First term looks like linear convergence, but second term does not go to zero.

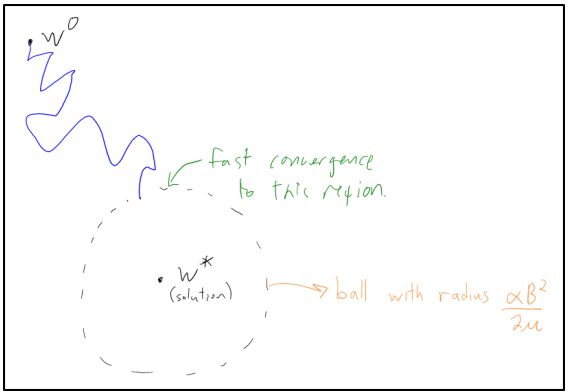


Stochastic Gradient with Constant Step Size

- Our bound on expected distance:

$$\mathbb{E}[\|x^t - x^*\|^2] \leq (1 - 2\alpha\mu)^t \|x^0 - x^*\|^2 + \frac{\alpha B^2}{2\mu}.$$

- First term looks like linear convergence, but second term does not go to zero.

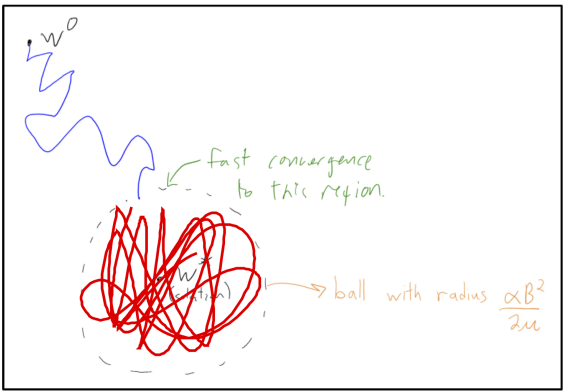


Stochastic Gradient with Constant Step Size

- Our bound on expected distance:

$$\mathbb{E}[\|x^t - x^*\|^2] \leq (1 - 2\alpha\mu)^t \|x^0 - x^*\|^2 + \frac{\alpha B^2}{2\mu}.$$

- First term looks like linear convergence, but second term does not go to zero.



Stochastic Gradient with Decreasing Step Size

- To get convergence, we need a decreasing step size.
  - Region that we converge to shrinks over time.
  - But it can't shrink too quickly or we may never reach x^*.

Stochastic Gradient with Decreasing Step Size

- To get convergence, we need a decreasing step size.
  - Region that we converge to shrinks over time.
  - But it can't shrink too quickly or we may never reach x^*.
  - Classic approach is to choose α_t such that

$$\sum_{t=1}^{\infty} \alpha_t = \infty, \quad \sum_{t=1}^{\infty} \alpha_t^2 < \infty,$$

    which suggests setting $\alpha_t = O(1/t)$.

Stochastic Gradient with Decreasing Step Size

- To get convergence, we need a decreasing step size.
  - Region that we converge to shrinks over time.
  - But it can't shrink too quickly or we may never reach x^*.
  - Classic approach is to choose α_t such that

$$\sum_{t=1}^{\infty} \alpha_t = \infty, \quad \sum_{t=1}^{\infty} \alpha_t^2 < \infty,$$

    which suggests setting $\alpha_t = O(1/t)$.
  - We can obtain convergence rates with decreasing steps:
    - If $\alpha_t = \frac{1}{\mu t}$ we can show

$$\begin{aligned} \mathbb{E}[f(\bar{x}^t) - f(x^*)] &= O(\log(t)/t) && \text{(non-smooth } f\text{)} \\ &= O(1/t) && \text{(smooth } f\text{)} \end{aligned}$$

      for the average iteration $\bar{x}^t = \frac{1}{k}\sum_{k=1}^{T} x_{k-1}$.
    - Note that $O(1/t)$ error implies $O(1/\epsilon)$ iterations required.

(pause)

What is the best subgradient?

- We analyzed the subgradient method,

$$x^{t+1} = x^t - \alpha_t g_t, \text{ where } g_t \in \partial f(x^t),$$

under any choice of subgradient.

What is the best subgradient?

- We analyzed the subgradient method,

$$x^{t+1} = x^t - \alpha_t g_t, \text{ where } g_t \in \partial f(x^t),$$

  under any choice of subgradient.
- But what is the "best" subgradient to use?
  - Convex functions have directional derivatives everywhere.
  - Direction $-z^t$ that minimizes directional derivative is minimum-norm subgradient,

$$z^t = \underset{z \in \partial f(x^t)}{\text{argmin}} \; ||z||$$

  - This is the steepest descent direction for non-smooth convex optimization problems.

What is the best subgradient?

- We analyzed the subgradient method,

$$x^{t+1} = x^t - \alpha_t g_t, \text{ where } g_t \in \partial f(x^t),$$

  under any choice of subgradient.
- But what is the "best" subgradient to use?
  - Convex functions have directional derivatives everywhere.
  - Direction $-z^t$ that minimizes directional derivative is minimum-norm subgradient,

$$z^t = \underset{z \in \partial f(x^t)}{\text{argmin}} \; ||z||$$

  - This is the steepest descent direction for non-smooth convex optimization problems.
  - You can compute this for L1-regularization, but not many other problems.
  - Basis for best L1-regularization methods, combined (carefully) with Newton.

Stochastic Subgradient with Sparse Features

- For many datasets, our feature vectors x_i are very sparse:

"CPSC	"Expedia"	"vicodin"	<recipient name>	...
1	0	0	0	...
0	1	0	0	...
0	0	1	0	...
0	1	0	1	...
1	0	1	1	...

- Consider case where d is huge but each row x_i has at most k non-zeroes:
  - The $O(d)$ cost of stochastic subgradient might be too high.
  - We can often modify stochastic subgradient to have $O(k)$ cost.

Digression: Operations on Sparse Vectors

- Consider a vector $g \in \mathbb{R}^d$ with at most k non-zeroes:

$$g^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 0 & -0.5 & 0 & 0 & 0 \end{bmatrix}.$$

- If $k << d$, we can store the vector using $O(k)$ storage instead of $O(d)$:

Digression: Operations on Sparse Vectors

- Consider a vector $g \in \mathbb{R}^d$ with at most k non-zeroes:

$$g^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 0 & -0.5 & 0 & 0 & 0 \end{bmatrix}.$$

- If $k << d$, we can store the vector using $O(k)$ storage instead of $O(d)$:
  - Store the non-zero values:

$$g_{\text{value}}^T = \begin{bmatrix} 1 & 2 & -0.5 \end{bmatrix}.$$

  - Store a pointer to where the non-zero values go:

$$g_{\text{point}}^T = \begin{bmatrix} 4 & 5 & 7 \end{bmatrix}.$$

Digression: Operations on Sparse Vectors

- Consider a vector $g \in \mathbb{R}^d$ with at most k non-zeroes:

$$g^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 0 & -0.5 & 0 & 0 & 0 \end{bmatrix}.$$

- If $k << d$, we can store the vector using $O(k)$ storage instead of $O(d)$:
  - Store the non-zero values:

$$g_{\text{value}}^T = \begin{bmatrix} 1 & 2 & -0.5 \end{bmatrix}.$$

  - Store a pointer to where the non-zero values go:

$$g_{\text{point}}^T = \begin{bmatrix} 4 & 5 & 7 \end{bmatrix}.$$

- With this representation, we can do standard vector operations in $O(k)$:
  - Compute αg in $O(k)$ by computing αg_{value}.
  - For dense w, set $w = (w - g)$ in $O(k)$ by subracting g_{value} from w at positions g_{point}

Stochastic Subgradient with Sparse Features

- Consider optimizing the hinge-loss,

$$\underset{w \in \mathbb{R}^d}{\text{argmin}} \frac{1}{n} \sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x_i)\},$$

when d is huge but each row has at most k non-zeroes.

Stochastic Subgradient with Sparse Features

- Consider optimizing the hinge-loss,

$$\operatorname*{argmin}_{w \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x_i)\},$$

  when d is huge but each row has at most k non-zeroes.

- A stochastic subgradient method could use

$$w^{t+1} = w^t - \alpha_t g_{i_t}, \text{ where } g_i = \begin{cases} -y_i x_i & \text{if } 1 - y_i(w^T x_i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Stochastic Subgradient with Sparse Features

- Consider optimizing the hinge-loss,

$$\underset{w \in \mathbb{R}^d}{\text{argmin}} \frac{1}{n} \sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x_i)\},$$

  when d is huge but each row has at most k non-zeroes.

- A stochastic subgradient method could use

$$w^{t+1} = w^t - \alpha_t g_{i_t}, \text{ where } g_i = \begin{cases} -y_i x_i & \text{if } 1 - y_i(w^T x_i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Notice that g_{i_t} has at most k non-zeroes:
  - Computing $\alpha_t g_{i_t}$ costs $O(k)$: multiply α_t by non-zeroes.
  - Computing $w^t - \alpha_t g_{i_t}$ costs $O(k)$: subtract non-zeroes.
- So stochastic subgradient is fast if k is small even if d is large.

Stochastic Subgradient with Sparse Features

- Consider the L2-regularized hinge-loss in the same setting,

$$\operatorname*{argmin}_{w \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x_i)\} + \frac{\lambda}{2}\|w\|^2,$$

using a stochastic subgradient method,

$$w^{t+1} = w^t - \alpha_t g_{i_t} - \alpha_t \lambda w^t, \text{ where } g_{i_t} \text{ is same as before}$$

Stochastic Subgradient with Sparse Features

- Consider the L2-regularized hinge-loss in the same setting,

$$\underset{w\in\mathbb{R}^d}{\text{argmin}} \frac{1}{n}\sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x_i)\} + \frac{\lambda}{2}\|w\|^2,$$

  using a stochastic subgradient method,

$$w^{t+1} = w^t - \alpha_t g_{i_t} - \alpha_t \lambda w^t, \text{ where } g_{i_t} \text{ is same as before}$$

- While g_{i_t} has at most k non-zeros, w^t could have d non-zeroes:
  - So adding L2-regularization increases cost from $O(k)$ to $O(d)$?

Stochastic Subgradient with Sparse Features

- Consider the L2-regularized hinge-loss in the same setting,

$$\operatorname*{argmin}_{w \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x_i)\} + \frac{\lambda}{2}\|w\|^2,$$

  using a stochastic subgradient method,

$$w^{t+1} = w^t - \alpha_t g_{i_t} - \alpha_t \lambda w^t, \text{ where } g_{i_t} \text{ is same as before}$$

- While g_{i_t} has at most k non-zeros, w^t could have d non-zeroes:
  - So adding L2-regularization increases cost from $O(k)$ to $O(d)$?
- To use L2-regularization and keep $O(k)$ cost, re-write iteration as

$$\begin{aligned} w^{t+1} &= w^t - \alpha_t g_{i_t} - \alpha_t \lambda w^t \\ &= \underbrace{(1 - \alpha_t \lambda) w^t}_{\text{changes scale of } w^t} - \underbrace{\alpha_t g_{i_t}}_{\text{sparse update}}. \end{aligned}$$

Stochastic Subgradient with Sparse Features

- Let's write the update as two steps

$$w^{t+\frac{1}{2}} = (1 - \alpha_t \lambda) w^t, \quad w^{t+1} = w^{t+\frac{1}{2}} - \alpha_t g_{i_t}.$$

Stochastic Subgradient with Sparse Features

- Let's write the update as two steps

$$w^{t+\frac{1}{2}} = (1 - \alpha_t \lambda) w^t, \quad w^{t+1} = w^{t+\frac{1}{2}} - \alpha_t g_{i_t}.$$

- We can implement both steps in $O(k)$ if we re-parameterize as

$$w^t = \beta^t v^t,$$

for some scalar β^t and vector v^t.

Stochastic Subgradient with Sparse Features

- Let's write the update as two steps

$$w^{t+\frac{1}{2}} = (1 - \alpha_t\lambda)w^t, \quad w^{t+1} = w^{t+\frac{1}{2}} - \alpha_t g_{i_t}.$$

- We can implement both steps in $O(k)$ if we re-parameterize as

$$w^t = \beta^t v^t,$$

for some scalar β^t and vector v^t.

- For the first step we need

$$\beta^{t+\frac{1}{2}} v^{t+\frac{1}{2}} = (1 - \alpha_t\lambda)\beta^t v^t,$$

which we can satisfy in $O(1)$ using $\beta^{t+\frac{1}{2}} = (1 - \alpha_t\lambda)\beta^t$ and $v^{t+\frac{1}{2}} = v^t$.

Stochastic Subgradient with Sparse Features

- Let's write the update as two steps

$$w^{t+\frac{1}{2}} = (1 - \alpha_t\lambda)w^t, \quad w^{t+1} = w^{t+\frac{1}{2}} - \alpha_t g_{i_t}.$$

- We can implement both steps in $O(k)$ if we re-parameterize as

$$w^t = \beta^t v^t,$$

for some scalar β^t and vector v^t.

- For the first step we need

$$\beta^{t+\frac{1}{2}} v^{t+\frac{1}{2}} = (1 - \alpha_t\lambda)\beta^t v^t,$$

which we can satisfy in $O(1)$ using $\beta^{t+\frac{1}{2}} = (1 - \alpha_t\lambda)\beta^t$ and $v^{t+\frac{1}{2}} = v^t$.

- For the second step we need

$$\beta^{t+1} v^{t+1} = \beta^{t+\frac{1}{2}} v^{t+\frac{1}{2}} - \alpha_t g_{i_t}.$$

which we can satisfy in $O(k)$ using $\beta^{t+1} = \beta^{t+\frac{1}{2}}$ and $v^{t+1} = v^{t+\frac{1}{2}} - \frac{\alpha_t}{\beta^{t+\frac{1}{2}}} g_{i_t}$.

Stochastic Subgradient with Sparse Features

- So we can implement the subgradient method with L2-regularization,

$$w^{t+1} = w^t - \alpha_t g_{i_t} - \alpha_t \lambda w^t,$$

in $O(k)$ by using the $w^t = \beta^t v^t$ representation and the update

$$\beta^{t+1} = (1 - \alpha_t \lambda)\beta^t, \quad v^{t+1} = v^t - \frac{\alpha_t}{\beta^{t+1}} g_{i_t}.$$

assuming that computing g_{i_t} can be done in $O(k)$ given β^t and v^t.

Stochastic Subgradient with Sparse Features

- So we can implement the subgradient method with L2-regularization,

$$w^{t+1} = w^t - \alpha_t g_{i_t} - \alpha_t \lambda w^t,$$

in $O(k)$ by using the $w^t = \beta^t v^t$ representation and the update

$$\beta^{t+1} = (1 - \alpha_t \lambda)\beta^t, \quad v^{t+1} = v^t - \frac{\alpha_t}{\beta^{t+1}} g_{i_t}.$$

assuming that computing g_{i_t} can be done in $O(k)$ given β^t and v^t.

- There exists efficient sparse updates in other scenarios too:
  - Duchi & Singer [2009]: L1-regularization proximal operator ("lazy updates").
  - Xu [2010]: L2-regularization and iterate average $\bar{w}^t$.

Stochastic Subgradient Methods in Practice

- Last time we argued that α_t must go to zero for convergence.
- Theory says using $\alpha_t = 1/\mu t$ and averaging is close to optimal:

Stochastic Subgradient Methods in Practice

- Last time we argued that α_t must go to zero for convergence.
- Theory says using $\alpha_t = 1/\mu t$ and averaging is close to optimal:
  - Except for some special cases, you should not do this.
    - Usually $\mu = O(1/n)$ or $O(1/\sqrt{n})$ so initial steps are huge.
    - Later steps are tiny: $1/t$ gets small very quickly.
    - Convergence rate slows dramatically if μ isn't accurate.
    - No adaptation to "easier" problems than worst case.

Stochastic Subgradient Methods in Practice

- Last time we argued that α_t must go to zero for convergence.
- Theory says using $\alpha_t = 1/\mu t$ and averaging is close to optimal:
  - Except for some special cases, you should not do this.
    - Usually $\mu = O(1/n)$ or $O(1/\sqrt{n})$ so initial steps are huge.
    - Later steps are tiny: $1/t$ gets small very quickly.
    - Convergence rate slows dramatically if μ isn't accurate.
    - No adaptation to "easier" problems than worst case.
- Tricks that can improve theoretical and practical properties:
  1. Use smaller initial step-sizes, that go to zero more slowly:

$$\alpha_t = \gamma/\sqrt{t} \quad \text{or } \alpha_t = \gamma.$$

  2. Take a (weighted) average of the iterations or gradients:

$$\bar{x}^t = \sum_{i=1}^{t} \omega_t z^t,$$

where ω_t is weight at iteration t.

Stochastic Subgradient Methods in Practice

- Last time we argued that α_t must go to zero for convergence.
- Theory says using $\alpha_t = 1/\mu t$ and averaging is close to optimal:
  - Except for some special cases, you should not do this.
    - Usually $\mu = O(1/n)$ or $O(1/\sqrt{n})$ so initial steps are huge.
    - Later steps are tiny: $1/t$ gets small very quickly.
    - Convergence rate slows dramatically if μ isn't accurate.
    - No adaptation to "easier" problems than worst case.
- Tricks that can improve theoretical and practical properties:
  1. Use smaller initial step-sizes, that go to zero more slowly:

$$\alpha_t = \gamma/\sqrt{t} \quad \text{or } \alpha_t = \gamma.$$

  2. Take a (weighted) average of the iterations or gradients:

$$\bar{x}^t = \sum_{i=1}^{t} \omega_t z^t,$$

     where ω_t is weight at iteration t.
- These tricks usually help, but tuning is often required:
  - stochastic subgradient is not a black box.

Speeding up Stochastic Subgradient Methods

Results that support using large steps and averaging:

- Averaging later iterations achieves $O(1/t)$ in non-smooth case.
- Gradient averaging improves constants in analysis.
- $\alpha_t = O(1/t^\beta)$ for $\beta \in (0.5, 1)$ more robust than $\alpha_t = O(1/t)$.

Speeding up Stochastic Subgradient Methods

Results that support using large steps and averaging:

- Averaging later iterations achieves $O(1/t)$ in non-smooth case.
- Gradient averaging improves constants in analysis.
- $\alpha_t = O(1/t^\beta)$ for $\beta \in (0.5, 1)$ more robust than $\alpha_t = O(1/t)$.
- Constant step size ($\alpha_t = \alpha$) achieves linear rate to accuracy $O(\alpha)$.
- In smooth case, iterate averaging is asymptotically optimal:
  - Achieves same rate as optimal stochastic Newton method.

Stochastic Newton Methods?

- Should we use Nesterov/Newton-like stochastic methods?
  - These do not improve the $O(1/\epsilon)$ convergence rate.

Stochastic Newton Methods?

- Should we use Nesterov/Newton-like stochastic methods?
  - These do not improve the $O(1/\epsilon)$ convergence rate.
- But some positive results exist.
  - Improves performance at start or if noise is small.
  - Newton-like AdaGrad method,

$$x^{t+1} = x^t + \alpha D \nabla f_{i_t}(x^t), \quad \text{with } D_{jj} = \sqrt{\sum_{k=1}^{t} \|\nabla_j f_{i_k}(x^t)\|}.$$

    - improves regret but not optimization error.
  - Two-phase Newton-like method achieves $O(1/\epsilon)$ without strong-convexity.

(pause)

Big-N Problems

- Recall our standard optimization framework,

$$\min_{x \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} f_i(x) \quad + \quad r(x)$$

$$\text{data fitting term} \quad + \quad \text{regularizer}$$

Big-N Problems

- Recall our standard optimization framework,

$$\min_{x \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} f_i(x) \quad + \quad r(x)$$

$$\text{data fitting term} \quad + \quad \text{regularizer}$$

- Stochastic methods:
  - $O(1/t)$ convergence but requires 1 gradient per iterations.
  - Rates are unimprovable for general stochastic objectives.

Big-N Problems

- Recall our standard optimization framework,

$$\min_{x \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} f_i(x) \quad + \quad r(x)$$

$$\text{data fitting term} \quad + \quad \text{regularizer}$$

- Stochastic methods:
  - $O(1/t)$ convergence but requires 1 gradient per iterations.
  - Rates are unimprovable for general stochastic objectives.
- Deterministic methods:
  - $O(\rho^t)$ convergence but requires N gradients per iteration.
  - The faster rate is possible because N is finite.

Big-N Problems

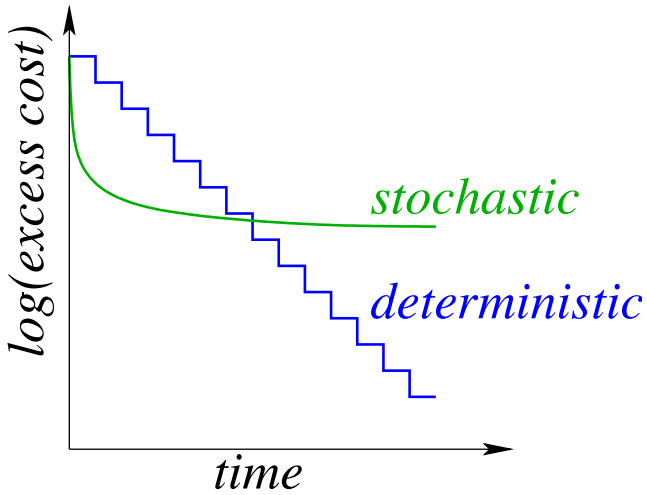
- Recall our standard optimization framework,

$$\min_{x \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} f_i(x) \quad + \quad r(x)$$

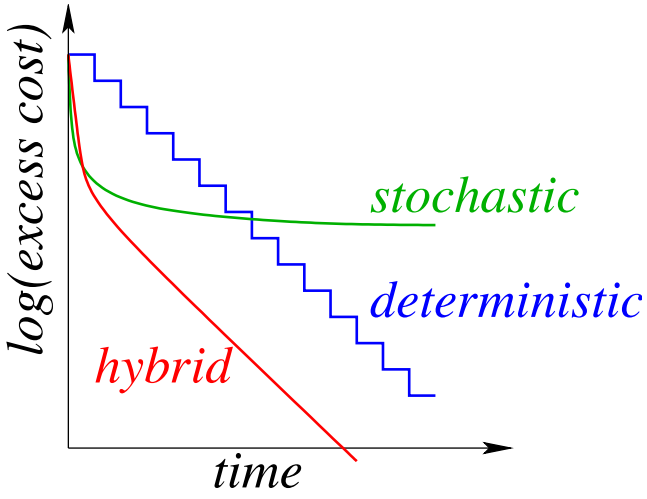
$$\text{data fitting term} \quad + \quad \text{regularizer}$$

- Stochastic methods:
  - $O(1/t)$ convergence but requires 1 gradient per iterations.
  - Rates are unimprovable for general stochastic objectives.
- Deterministic methods:
  - $O(\rho^t)$ convergence but requires N gradients per iteration.
  - The faster rate is possible because N is finite.
- For minimizing finite sums, can we design a better method?

Motivation for Hybrid Methods



Motivation for Hybrid Methods



Hybrid Deterministic-Stochastic

- Approach 1: control the sample size.

Hybrid Deterministic-Stochastic

- Approach 1: control the sample size.
- The FG method uses all N gradients,

$$\nabla f(x^t) = \frac{1}{N}\sum_{i=1}^{N} \nabla f_i(x^t).$$

- The SG method approximates it with 1 sample,

$$\nabla f_{i_t}(x^t) \approx \frac{1}{N}\sum_{i=1}^{N} \nabla f_i(x^t).$$

Hybrid Deterministic-Stochastic

- Approach 1: control the sample size.
- The FG method uses all N gradients,

$$\nabla f(x^t) = \frac{1}{N}\sum_{i=1}^{N} \nabla f_i(x^t).$$

- The SG method approximates it with 1 sample,

$$\nabla f_{i_t}(x^t) \approx \frac{1}{N}\sum_{i=1}^{N} \nabla f_i(x^t).$$

- A common variant is to use larger sample $\mathcal{B}^t$,

$$\frac{1}{|\mathcal{B}^t|}\sum_{i\in\mathcal{B}^t} \nabla f_i(x^t) \approx \frac{1}{N}\sum_{i=1}^{N} \nabla f_i(x^t).$$

Approach 1: Batching

- The SG method with a sample $\mathcal{B}^t$ uses iterations

$$x^{t+1} = x^t - \frac{\alpha^t}{|\mathcal{B}^t|} \sum_{i \in \mathcal{B}^t} f_i(x^t).$$

- For a fixed sample size $|\mathcal{B}^t|$, the rate is sublinear.

Approach 1: Batching

- The SG method with a sample $\mathcal{B}^t$ uses iterations

$$x^{t+1} = x^t - \frac{\alpha^t}{|\mathcal{B}^t|} \sum_{i \in \mathcal{B}^t} f_i(x^t).$$

- For a fixed sample size $|\mathcal{B}^t|$, the rate is sublinear.
- **Gradient error decreases as sample size $|\mathcal{B}^t|$ increases.**

Approach 1: Batching

- The SG method with a sample $\mathcal{B}^t$ uses iterations

$$x^{t+1} = x^t - \frac{\alpha^t}{|\mathcal{B}^t|} \sum_{i \in \mathcal{B}^t} f_i(x^t).$$

- For a fixed sample size $|\mathcal{B}^t|$, the rate is sublinear.
- **Gradient error decreases as sample size $|\mathcal{B}^t|$ increases.**
- Common to gradually increase the sample size $|\mathcal{B}^t|$.

  [Bertsekas & Tsitsiklis, 1996]

Approach 1: Batching

- The SG method with a sample $\mathcal{B}^t$ uses iterations

$$x^{t+1} = x^t - \frac{\alpha^t}{|\mathcal{B}^t|} \sum_{i \in \mathcal{B}^t} f_i(x^t).$$

- For a fixed sample size $|\mathcal{B}^t|$, the rate is sublinear.
- **Gradient error decreases as sample size $|\mathcal{B}^t|$ increases.**
- Common to gradually increase the sample size $|\mathcal{B}^t|$.
  [Bertsekas & Tsitsiklis, 1996]
- We can choose $|\mathcal{B}^t|$ to achieve a linear convergence rate:
  - Early iterations are cheap like SG iterations.
  - Later iterations can use a Newton-like method.

Stochastic Average Gradient

- Growing $|\mathcal{B}^t|$ eventually requires O(N) iteration cost.
- **Can we have a rate of $O(\rho^t)$ with only 1 gradient evaluation per iteration?**

Stochastic Average Gradient

- Growing $|\mathcal{B}^t|$ eventually requires O(N) iteration cost.
- **Can we have a rate of $O(\rho^t)$ with only 1 gradient evaluation per iteration?**
  - YES!

Stochastic Average Gradient

- Growing $|\mathcal{B}^t|$ eventually requires O(N) iteration cost.
- **Can we have a rate of $O(\rho^t)$ with only 1 gradient evaluation per iteration?**
  - YES! The stochastic average gradient (SAG) algorithm:
    - Randomly select i_t from $\{1, 2, \ldots, N\}$ and compute $f'_{i_t}(x^t)$.

$$x^{t+1} = x^t - \frac{\alpha^t}{N} \sum_{i=1}^{N} \nabla f_i(x^t)$$

Stochastic Average Gradient

- Growing $|\mathcal{B}^t|$ eventually requires O(N) iteration cost.
- **Can we have a rate of $O(\rho^t)$ with only 1 gradient evaluation per iteration?**
  - YES! The stochastic average gradient (SAG) algorithm:
    - Randomly select i_t from $\{1, 2, \ldots, N\}$ and compute $f'_{i_t}(x^t)$.

$$x^{t+1} = x^t - \frac{\alpha^t}{N} \sum_{i=1}^{N} \nabla f_i(x^t)$$

Stochastic Average Gradient

- Growing $|\mathcal{B}^t|$ eventually requires O(N) iteration cost.
- **Can we have a rate of $O(\rho^t)$ with only 1 gradient evaluation per iteration?**
  - YES! The stochastic average gradient (SAG) algorithm:
    - Randomly select i_t from $\{1, 2, \ldots, N\}$ and compute $f'_{i_t}(x^t)$.

$$x^{t+1} = x^t - \frac{\alpha^t}{N} \sum_{i=1}^{N} y_i^t$$

    - **Memory**: $y_i^t = \nabla f_i(x^t)$ from the last t where i was selected.
      [Le Roux et al., 2012]

Stochastic Average Gradient

- Growing $|\mathcal{B}^t|$ eventually requires O(N) iteration cost.
- **Can we have a rate of $O(\rho^t)$ with only 1 gradient evaluation per iteration?**
  - YES! The stochastic average gradient (SAG) algorithm:
    - Randomly select i_t from $\{1, 2, \ldots, N\}$ and compute $f'_{i_t}(x^t)$.

$$x^{t+1} = x^t - \frac{\alpha^t}{N} \sum_{i=1}^{N} y_i^t$$

    - **Memory**: $y_i^t = \nabla f_i(x^t)$ from the last t where i was selected.
      [Le Roux et al., 2012]
  - Stochastic variant of increment average gradient (IAG).
    [Blatt et al., 2007]

Stochastic Average Gradient

- Growing $|\mathcal{B}^t|$ eventually requires O(N) iteration cost.
- **Can we have a rate of $O(\rho^t)$ with only 1 gradient evaluation per iteration?**
  - YES! The stochastic average gradient (SAG) algorithm:
    - Randomly select i_t from $\{1, 2, \ldots, N\}$ and compute $f'_{i_t}(x^t)$.

$$x^{t+1} = x^t - \frac{\alpha^t}{N} \sum_{i=1}^{N} y_i^t$$

    - **Memory**: $y_i^t = \nabla f_i(x^t)$ from the last t where i was selected.
      [Le Roux et al., 2012]
  - Stochastic variant of increment average gradient (IAG).
    [Blatt et al., 2007]
  - Assumes gradients of non-selected examples don't change.
  - Assumption becomes accurate as $||x^{t+1} - x^t|| \to 0$.

Convergence Rate of SAG

- If each f_i' is $L-$continuous and f is strongly-convex, with $\alpha_t = 1/16L$ SAG has

$$\mathbb{E}[f(x^t) - f(x^*)] \leqslant \left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^t C,$$

where

$$C = [f(x^0) - f(x^*)] + \frac{4L}{N}\|x^0 - x^*\|^2 + \frac{\sigma^2}{16L}.$$

Convergence Rate of SAG

- If each f_i' is $L-$continuous and f is strongly-convex, with $\alpha_t = 1/16L$ SAG has

$$\mathbb{E}[f(x^t) - f(x^*)] \leqslant \left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^t C,$$

where

$$C = [f(x^0) - f(x^*)] + \frac{4L}{N}\|x^0 - x^*\|^2 + \frac{\sigma^2}{16L}.$$

- Linear convergence rate but only 1 gradient per iteration.
  - For well-conditioned problems, constant reduction per pass:

$$\left(1 - \frac{1}{8N}\right)^N \leq \exp\left(-\frac{1}{8}\right) = 0.8825.$$

  - For ill-conditioned problems, almost same as deterministic method (but N times faster).

Rate of Convergence Comparison

- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:

Rate of Convergence Comparison

- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:
  - Gradient method has rate $\left(\frac{L-\mu}{L+\mu}\right)^2 = 0.99998$.

Rate of Convergence Comparison

- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:
  - Gradient method has rate $\left(\frac{L-\mu}{L+\mu}\right)^2 = 0.99998$.
  - Accelerated gradient method has rate $\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.99761$.

Rate of Convergence Comparison

- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:
  - Gradient method has rate $\left(\frac{L-\mu}{L+\mu}\right)^2 = 0.99998$.
  - Accelerated gradient method has rate $\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.99761$.
  - SAG (N iterations) has rate $\left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^N = 0.88250$.

Rate of Convergence Comparison

- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:
  - Gradient method has rate $\left(\frac{L-\mu}{L+\mu}\right)^2 = 0.99998$.
  - Accelerated gradient method has rate $\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.99761$.
  - SAG (N iterations) has rate $\left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^N = 0.88250$.
  - *Fastest possible* first-order method: $\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.99048$.

Rate of Convergence Comparison

- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:
  - Gradient method has rate $\left(\frac{L-\mu}{L+\mu}\right)^2 = 0.99998$.
  - Accelerated gradient method has rate $\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.99761$.
  - SAG (N iterations) has rate $\left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^N = 0.88250$.
  - *Fastest possible* first-order method: $\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.99048$.
- SAG beats two lower bounds:
  - Stochastic gradient bound (of $O(1/t)$).
  - Deterministic gradient bound (for typical L, μ, and N).

Rate of Convergence Comparison

- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:
  - Gradient method has rate $\left(\frac{L-\mu}{L+\mu}\right)^2 = 0.99998$.
  - Accelerated gradient method has rate $\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.99761$.
  - SAG (N iterations) has rate $\left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^N = 0.88250$.
  - *Fastest possible* first-order method: $\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.99048$.
- SAG beats two lower bounds:
  - Stochastic gradient bound (of $O(1/t)$).
  - Deterministic gradient bound (for typical L, μ, and N).
- Number of f_i' evaluations to reach ϵ:

Rate of Convergence Comparison

- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:
  - Gradient method has rate $\left(\frac{L-\mu}{L+\mu}\right)^2 = 0.99998$.
  - Accelerated gradient method has rate $\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.99761$.
  - SAG (N iterations) has rate $\left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^N = 0.88250$.
  - *Fastest possible* first-order method: $\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.99048$.
- SAG beats two lower bounds:
  - Stochastic gradient bound (of $O(1/t)$).
  - Deterministic gradient bound (for typical L, μ, and N).
- Number of f_i' evaluations to reach ϵ:
  - Stochastic: $O(\frac{L}{\mu}(1/\epsilon))$.

Rate of Convergence Comparison

- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:
  - Gradient method has rate $\left(\frac{L-\mu}{L+\mu}\right)^2 = 0.99998$.
  - Accelerated gradient method has rate $\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.99761$.
  - SAG (N iterations) has rate $\left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^N = 0.88250$.
  - *Fastest possible* first-order method: $\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.99048$.
- SAG beats two lower bounds:
  - Stochastic gradient bound (of $O(1/t)$).
  - Deterministic gradient bound (for typical L, μ, and N).
- Number of f_i' evaluations to reach ϵ:
  - Stochastic: $O(\frac{L}{\mu}(1/\epsilon))$.
  - Gradient: $O(N\frac{L}{\mu}\log(1/\epsilon))$.

Rate of Convergence Comparison

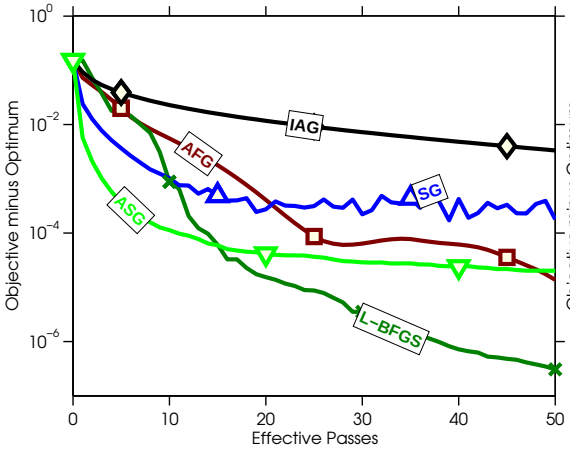
- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:
  - Gradient method has rate $\left(\frac{L-\mu}{L+\mu}\right)^2 = 0.99998$.
  - Accelerated gradient method has rate $\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.99761$.
  - SAG (N iterations) has rate $\left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^N = 0.88250$.
  - *Fastest possible* first-order method: $\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.99048$.
- SAG beats two lower bounds:
  - Stochastic gradient bound (of $O(1/t)$).
  - Deterministic gradient bound (for typical L, μ, and N).
- Number of f_i' evaluations to reach ϵ:
  - Stochastic: $O(\frac{L}{\mu}(1/\epsilon))$.
  - Gradient: $O(N\frac{L}{\mu}\log(1/\epsilon))$.
  - Accelerated: $O(N\sqrt{\frac{L}{\mu}}\log(1/\epsilon))$.

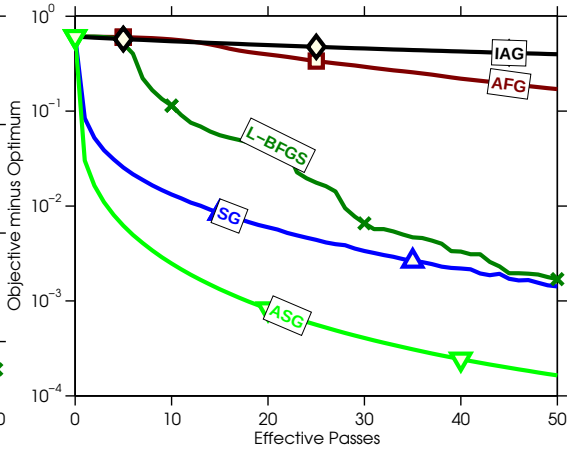
Rate of Convergence Comparison

- Assume that $N = 700000$, $L = 0.25$, $\mu = 1/N$:
  - Gradient method has rate $\left(\frac{L-\mu}{L+\mu}\right)^2 = 0.99998$.
  - Accelerated gradient method has rate $\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.99761$.
  - SAG (N iterations) has rate $\left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^N = 0.88250$.
  - *Fastest possible* first-order method: $\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.99048$.
- SAG beats two lower bounds:
  - Stochastic gradient bound (of $O(1/t)$).
  - Deterministic gradient bound (for typical L, μ, and N).
- Number of f_i' evaluations to reach ϵ:
  - Stochastic: $O(\frac{L}{\mu}(1/\epsilon))$.
  - Gradient: $O(N\frac{L}{\mu}\log(1/\epsilon))$.
  - Accelerated: $O(N\sqrt{\frac{L}{\mu}}\log(1/\epsilon))$.
  - SAG: $O(\max\{N, \frac{L}{\mu}\}\log(1/\epsilon))$.

Comparing Deterministic and Stochatic Methods

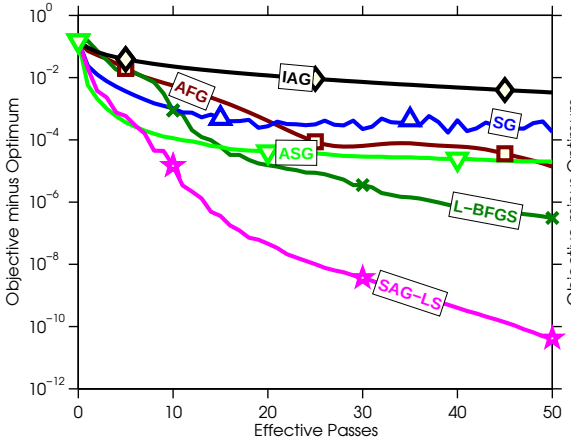
- quantum ($n = 50000$, $p = 78$) and rcv1 ($n = 697641$, $p = 47236$)

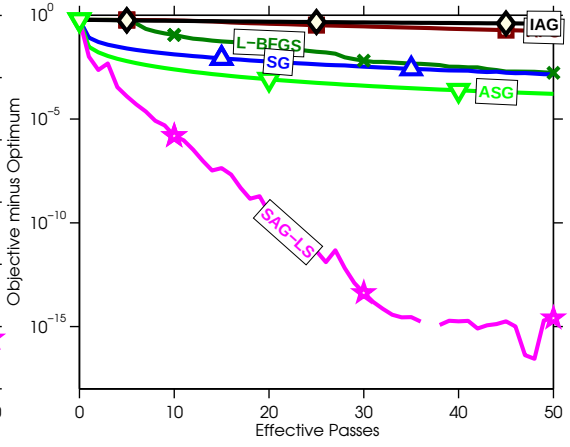




SAG Compared to FG and SG Methods

- quantum ($n = 50000$, $p = 78$) and rcv1 ($n = 697641$, $p = 47236$)





Other Linearly-Convergent Stochastic Methods

- Subsequent stochastic algorithms with linear rates:
  - Stochastic dual coordinate ascent [Shalev-Schwartz & Zhang, 2013]
  - Incremental surrogate optimization [Mairal, 2013].
  - Stochastic variance-reduced gradient (SVRG)
    [Johnson & Zhang, 2013, Konecny & Richtarik, 2013, Mahdavi et al., 2013, Zhang et al., 2013]
  - SAGA [Defazio et al., 2014]

Other Linearly-Convergent Stochastic Methods

- Subsequent stochastic algorithms with linear rates:
  - Stochastic dual coordinate ascent [Shalev-Schwartz & Zhang, 2013]
  - Incremental surrogate optimization [Mairal, 2013].
  - Stochastic variance-reduced gradient (SVRG)
    [Johnson & Zhang, 2013, Konecny & Richtarik, 2013, Mahdavi et al., 2013, Zhang et al., 2013]
  - SAGA [Defazio et al., 2014]
- SVRG has a much lower memory requirement (later in talk).
- There are also projected/proximal/ADMM extensions.

SAG Implementation Issues

- Basic SAG algorithm:
  - while(1)
  - Sample i from $\{1, 2, \ldots, N\}$.
  - Compute $f_i'(x)$.
  - $d = d - y_i + f_i'(x)$.
  - $y_i = f_i'(x)$.
  - $x = x - \frac{\alpha}{N} d$.

SAG Implementation Issues

- Basic SAG algorithm:
  - while(1)
  - Sample i from $\{1, 2, \ldots, N\}$.
  - Compute $f_i'(x)$.
  - $d = d - y_i + f_i'(x)$.
  - $y_i = f_i'(x)$.
  - $x = x - \frac{\alpha}{N} d$.
- Practical variants of the basic algorithm allow:
  - Regularization.
  - Sparse gradients.
  - Automatic step-size selection.
    - Common to use adaptive step-size procedure to estimate L.
  - Termination criterion.
    - Can use $\|x^{t+1} - x^t\|/\alpha = \frac{1}{n} d \approx \|\nabla f(x^t)\|$ to decide when to stop.
  - Acceleration [Lin et al., 2015].

SAG Implementation Issues

- Basic SAG algorithm:
  - while(1)
  - Sample i from $\{1, 2, \ldots, N\}$.
  - Compute $f_i'(x)$.
  - $d = d - y_i + f_i'(x)$.
  - $y_i = f_i'(x)$.
  - $x = x - \frac{\alpha}{N} d$.
- Practical variants of the basic algorithm allow:
  - Regularization.
  - Sparse gradients.
  - Automatic step-size selection.
    - Common to use adaptive step-size procedure to estimate L.
  - Termination criterion.
    - Can use $\|x^{t+1} - x^t\|/\alpha = \frac{1}{n} d \approx \|\nabla f(x^t)\|$ to decide when to stop.
  - Acceleration [Lin et al., 2015].
  - Adaptive non-uniform sampling [Schmidt et al., 2013].

Reshuffling and Non-Uniform Sampling

- Does re-shuffling and doing full passes work better?

Reshuffling and Non-Uniform Sampling

- Does re-shuffling and doing full passes work better?
  - For classic SG: Maybe?
    - Noncommutative arithmetic-geometric mean inequality conjecture.

[Recht & Ré, 2012]

Reshuffling and Non-Uniform Sampling

- Does re-shuffling and doing full passes work better?
  - For classic SG: Maybe?
    - Noncommutative arithmetic-geometric mean inequality conjecture.

[Recht & Ré, 2012]

  - For SAG: NO.
  - Performance is intermediate between IAG and SAG.

Reshuffling and Non-Uniform Sampling

- Does re-shuffling and doing full passes work better?
  - For classic SG: Maybe?
    - Noncommutative arithmetic-geometric mean inequality conjecture.

      [Recht & Ré, 2012]
  - For SAG: NO.
  - Performance is intermediate between IAG and SAG.
- Can non-uniform sampling help?

Reshuffling and Non-Uniform Sampling

- Does re-shuffling and doing full passes work better?
  - For classic SG: Maybe?
    - Noncommutative arithmetic-geometric mean inequality conjecture.

      [Recht & Ré, 2012]
  - For SAG: NO.
  - Performance is intermediate between IAG and SAG.
- Can non-uniform sampling help?
  - For classic SG methods, can only improve constants.

Reshuffling and Non-Uniform Sampling

- Does re-shuffling and doing full passes work better?
  - For classic SG: Maybe?
    - Noncommutative arithmetic-geometric mean inequality conjecture.

[Recht & Ré, 2012]

  - For SAG: NO.
  - Performance is intermediate between IAG and SAG.
- Can non-uniform sampling help?
  - For classic SG methods, can only improve constants.
  - For SAG, bias sampling towards Lipschitz constants L_i,

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L_i \|x - y\|.$$

improves rate to depend on L_{mean} instead of L_{max}.

(with bigger step size)

Reshuffling and Non-Uniform Sampling

- Does re-shuffling and doing full passes work better?
  - For classic SG: Maybe?
    - Noncommutative arithmetic-geometric mean inequality conjecture.

      [Recht & Ré, 2012]
  - For SAG: NO.
  - Performance is intermediate between IAG and SAG.
- Can non-uniform sampling help?
  - For classic SG methods, can only improve constants.
  - For SAG, bias sampling towards Lipschitz constants L_i,

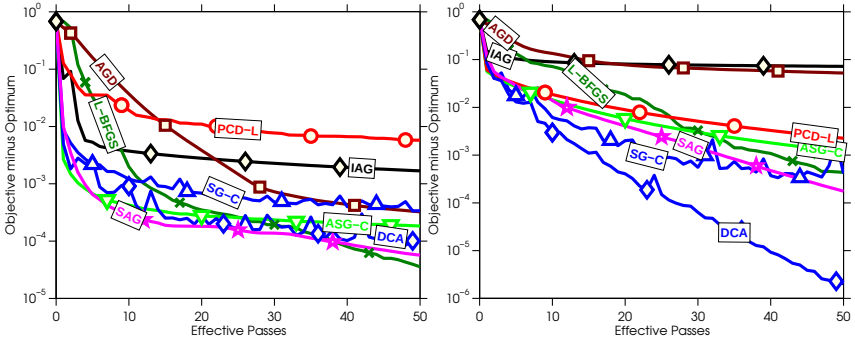
    $$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L_i \|x - y\|.$$

    improves rate to depend on L_{mean} instead of L_{max}.

    (with bigger step size)
  - Adaptively estimate L_i as you go. (see paper/code).
  - Slowly learns to ignore well-classified examples.

SAG with Adaptive Non-Uniform Sampling

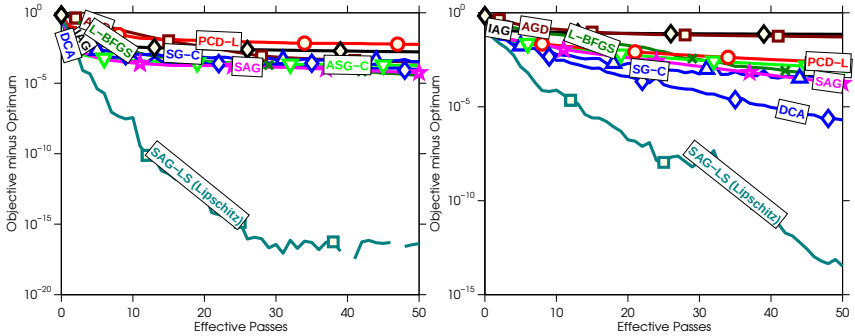
- protein ($n = 145751$, $p = 74$) and sido ($n = 12678$, $p = 4932$)



- Datasets where SAG had the worst relative performance.

SAG with Non-Uniform Sampling

- protein ($n = 145751$, $p = 74$) and sido ($n = 12678$, $p = 4932$)



- Adaptive non-uniform sampling helps a lot.

SAG with Mini-Batches

- Reasons to use mini-batches with SAG:
  1. Parallelize gradient calculation.
  2. Decrease memory (only store gradient of the mini-batch).

SAG with Mini-Batches

- Reasons to use mini-batches with SAG:
  1. Parallelize gradient calculation.
  2. Decrease memory (only store gradient of the mini-batch).
  3. Increase convergence rate.
     (classic SG methods: only changes constant)

SAG with Mini-Batches

- Reasons to use mini-batches with SAG:
  1. Parallelize gradient calculation.
  2. Decrease memory (only store gradient of the mini-batch).
  3. Increase convergence rate.
     (classic SG methods: only changes constant)
- Convergence rate depends on L for mini-batches:
  - $L(\mathcal{B}) \leq L(i)$, possibly by up to $|\mathcal{B}|$.
  - Allows bigger step-size, $\alpha = 1/L(\mathcal{B})$.
  - Place examples in batches to make $L(\mathcal{B})$ small.

Minimizing Finite Sums: Dealing with the Memory

- A major disadvantage of SAG is the memory requirement.

Minimizing Finite Sums: Dealing with the Memory

- A major disadvantage of SAG is the memory requirement.
- Besides mini-batches, structure in objective may avoid this:
  - For linear models where $f_i(w) = g(w^T x_i)$, then only require $O(n)$ memory:

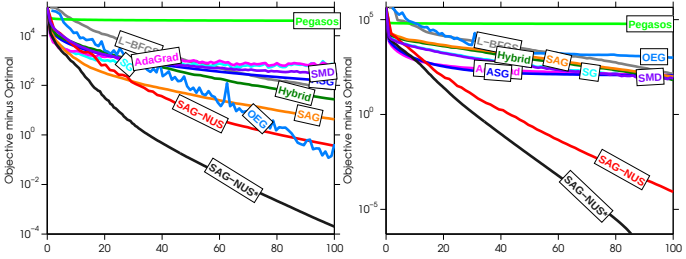
$$\nabla f_i(w) = \underbrace{g'(w^T x_i)}_{\text{scalar}} \underbrace{x_i}_{\text{data}}.$$

Minimizing Finite Sums: Dealing with the Memory

- A major disadvantage of SAG is the memory requirement.
- Besides mini-batches, structure in objective may avoid this:
  - For linear models where $f_i(w) = g(w^T x_i)$, then only require $O(n)$ memory:

$$\nabla f_i(w) = \underbrace{g'(w^T x_i)}_{\text{scalar}} \underbrace{x_i}_{\text{data}}.$$

  - For CRFs, only need to store marginals of parts.



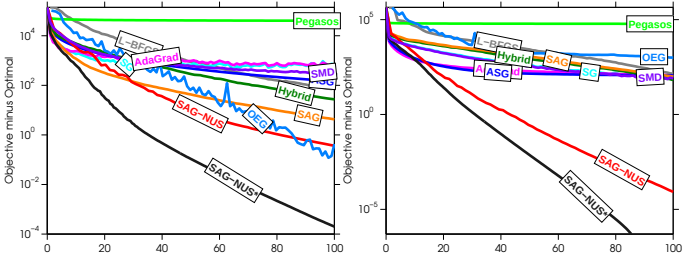
(optical character and named-entity recognition tasks)

Minimizing Finite Sums: Dealing with the Memory

- A major disadvantage of SAG is the memory requirement.
- Besides mini-batches, structure in objective may avoid this:
  - For linear models where $f_i(w) = g(w^T x_i)$, then only require $O(n)$ memory:

$$\nabla f_i(w) = \underbrace{g'(w^T x_i)}_{\text{scalar}} \underbrace{x_i}_{\text{data}}.$$

  - For CRFs, only need to store marginals of parts.



(optical character and named-entity recognition tasks)

- If the above don't work, use SVRG...

Stochastic Variance-Reduced Gradient

SVRG algorithm:

- Start with x_0
- for $s = 0, 1, 2 \ldots$
  - $d_s = \frac{1}{N} \sum_{i=1}^{N} f_i'(x_s)$
  - $x^0 = x_s$

Stochastic Variance-Reduced Gradient

SVRG algorithm:

- Start with x_0
- for $s = 0, 1, 2 \ldots$
  - $d_s = \frac{1}{N} \sum_{i=1}^{N} f_i'(x_s)$
  - $x^0 = x_s$
  - for $t = 1, 2, \ldots m$
    - Randomly pick $i_t \in \{1, 2, \ldots, N\}$
    - $x^t = x^{t-1} - \alpha_t (f_{i_t}'(x^{t-1}) - f_{i_t}'(x_s) + d_s)$.
  - $x_{s+1} = x^t$ for random $t \in \{1, 2, \ldots, m\}$.

Stochastic Variance-Reduced Gradient

SVRG algorithm:

- Start with x_0
- for $s = 0, 1, 2 \ldots$
  - $d_s = \frac{1}{N} \sum_{i=1}^{N} f_i'(x_s)$
  - $x^0 = x_s$
  - for $t = 1, 2, \ldots m$
    - Randomly pick $i_t \in \{1, 2, \ldots, N\}$
    - $x^t = x^{t-1} - \alpha_t (f_{i_t}'(x^{t-1}) - f_{i_t}'(x_s) + d_s)$.
  - $x_{s+1} = x^t$ for random $t \in \{1, 2, \ldots, m\}$.

Requires 2 gradients per iteration and occasional full passes,
but only requires storing d_s and x_s.

Stochastic Variance-Reduced Gradient

SVRG algorithm:

- Start with x_0
- for $s = 0, 1, 2 \ldots$
  - $d_s = \frac{1}{N} \sum_{i=1}^{N} f_i'(x_s)$
  - $x^0 = x_s$
  - for $t = 1, 2, \ldots m$
    - Randomly pick $i_t \in \{1, 2, \ldots, N\}$
    - $x^t = x^{t-1} - \alpha_t (f_{i_t}'(x^{t-1}) - f_{i_t}'(x_s) + d_s)$.
  - $x_{s+1} = x^t$ for random $t \in \{1, 2, \ldots, m\}$.

Requires 2 gradients per iteration and occasional full passes,
but only requires storing d_s and x_s.

Practical issues similar to SAG (acceleration versions, automatic step-size/termination, handles sparsity/regularization, non-uniform sampling, mini-batches).

(pause)

Stochastic Subgradient for Infinite Datasets?

- In analysis of stochastic subgradient, two assumptions on g_{i_t}:
  - Unbiased approximation of subgradient: $\mathbb{E}[g_{i_t}] = g_t$.
  - Variance is bounded: $\mathbb{E}[\|g_{i_t}\|^2] \leq B^2$.
- Unlike SAG, stochastic subgradient applies to general stochastic optimization:

$$\operatorname*{argmin}_{x \in \mathbb{R}^d} \mathbb{E}[f_i(x)].$$

Stochastic Subgradient for Infinite Datasets?

- In analysis of stochastic subgradient, two assumptions on g_{i_t}:
  - Unbiased approximation of subgradient: $\mathbb{E}[g_{i_t}] = g_t$.
  - Variance is bounded: $\mathbb{E}[\|g_{i_t}\|^2] \leq B^2$.
- Unlike SAG, stochastic subgradient applies to general stochastic optimization:

$$\operatorname*{argmin}_{x \in \mathbb{R}^d} \mathbb{E}[f_i(x)].$$

- We can use stochastic subgradient on IID samples from infinite dataset:
  - $O(1/\epsilon)$ rate still applies.

Stochastic vs. Deterministic for Stochastic Objectives

- Consider smooth/strongly-convex stochastic objectives,

$$\min_{x \in \mathbb{R}^D} \mathbb{E}[f_i(x)],$$

including the generalization error in machine learning.

Stochastic vs. Deterministic for Stochastic Objectives

- Consider smooth/strongly-convex stochastic objectives,

$$\min_{x \in \mathbb{R}^D} \mathbb{E}[f_i(x)],$$

  including the generalization error in machine learning.
- Error ϵ has two parts [Bottou & Bousquet, 2007]:

$$\epsilon = (\text{optimization error}) + (\text{estimation error}).$$

(for generalization error, also have model error)

Stochastic vs. Deterministic for Stochastic Objectives

- Consider smooth/strongly-convex stochastic objectives,

$$\min_{x \in \mathbb{R}^D} \mathbb{E}[f_i(x)],$$

  including the generalization error in machine learning.
- Error ϵ has two parts [Bottou & Bousquet, 2007]:

$$\epsilon = (\text{optimization error}) + (\text{estimation error}).$$

(for generalization error, also have model error)

- Consider two strategies:
  - Generate t samples, then minimize exactly (ERM):
    - Optimization error $= 0$.
    - Estimation error $= \tilde{O}(1/t)$.

Stochastic vs. Deterministic for Stochastic Objectives

- Consider smooth/strongly-convex stochastic objectives,

$$\min_{x \in \mathbb{R}^D} \mathbb{E}[f_i(x)],$$

  including the generalization error in machine learning.
- Error ϵ has two parts [Bottou & Bousquet, 2007]:

$$\epsilon = (\text{optimization error}) + (\text{estimation error}).$$

(for generalization error, also have model error)

- Consider two strategies:
  - Generate t samples, then minimize exactly (ERM):
    - Optimization error $= 0$.
    - Estimation error $= \tilde{O}(1/t)$.
  - Or just applying stochastic gradient as we go:
    - Optimization error $= O(1/t)$.
    - Estimation error $= \tilde{O}(1/t)$.

Stochastic vs. Deterministic for Stochastic Objectives

- So just go through your data once with stochastic gradient?

Stochastic vs. Deterministic for Stochastic Objectives

- So just go through your data once with stochastic gradient?
- "overwhelming empirical evidence shows that for almost all actual data, the ERM *is* better. However, we have no understanding of why this happens"

[Srebro & Sridharan, 2011]

Stochastic vs. Deterministic for Stochastic Objectives

- So just go through your data once with stochastic gradient?
- "overwhelming empirical evidence shows that for almost all actual data, the ERM *is* better. However, we have no understanding of why this happens"

  [Srebro & Sridharan, 2011]
- Constants matter in learning:
  - SG optimal in terms of sample size.
  - But not other quantities: L, μ, x^0.
  - We care about multiplying test error by 2!

Stochastic vs. Deterministic for Stochastic Objectives

- So just go through your data once with stochastic gradient?
- "overwhelming empirical evidence shows that for almost all actual data, the ERM *is* better. However, we have no understanding of why this happens"

  [Srebro & Sridharan, 2011]
- Constants matter in learning:
  - SG optimal in terms of sample size.
  - But not other quantities: L, μ, x^0.
  - We care about multiplying test error by 2!
- Growing-batch deterministic methods [Byrd et al., 2011].
- Or take t iterations of SAG on fixed $N < t$ samples.
  - Optimization accuracy decreases to $O(\rho^t)$.
  - Estimation accuracy increases to $\tilde{O}(1/N)$.

Stochastic vs. Deterministic for Stochastic Objectives

- So just go through your data once with stochastic gradient?
- "overwhelming empirical evidence shows that for almost all actual data, the ERM *is* better. However, we have no understanding of why this happens"

  [Srebro & Sridharan, 2011]
- Constants matter in learning:
  - SG optimal in terms of sample size.
  - But not other quantities: L, μ, x^0.
  - We care about multiplying test error by 2!
- Growing-batch deterministic methods [Byrd et al., 2011].
- Or take t iterations of SAG on fixed $N < t$ samples.
  - Optimization accuracy decreases to $O(\rho^t)$.
  - Estimation accuracy increases to $\tilde{O}(1/N)$.
- SAG obtains better bounds for difficult optimization problems.

Streaming SVRG

Streaming SVRG algorithm [Frostig et al., 2015]:

- Start with x_0 and initial sample size N

Streaming SVRG

Streaming SVRG algorithm [Frostig et al., 2015]:

- Start with x_0 and initial sample size N
- for $s = 0, 1, 2 \ldots$
  - $d_s = \frac{1}{N} \sum_{i=1}^{N} f_i'(x_s)$ for N fresh samples.
  - $x^0 = x_s$

Streaming SVRG

Streaming SVRG algorithm [Frostig et al., 2015]:

- Start with x_0 and initial sample size N
- for $s = 0, 1, 2 \ldots$
  - $d_s = \frac{1}{N} \sum_{i=1}^{N} f_i'(x_s)$ for N fresh samples.
  - $x^0 = x_s$
  - for $t = 1, 2, \ldots m$
    - Randomly pick 1 fresh sample.
    - $x^t = x^{t-1} - \alpha_t (f_{i_t}'(x^{t-1}) - f_{i_t}'(x_s) + d_s)$.
  - $x_{s+1} = x^t$ for random $t \in \{1, 2, \ldots, m\}$.
  - Increase samples size N.

Streaming SVRG

Streaming SVRG algorithm [Frostig et al., 2015]:

- Start with x_0 and initial sample size N
- for $s = 0, 1, 2 \dots$
  - $d_s = \frac{1}{N} \sum_{i=1}^{N} f_i'(x_s)$ for N fresh samples.
  - $x^0 = x_s$
  - for $t = 1, 2, \dots m$
    - Randomly pick 1 fresh sample.
    - $x^t = x^{t-1} - \alpha_t (f_{i_t}'(x^{t-1}) - f_{i_t}'(x_s) + d_s)$.
  - $x_{s+1} = x^t$ for random $t \in \{1, 2, \dots, m\}$.
  - Increase samples size N.

- Streaming SVRG is optimal in non-asymptotic regime.
- Same variance as ERM (only true for avg(SG) asymptotically).
- Second-order methods are not necessary.

Constant-Step SG under Strong Assumptions

- We can beat $O(1/t)$ under stronger assumptions.

Constant-Step SG under Strong Assumptions

- We can beat $O(1/t)$ under stronger assumptions.
- E.g., Schmidt & Le Roux [2013],

$$\|f_i'(x)\| \leq B\|f'(x)\|.$$

- Crazy assumption: assumes x^* minimizes f_i.

Constant-Step SG under Strong Assumptions

- We can beat $O(1/t)$ under stronger assumptions.
- E.g., Schmidt & Le Roux [2013],

$$\|f_i'(x)\| \leq B\|f'(x)\|.$$

- Crazy assumption: assumes x^* minimizes f_i.
- With $\alpha_t = \frac{1}{LB^2}$, stochastic gradient has

$$\mathbb{E}[f(x^t)] - f(x^*) \leq \left(1 - \frac{\mu}{LB^2}\right)^t [f(x^0) - f(x^*)].$$

- If you expect to over-fit, maybe constant α_t is enough?

Online Convex Optimization

- What if data is not IID?

Online Convex Optimization

- What if data is not IID?
- Addressed by online convex optimization (OCO) framework: [Zinkevich, 2003]
  - At time t, make a prediction x^t.

Online Convex Optimization

- What if data is not IID?
- Addressed by online convex optimization (OCO) framework: [Zinkevich, 2003]
  - At time t, make a prediction x^t.
  - Receive arbitrary convex loss f_t.
- OCO analyzes regret,

$$\sum_{k=1}^{t} f_t(x^t) - f_t(x^*),$$

comparing vs. best fixed x^* for any sequence $\{f_t\}$.

Online Convex Optimization

- What if data is not IID?
- Addressed by online convex optimization (OCO) framework: [Zinkevich, 2003]
  - At time t, make a prediction x^t.
  - Receive arbitrary convex loss f_t.
- OCO analyzes regret,

$$\sum_{k=1}^{t} f_t(x^t) - f_t(x^*),$$

  comparing vs. best fixed x^* for any sequence $\{f_t\}$.
- SG-style methods achieve optimal $O(\sqrt{t})$ regret.
- Strongly-convex losses: $O(\log(t))$ regret [Hazan et al., 2006].

Online Convex Optimization

- What if data is not IID?
- Addressed by online convex optimization (OCO) framework: [Zinkevich, 2003]
  - At time t, make a prediction x^t.
  - Receive arbitrary convex loss f_t.
- OCO analyzes regret,

$$\sum_{k=1}^{t} f_t(x^t) - f_t(x^*),$$

  comparing vs. best fixed x^* for any sequence $\{f_t\}$.
- SG-style methods achieve optimal $O(\sqrt{t})$ regret.
- Strongly-convex losses: $O(\log(t))$ regret [Hazan et al., 2006].
- Variants exist see features first [Cesa-Bianchi et al., 1993.
- Bandit setting: no gradients.

Summary

Summary

- Subgradients: generalize gradients for non-smooth convex functions.
- Subgradient method: optimal but very-slow general non-smooth method.
- Stochastic subgadient method: same rate but n times cheaper.

Summary

- Subgradients: generalize gradients for non-smooth convex functions.
- Subgradient method: optimal but very-slow general non-smooth method.
- Stochastic subgadient method: same rate but n times cheaper.
- Constant step-size: subgradient quickly converges to approximate solution.
- Decreasing step-size: subgradient slowly converges to exact solution.
- Practical stochastic subgradient methods:
  - Tricks like $\beta^t v^t$ allow training on huge sparse datasets.
  - Different step-size strategies and averaging significantly improve performance.

Summary

- Subgradients: generalize gradients for non-smooth convex functions.
- Subgradient method: optimal but very-slow general non-smooth method.
- Stochastic subgadient method: same rate but n times cheaper.
- Constant step-size: subgradient quickly converges to approximate solution.
- Decreasing step-size: subgradient slowly converges to exact solution.
- Practical stochastic subgradient methods:
  - Tricks like $\beta^t v^t$ allow training on huge sparse datasets.
  - Different step-size strategies and averaging significantly improve performance.
- Stochastic average gradient: $O(\log(1/\epsilon))$ iterations with 1 gradient per iteration.

Summary

- Subgradients: generalize gradients for non-smooth convex functions.
- Subgradient method: optimal but very-slow general non-smooth method.
- Stochastic subgadient method: same rate but n times cheaper.
- Constant step-size: subgradient quickly converges to approximate solution.
- Decreasing step-size: subgradient slowly converges to exact solution.
- Practical stochastic subgradient methods:
  - Tricks like $\beta^t v^t$ allow training on huge sparse datasets.
  - Different step-size strategies and averaging significantly improve performance.
- Stochastic average gradient: $O(\log(1/\epsilon))$ iterations with 1 gradient per iteration.
- Infinite Training Sets can be used with stochastic subgradient.
  - But recent results indicate it's sometimes better to apply SAG to finite sample.

- Next time: how to use (some) infinite sets of features.