CONVEX OPTIMIZATION CHEAT SEET

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See Nesterov's book for proofs of the below.
We say that a function f is convex if for all x and y on its domain and all $0 \leq \alpha \leq 1$ we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

If f is differentiable, equivalent definitions are that

$$\begin{aligned} &f(y) \geq f(x) + \langle f'(x), y - x\rangle, \\ &\langle f'(x) - f'(y), x - y)\rangle \geq 0. \end{aligned}$$

If f is twice-differentiable, an equivalent definition is that

$$\nabla^2 f(x) \succeq 0.$$

For a differentiable convex f, the following conditions are equivalent to the condition that the gradient f' is L-Lipschitz continuous:

$$\begin{aligned}
&\|f'(x) - f'(y)\| \leq L\|x - y\| \\
&f(y) \leq f(x) + \langle f'(x), y - x\rangle + \frac{L}{2}\|x - y\|^2 \\
&f(y) \geq f(x) + \langle f'(x), y - x\rangle + \frac{1}{2L}\|f'(x) - f'(y)\|^2 \\
&\langle f'(x) - f'(y), x - y\rangle \leq L\|x - y\|^2 \\
&\langle f'(x) - f'(y), x - y\rangle \geq \frac{1}{L}\|f'(x) - f'(y)\|^2 \\
&f(\alpha x + (1 - \alpha y)) \geq \alpha f(x) + (1-\alpha)f(y) - \frac{\alpha(1-\alpha)L}{2}\|x - y\|^2 \\
&f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) - \frac{\alpha(1-\alpha)}{2L}\|f'(x) - f'(y)\|^2.
\end{aligned}$$

You can define Lipschitz continuity under a different norm, and in this case the first condition becomes $\|f'(x) - f'(y)\|_q \leq L\|x - y\|_p$ where $\|\cdot\|_p$ and $\|\cdot\|_q$ are dual norms. For all the other inequalities, you replace all instances of $\|x - y\|$ with $\|x - y\|_p$ and $\|f'(x) - f'(y)\|$ with $\|f'(x) - f'(y)\|_q$.
For twice-differentiable f, any of the above are equivalent (under the Euclidean norm) to

$$\nabla^2 f(x) \preceq LI.$$

The following conditions are equivalent to the condition that a differentiable f is μ-strongly convex:

$$\begin{aligned}
&x \mapsto f(x) - \frac{\mu}{2}\|x\|^2 \text{ is convex} \\
&f(y) \geq f(x) + \langle f'(x), y - x\rangle + \frac{\mu}{2}\|x - y\|^2 \\
&\langle f'(x) - f'(y), x - y\rangle \geq \mu\|x - y\|^2 \\
&f(\alpha x + (1-\alpha)y)) \leq \alpha f(x) + (1-\alpha)f(y) - \frac{\alpha(1-\alpha)\mu}{2}\|x - y\|^2.
\end{aligned}$$

The following are not equivalent to μ-strong convexity but are implied by it:

$$\begin{aligned}
&\|f'(x) - f'(y)\| \geq \mu\|x - y\| \\
&f(y) \leq f(x) + \langle f'(x), y - x\rangle + \frac{1}{2\mu}\|f'(x) - f'(y)\|^2 \\
&\langle f'(x) - f'(y), x - y\rangle \leq \frac{1}{\mu}\|f'(x) - f'(y)\|^2.
\end{aligned}$$

For a twice-differentiable f strong-convexity is equivalent to:

$$\nabla^2 f(x) \succeq \mu I.$$

If f is μ-strongly convex and f' is L-Lipschitz continuous then we have

$$\langle f'(x) - f'(y), x - y\rangle \geq \frac{\mu L}{\mu + L}\|x - y\|^2 + \frac{1}{\mu + L}\|f'(x) - f'(y)\|^2.$$