Linearly-Convergent Stochastic-Gradient Methods

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Context: Machine Learning for "Big Data"

- **Large-scale machine learning**: large N, large P
  - N: number of observations (inputs)
  - P: dimension of each observation

- **Examples**: vision, bioinformatics, speech, language, etc.
  - Pascal large-scale datasets: $N = 5 \cdot 10^5, P = 10^3$
  - ImageNet: $N = 10^7$
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- Main computational challenge:
  - Design algorithms for very large N and P.

Example: Supervised Machine Learning

- **Data**: n observations (a_i, b_i), $i = 1, \dots, N$.
- Prediction as linear function $x^T a_i$ of features $a_i \in \mathbb{R}^P$.
- **Regularized empirical risk minimization**: find x^* solution of

$$\min_{x \in \mathbb{R}^P} \frac{1}{N} \sum_{i=1}^{N} \ell(b_i, x^T a_i) \quad + \quad \lambda r(x)$$

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- Applications to any data-oriented field:
  - Vision, bioinformatics, speech, natural language, web.
- Main practical challenges:
  - Designing/learning good features a_i.
  - Efficiently solving the problem when N or P are very large.

Big-N Problems

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- We will focus on strongly-convex functions g.
- Simplest example is ℓ_2-regularized least-squares,

$$f_i(x) := (a_i^T x - b_i)^2 + \frac{\lambda}{2}\|x\|^2.$$

- Other examples include any ℓ_2-regularized convex loss:
  - logistic regression, smooth SVMs, CRFs, etc.

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  - Random selection of $i(t)$ from $\{1, 2, \ldots, N\}$,

$$x_{t+1} = x_t - \alpha_t f_{i(t)}(x_t).$$

  - Iteration cost is independent of N.
  - Sublinear convergence rate: $O(1/t)$.

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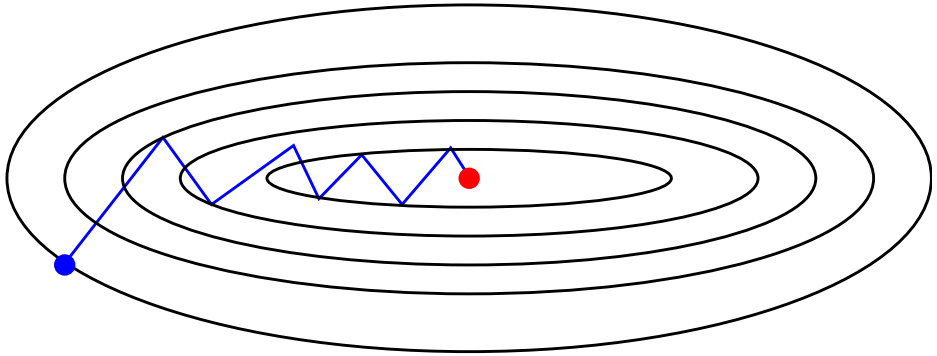

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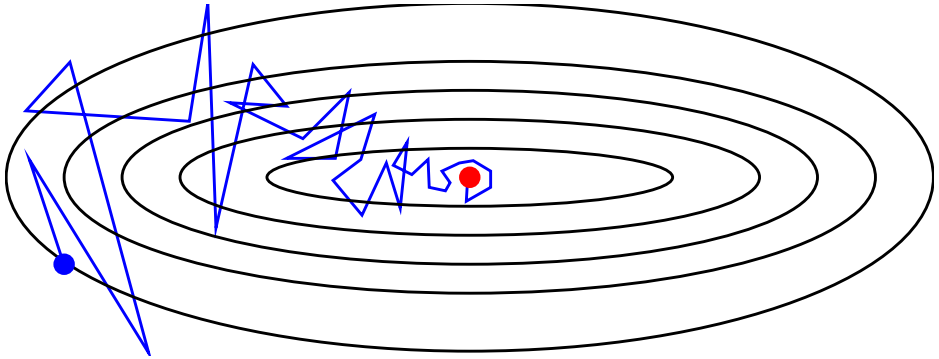
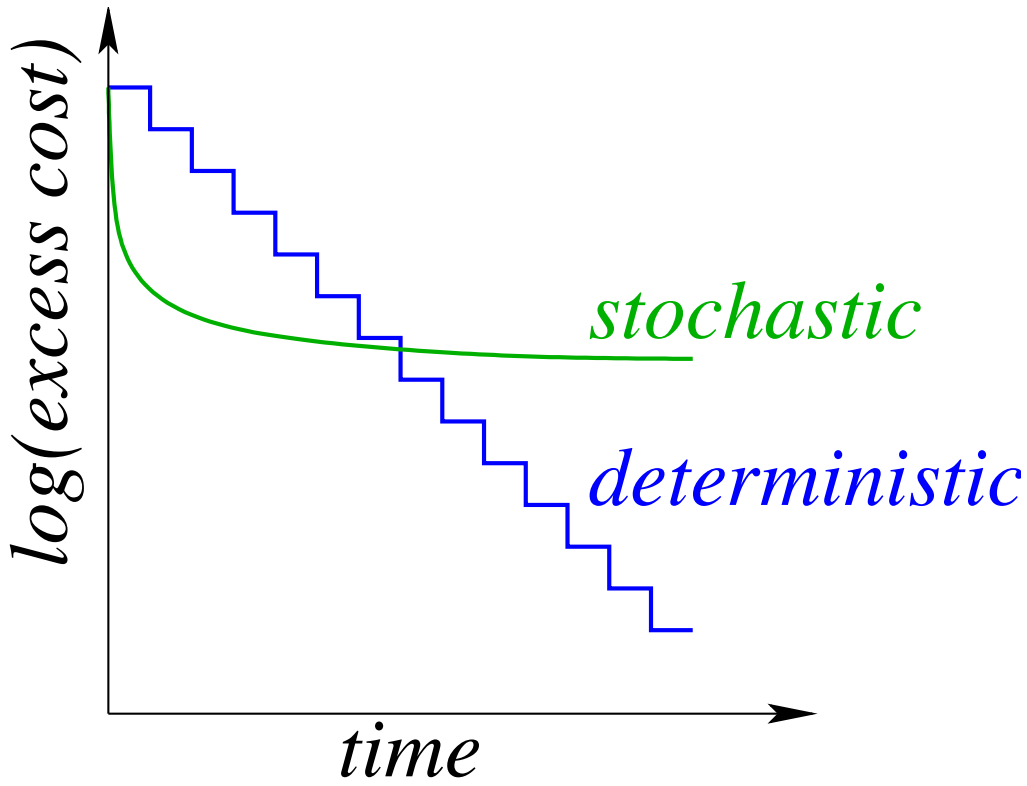


Motivation for New Methods

- FG method has $O(N)$ cost with linear rate.
- SG method has $O(1)$ cost with sublinear rate.



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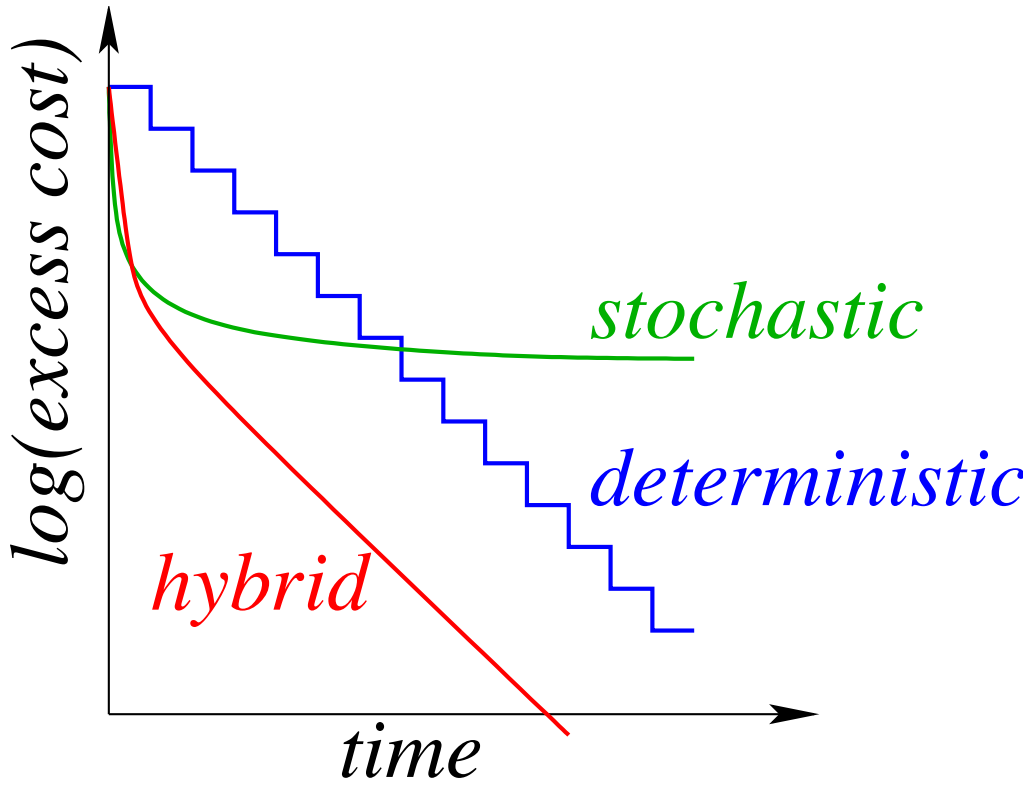

- Goal is linear rate with reduced cost.

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- **Hybrid Methods, Incremental Average Gradient**:

  [Bertsekas, 1997, Blatt et al., 2007]

  - Linear rate, but iterations make full passes through the data.

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   - Iteration cost grows from $O(1)$ to $O(N)$.

2. SAG algorithm: sequence of estimates converging to $g'(x^t)$ as $\|x^t - x^{t-1}\| \to 0$.
   - Linear convergence rate.
   - Iteration cost is $O(1)$.

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- A common variant is to use larger sample $\mathcal{B}^t$,

$$\frac{1}{|\mathcal{B}^t|}\sum_{i\in\mathcal{B}^t} f_i'(x^t) \approx \frac{1}{N}\sum_{i=1}^{N} f_i(x^t).$$

SG methods with a larger subsample

- The SG method with a sample $\mathcal{B}^t$ uses iterations

$$x^{t+1} = x^t - \frac{\alpha^t}{|\mathcal{B}^t|} \sum_{i \in \mathcal{B}^t} f_i(x^t).$$

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- Early iterations are cheap like SG iterations.

Convergence of gradient method with bounded error

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- We analyze how $\|e^t\|$ affects the convergence rate.

Convergence of gradient method with bounded error

Proposition 1. *If the sequence* $\{\mathbb{E}[\|e^t\|^2]\}$ *is in* $O(\gamma^t)$, *then*

$$\mathbb{E}[g(x^t) - g(x^*)] \leqslant (1 - \mu/L)^t[g(x^0) - g(x^*)] + O(\rho^t),$$

where $\rho = \max\{\gamma, 1 - \mu/L + \epsilon\}$ *for any* $\epsilon \geqslant 0$.

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- We also obtain a bound on the iterates because

$$\frac{\mu}{2}\|x^t - x^*\|^2 \leqslant g(x^t) - g(x^*) \leqslant \frac{L}{2}\|x^t - x^*\|^2.$$

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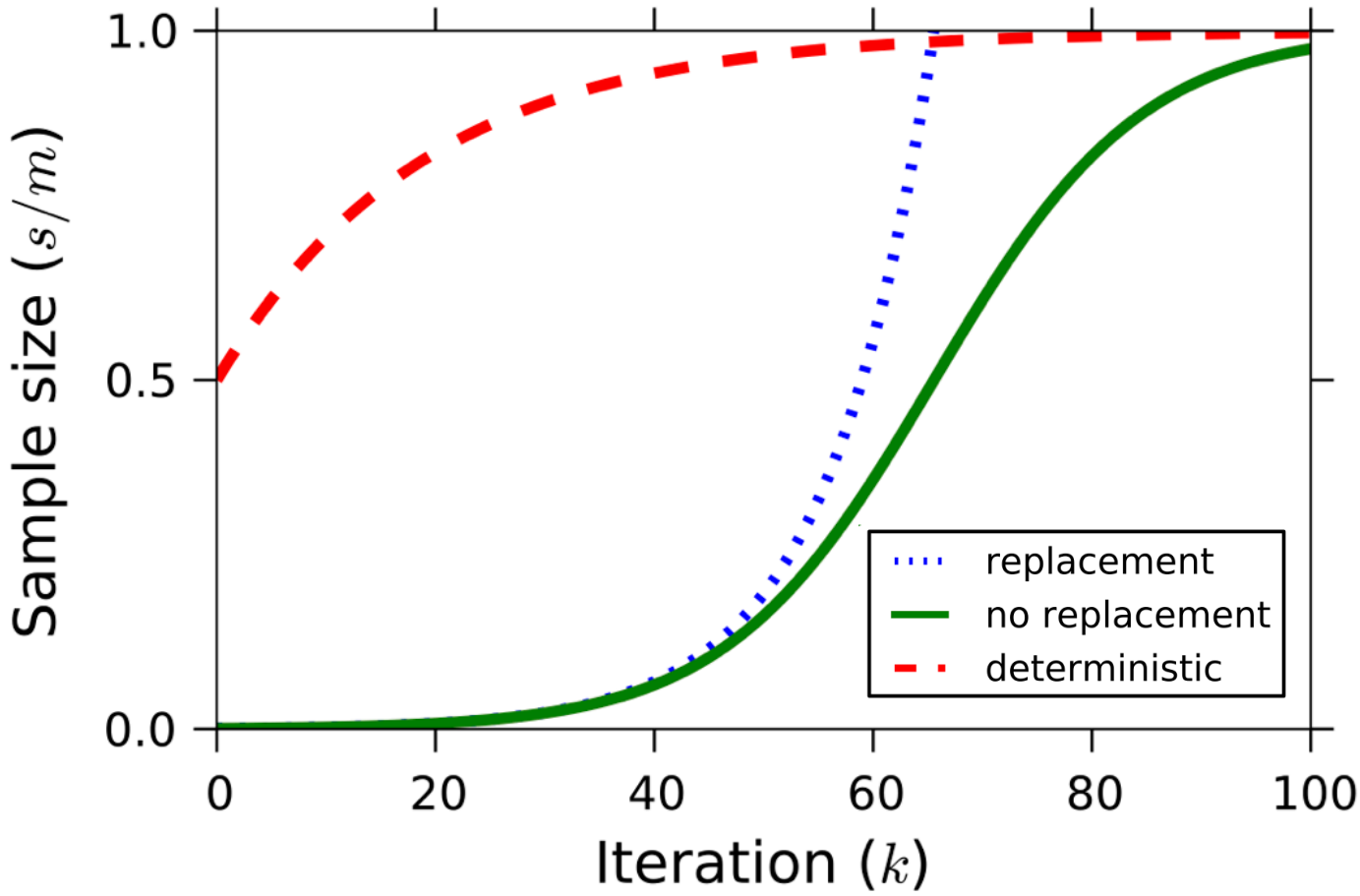
Sample Size needed for Linear Rate



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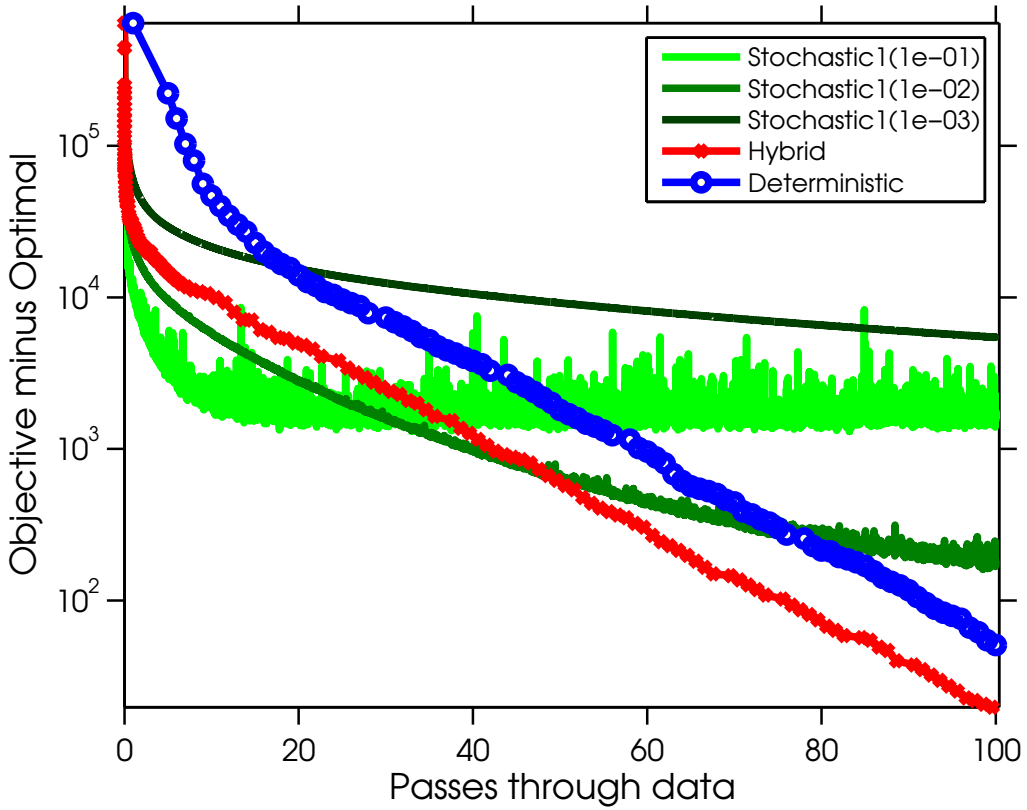
- We made a practical implementation:
  - *L-BFGS* Hessian approximation.
  - Armijo line-search *on the batch*.
  - Eventually reduces to standard quasi-Newton method.

Evaluation on Chain-Structured CRFs

Results on chain-structured conditional random field:



Hybrid uses $|\mathcal{B}^{t+1}| = \lceil \min\{1.1 \cdot |\mathcal{B}^t| + 1, N\} \rceil$.

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  - Stochastic variant of increment average gradient (IAG). [Blatt et al. 2007]
  - Assumes that gradients of other examples don't change.

Convergence Rate of SAG

- Assume each f_i' is $L-$continuous, g is μ-strongly convx.

Theorem. *With* $\alpha_t = \frac{1}{16L}$ *the SAG iterations satisfy*

$$\mathbb{E}[g(x^t) - g(x^*)] \leqslant \left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^t C,$$

with

$$C = [g(x^0) - g(x^*)] + \frac{4L}{N}\|x^0 - x^*\|^2 + \frac{\sigma^2}{16L}.$$

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- **Linear convergene with iteration cost independent of** N**.**
- *"despite 60 years of extensive research on SG methods, with a significant portion of the applications focusing on finite datasets, we are not aware of any other SG method that achieves a linear convergence rate while preserving the iteration cost of standard SG methods."*

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- Further, this rate is "very fast":
  - For well-conditioned problems, constant reduction per pass:

$$\left(1 - \frac{1}{8N}\right)^N \leq \exp\left(-\frac{1}{8}\right) = 0.8825.$$

  - For ill-conditioned problems, almost same as deterministic method. (but N times faster)

Rate of Convergence Comparison

- Assume that $L = 100$, $\mu = .01$, and $n = 80000$:

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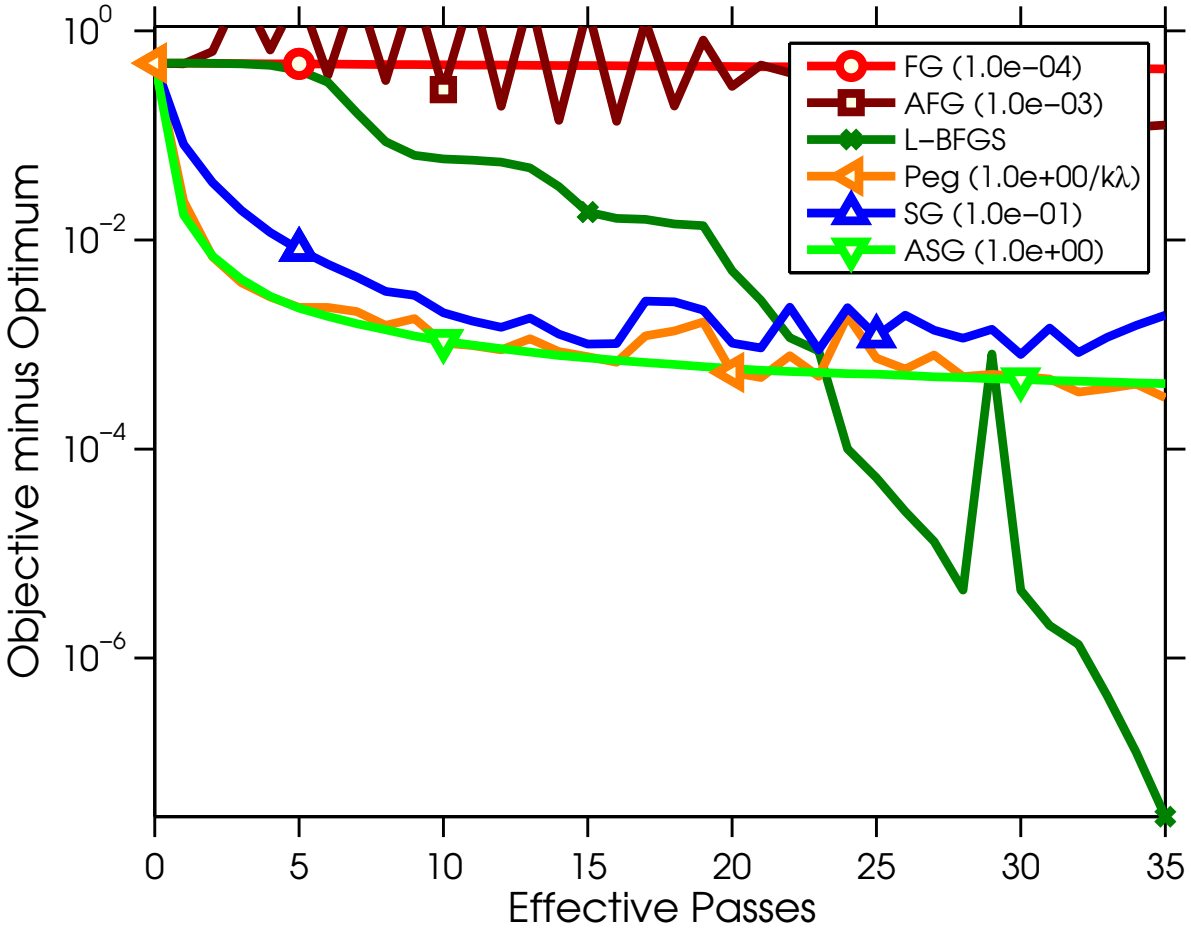
- $O(1/k)$ rate in convex case (vs. $O(1/\sqrt{k})$ for SG methods).
- Algorithm is *adaptive to* μ around optimum.
- Can improve C by clever initialization of x^0 and y_i^0.

Experiments with SAG algorithm

- rcv1 data set ($n = 20242$, $p = 47236$)



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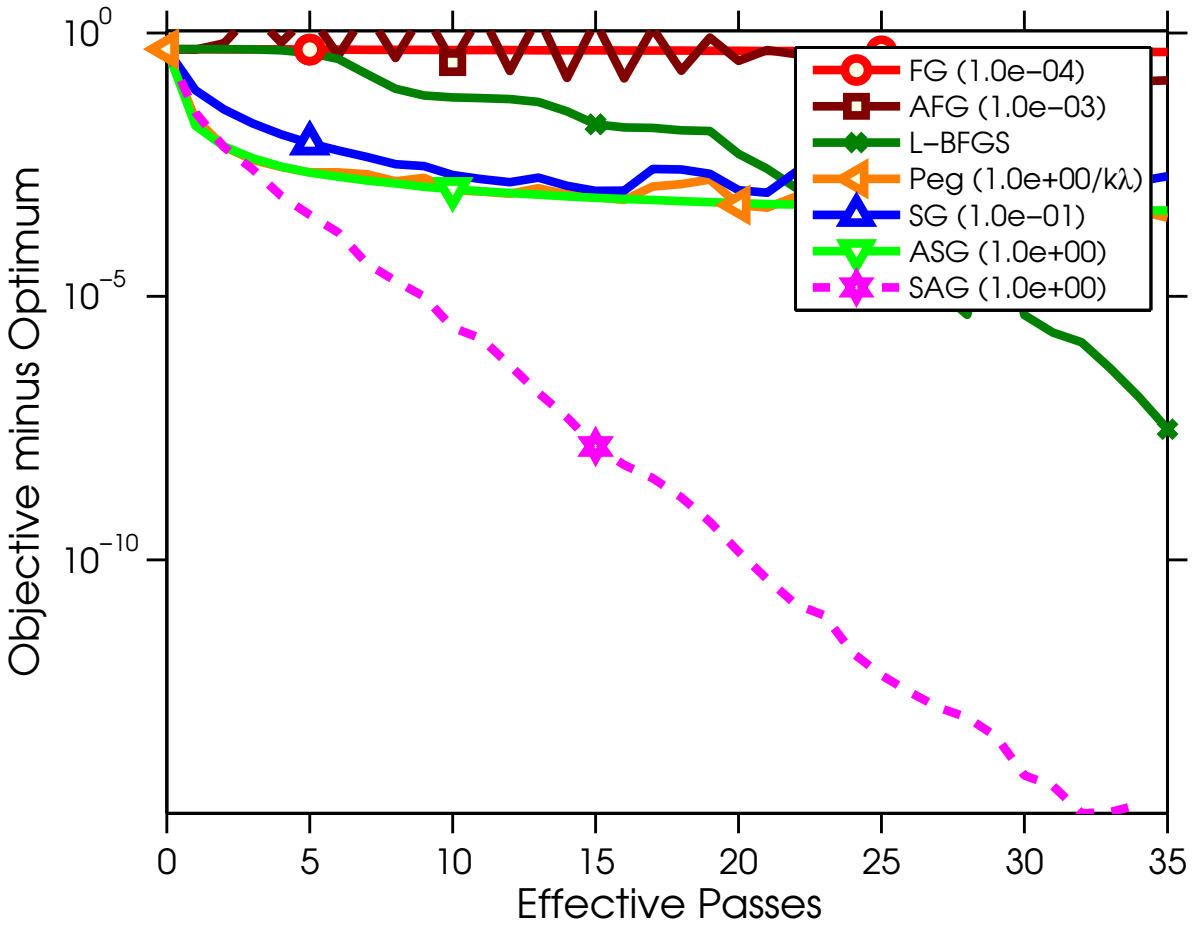

Reducing memory requirements

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$$x^{t+1} = x^t - \frac{\alpha_t}{n} \sum_{i=1}^{N} y_i^t,$$

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- Thanks for coming!