First-Order Optimization Algorithms for Machine Learning

Stochastic Subgradient

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Last time: Stochastic Gradient Descent

- We discussed minimizing finite sums,

$$f(w) = \frac{1}{n}\sum_{i=1}^{n} f_i(w),$$

when n is very large.

- We discussed the stochastic gradient method,

$$w^{k+1} = w^k - \alpha_k \nabla f_{i_k}(w^k),$$

where i_k is chosen uniformly from $\{1, 2, \ldots, n\}$.

- Iterations are n-times cheaper than gradient descent.
  - But convergence rate is much slower than gradient descent.
    - And tricks like momentum/Newton/adaptive do not close the gap.

Stochastic vs. Deterministic for Non-Smooth

- The story changes for non-smooth problems.
- Consider the binary support vector machine (SVM) objective:

$$f(w) = \sum_{i=1}^{n} \max\{0, 1 - y_i(w^\top x_i)\} + \frac{\lambda}{2}\|w\|^2.$$

- Rates for subgradient methods for non-smooth objectives:

Assumption	Deterministic	Stochastic
Convex	$O(1/\epsilon^2)$	$O(1/\epsilon^2)$
Strongly	$O(1/\epsilon)$	$O(1/\epsilon)$

- So for non-smooth problems (without nice structure as in proximal-gradient):
  - Deterministic methods are not faster than stochastic method.
  - So use stochastic subgradient (iterations are n times faster).

Subgradient Method

- The basic subgradient method:

$$w^{k+1} = w^k - \alpha_k g_k,$$

  for some $g_k \in \partial f(w^k)$.
- Decreases distance to solution for small enough α_k (for convex f).

- The basic stochastic subgradient method:

$$w^{k+1} = w^k - \alpha_k g_{i_k},$$

  for some $g_{i_k} \in \partial f_{i_k}(w^k)$ for some random $i_k \in \{1, 2, \dots, n\}$.

- Stochastic subgradient is n times faster with similar convergence properties.
- Decreases expected distance to solution for small enough α_k (for convex f).

Convergence Rate of Stochastic Gradient Method

- We'll first show progress bound for stochastic gradient assuming ∇f is Lipschitz.
  - We'll come back to the non-smooth case.

- Recall the the descent lemma applied to w^{k+1} and w^k,

$$f(w^{k+1}) \leq f(w^k) + \nabla f(w^k)^\top (w^{k+1} - w^k) + \frac{L}{2}\|w^{k+1} - w^k\|^2.$$

- Plugging in stochastic gradient iteration $(w^{k+1} - w^k) = -\alpha_k \nabla f_{i_k}(w^k)$ gives

$$f(w^{k+1}) \leq f(w^k) - \alpha_k \nabla f(w^k)^\top \nabla f_{i_k}(w^k) + \alpha_k^2 \frac{L}{2}\|\nabla f_{i_k}(w^k)\|^2.$$

Convergence Rate of Stochastic Gradient Method

- So far any choice of α_k and i_k we have

$$f(w^{k+1}) \leq f(w^k) - \alpha_k \nabla f(w^k)^\top \nabla f_{i_k}(w^k) + \alpha_k^2 \frac{L}{2} \|\nabla f_{i_k}(w^k)\|^2.$$

- Let's take the expectation with respect to i_k assuming $p(i_k = i) = 1/n$,

$$\begin{aligned}\mathbb{E}[f(w^{k+1})] &\leq \mathbb{E}[f(w^k) - \alpha_k \nabla f(w^k)^\top \nabla f_{i_k}(w^k) + \alpha_k^2 \frac{L}{2} \|\nabla f_{i_k}(w^k)\|^2] \\ &= f(w^k) - \alpha_k \nabla f(w^k)^\top \mathbb{E}[\nabla f_{i_k}(w^k)] + \alpha_k^2 \frac{L}{2} \mathbb{E}[\|\nabla f_{i_k}(w^k)\|^2],\end{aligned}$$

  where the second line uses linearity of expectation (and α_k not depending on i_k).
- We know that $\mathbb{E}[\nabla f_{i_k}(w^k)] = \nabla f(w^k)$ (unbiased) so this gives

$$\mathbb{E}[f(w^{k+1})] \leq f(w^k) - \alpha_k \underbrace{\|\nabla f(w^k)\|^2}_{\text{good}} + \alpha_k^2 \underbrace{\frac{L}{2} \mathbb{E}[\|\nabla f_{i_k}(w^k)\|^2]}_{\text{bad}}.$$

Convergence Rate of Stochastic Gradient Method

- So a progress bound for stochastic gradient is

$$\mathbb{E}[f(w^{k+1})] \leq f(w^k) - \alpha_k \underbrace{\|\nabla f(w^k)\|^2}_{\text{good}} + \alpha_k^2 \underbrace{\frac{L}{2}\mathbb{E}[\|\nabla f_{i_k}(w^k)\|^2]}_{\text{bad}}.$$

- "Good" term looks like usual measure of progress: big gradient $\rightarrow$ big progress.
- "Bad" term is the problem: less progress if gradients are very different.
  - And now choosing $\alpha_k = 1/L$ might not be small enough.
  - But we can control badness: if α_k is small then $\alpha_k >> \alpha_k^2$.

- Step-size α_k controls how fast we move towards solution.
- And squared step-size α_k^2 controls how much variance moves us away.
  - This term will destroy linear convergence.

Stochastic Gradient Convergence Assumptions

- We're going to analyze stochastic gradient rate under these assumptions:
  - f is bounded below (not necessarily convex).
  - ∇f is L-Lipschitz continuous.
  - $\mathbb{E}[\|\nabla f_i(w)\|^2] \leq \sigma^2$ for some constant σ^2 and all w ("variance" is bounded).
    - This bounds the worst-case effect of the "bad term".

- Possible to relax noise bound to more-realistic $\mathbb{E}[\|\nabla f_i(w^k) - \nabla f(w^k)\|^2] \leq \sigma^2$.
  - Just get some extra terms in the result.

- Possible to show similar results for non-smooth functions.
  - Need something stronger than "bounded below" ("weakly convexity" or "tame").
  - 2018: first result that applied to ReLU neural networks.

Convergence Rate of Stochastic Gradient Method

- Let's use the "variance" bound inside previous bound,

$$\mathbb{E}[f(w^{k+1})] \leq f(w^k) - \alpha_k \|\nabla f(w^k)\|^2 + \alpha_k^2 \frac{L}{2} \mathbb{E}[\|\nabla f_{i_k}(w^k)\|^2]$$
$$\leq f(w^k) - \alpha_k \|\nabla f(w^k)\|^2 + \alpha_k^2 \frac{L\sigma^2}{2}$$

- As before, re-arrange to get the gradient norm on the left side,

$$\alpha_k \|\nabla f(w^k)\|^2 \leq f(w^k) - \mathbb{E}[f(w^{k+1})] + \alpha_k^2 \frac{L\sigma^2}{2}.$$

- Sum this up (and use iterated expectation) to get

$$\sum_{k=1}^{t} \alpha_{k-1} \mathbb{E}\|\nabla f(w^{k-1})\|^2 \leq \sum_{k=1}^{t} [\mathbb{E} f(w^{k-1}) - \mathbb{E} f(w^k)] + \sum_{k=1}^{t} \alpha_{k-1}^2 \frac{L\sigma^2}{2}.$$

Convergence Rate of Stochastic Gradient Method

- The bound from the previous slide:

$$\sum_{k=1}^{t} \alpha_{k-1} \underbrace{\mathbb{E}\|\nabla f(w^{k-1})\|^2}_{\text{bound by min}} \leq \sum_{k=1}^{t} \underbrace{[\mathbb{E}f(w^{k-1}) - \mathbb{E}f(w^k)]}_{\text{telescope}} + \sum_{k=1}^{t} \alpha_{k-1}^2 \underbrace{\frac{L\sigma^2}{2}}_{\text{no } k}.$$

- Applying the above operations gives

$$\min_{k=0,1,\ldots,t-1} \{\mathbb{E}\|\nabla f(w^k)\|^2\} \sum_{k=0}^{t-1} \alpha_k \leq f(w^0) - \mathbb{E}f(w^t) + \frac{L\sigma^2}{2} \sum_{k=0}^{t-1} \alpha_k^2.$$

- Using $\mathbb{E}f(w^k) \geq f^*$ and dividing both sides by $\sum_k \alpha_{k-1}$ gives

$$\min_{k=0,1,\ldots,t-1} \{\mathbb{E}\|\nabla f(w^k)\|^2\} \leq \frac{f(w^0) - f^*}{\sum_{k=0}^{t-1} \alpha_k} + \frac{L\sigma^2}{2} \frac{\sum_{k=0}^{t-1} \alpha_k^2}{\sum_{k=0}^{t-1} \alpha_k}.$$

Convergence Rate of Stochastic Gradient Method

- The final bound (bonus slides show how you can avoid min using random iterate):

$$\min_{k=0,1,\ldots,t-1} \{\mathbb{E}\|\nabla f(w^k)\|^2\} \leq \frac{f(w^0) - f^*}{\sum_{k=0}^{t-1} \alpha_k} + \frac{L\sigma^2}{2} \frac{\sum_{k=0}^{t-1} \alpha_k^2}{\sum_{k=0}^{t-1} \alpha_k}.$$

- If $\sigma^2 = 0$, then we could use a constant step-size and would get a $O(1/t)$ rate.
  - Same as regular gradient descent (though $\sigma^2 = 0$ doesn't really make sense).
- But due to stochasticity, convergence rate is determined by $\sum_k \alpha_k^2 / \sum_k \alpha_k$.

- Classic decreasing step-sizes: set $\alpha_k = \alpha/k$ for some α.
  - Gives $\sum_k \alpha_k = O(\log(t))$ and $\sum_k \alpha_k^2 = O(1)$, so error at t is $O(1/\log(t))$.

- Bigger decreasing step-sizes: set $\alpha_k = \alpha/\sqrt{k}$ for some α.
  - Gives $\sum_k \alpha_k = O(\sqrt{k})$ and $\sum_k \alpha_k^2 = O(\log(k))$, so error at t is $O(\log(t)/\sqrt{t})$.

- Constant step-sizes: set $\alpha_k = \alpha$ for some α.
  - Gives $\sum_k \alpha_k = k\alpha$ and $\sum_k \alpha_k^2 = k\alpha^2$, so error at t is $O(1/\alpha t) + O(\alpha)$.

Outline

Convergence of Stochastic [Sub]Gradient under Strong Convexity

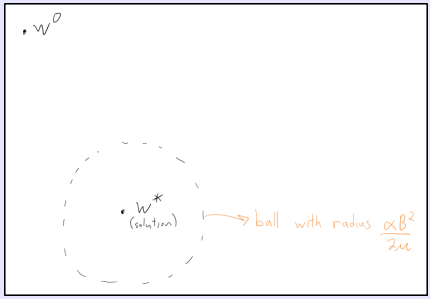
- You can get faster rates if f is strongly-convex:
  - With decreasing $\alpha_k = 1/\mu k$ you get $O(1/t)$ for t iterations (but not linear).
    - But be careful, if you over-estimate μ rate can be much worse.
    - Also, initial steps are huge (this approach only seems to work for binary SVMs).
  - With constant $\alpha_k = \alpha < 1/2\mu$ you get $O(\rho(\alpha)^k) + O(\alpha)$ for t iterations.
    - Linear convergence up to some accuracy proportional to α for sufficiently small α.
- For non-smooth strongly-convex f you get similar results:
  - Setting $\alpha_k = 1/\mu k$ gives $O(\log(t)/t)$.
    - Can improve to $O(1/t)$ by using averaging of the last $t/2$ values of w^k.
  - Setting $\alpha_k = \alpha < 1/2\mu$ still gives $\mathbb{E}[\|w^k - w^*\|^2] = O(\rho(\alpha)^k) + O(\alpha)$.
    - Looks like linear convergence if far from solution (or gradients are similar).
    - No progress if close to solution or have high variance in gradients.

Stochastic Subradient with Constant Step Size

- Expected distance with constant step-size and strong convexity (see bonus):

$$\mathbb{E}[\|w^k - w^*\|^2] \leq (1 - 2\alpha\mu)^k \|w^0 - w^*\|^2 + \frac{\alpha\sigma^2}{2\mu},$$

- First term looks like linear convergence, but second term does not go to zero.



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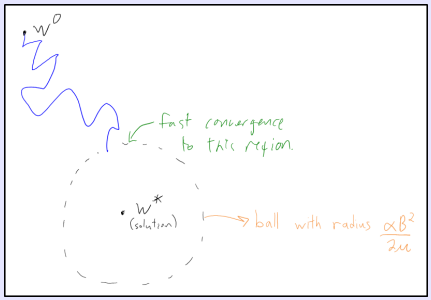

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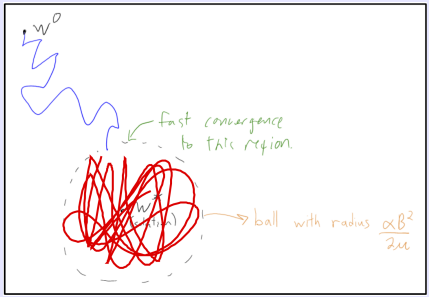
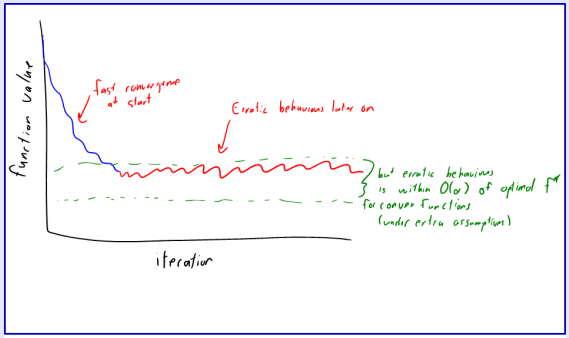


- Theory justifies "divide the step-size in half if it looks like it's stalled" heuristic.
  - Halving α divides radius of the ball around w^* in half (similar for non-convex).

Stochastic Subradient with Constant Step Size

- If ∇f is also Lipschitz we can show

$$\mathbb{E}[f(w^k) - f(w^*)] \leq (1 - 2\alpha\mu)^k (f(w^0) - f(w^*)) + \frac{L\alpha\sigma^2}{4\mu}.$$



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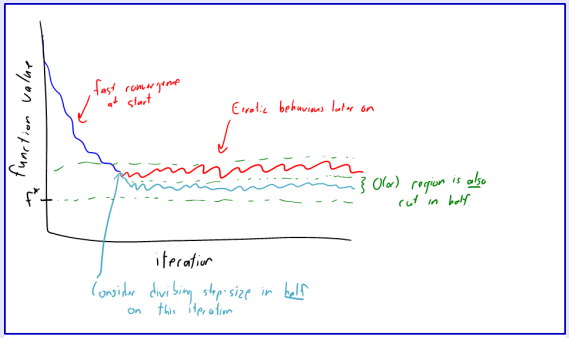


Digression: Sparse Features

- For many datasets, our feature vectors x^i are very sparse:

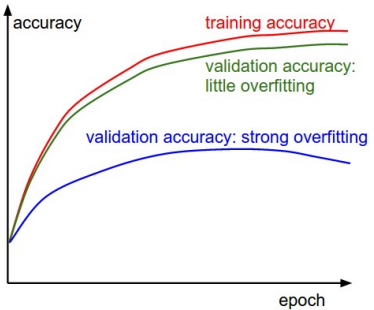
"CPSC"	"Expedia"	"vicodin"	<recipient name>	...
1	0	0	0	...
0	1	0	0	...
0	0	1	0	...
0	1	0	1	...
1	0	1	1	...

- Consider case where d is huge but each row x^i has at most z non-zeroes:
  - The $O(d)$ cost of stochastic subgradient might be too high.
  - We can often modify stochastic subgradient to have $O(z)$ cost.

- See bonus slides for details on this issue, and how to handle regularization.
  - Various forms of "lazy updates" to deal with non-sparse gradient of regularizer.

Early Stopping

- It's hard to decide when to stop stochastic gradient.
- Common heuristic is "early stopping":
  - Every m iterations, stop and compute the validation error.
  - Stop if the validation error starts increasing.



http://cs231n.github.io/neural-networks-3

- This can be viewed as a form regularization ("stop overfitting before it happens").

Stochastic Nesterov/Newton Methods?

- Should we use Nesterov/Newton-like stochastic methods?
  - These do not improve the $O(1/\epsilon)$ convergence rate.

- In fact, there is a negative result due to Polyak and Ruppert:
  - Classic result is that scaling by $\nabla^2 f(w^*)$ gives optimal asymptotic rate.
  - You can get same rate without Hessian, by just averaging the later iterations:

$$\bar{w}^t = \frac{1}{t-k}\sum_{k'=k}^{t} w^{k'},$$

- Practical averaging strategies:
  - Could weight all iterations equally.
  - Could ignore first half of the iterations then weight equally.
  - Could weight proportional to k.

Stochastic Nesterov/Newton Methods?

- Some positive results regarding stochastic Nesterov/Newton:
  - Nesterov/Newton can improve dependence on L and μ.
    - May be faster ifcondition number L/μ is large and noise σ^2 is small.
  - Two-phase Newton-like method achieves $O(1/\epsilon)$ without strong-convexity.
  - AdaGrad method,

$$w^{k+1} = w^k + \alpha D^{-1} g_{i_k}, \quad \text{with diagonal } D_{jj} = \sqrt{\delta + \sum_{k'=0}^{k} (\nabla_j f_{i_{k'}}(w^{k'}))^2},$$

    improves "regret" but not optimization error (we'll cover regret later).

  - Some heuristic extensions of AdaGrad:
    - RMSprop: variant of AdaGrad where step-size does not go to zero.
    - Adam: variant where momentum is added.
    - These methods act more like a constant step-size, and do not converge in general.

Active-Set Identification and Regularized Dual Averaging

- You can perform a proximal stochastic sub-gradient iteration,

$$w^{k+\frac{1}{2}} = w^k - \alpha_k g_{i_k}$$

$$w^{k+1} = \underset{v \in \mathbb{R}^d}{\text{argmin}} \left\{ \frac{1}{2}\|v - w^{k+\frac{1}{2}}\|^2 + \alpha_k r(v) \right\}.$$

- Does not converge faster than SGD and does not identify active set.
  - Smoothness does not help in the general stochastic setting.
  - With L1-regularization, all w_j^k become non-zero infinitely-often.
- Variant with the active set property (but same rate) is regularized dual averaging,

$$w^{k+\frac{1}{2}} = w^0 - \frac{\alpha_k}{k} \sum_{t=1}^{k} g_{i_t}$$

$$w^{k+1} = \underset{v \in \mathbb{R}^d}{\text{argmin}} \left\{ \frac{1}{2}\|v - w^{k+\frac{1}{2}}\|^2 + \alpha_k r(v) \right\}.$$

Summary

- Stochastic gradient convergence rate:
  - Decreasing step-size: subgradient slowly converges to exact solution.
    - Same rate as deterministic subgradient but n-times cheaper iterations.
- Practical aspects of stochastic gradient methods:
  - Constant step-size: subgradient quickly converges to approximate solution.
  - Sparse datasets, early stopping, iterate averaging.
  - Negative and positive results regarding second-order methods.
  - Does not identify active set, but gradient averaging can fix this.
- Next time: new stochastic methods with linear convergence rates..

Random Iterate for Non-Convex Rate not depending on Min

- The bound we had earlier, but dividing both sides by $\sum_{k=0}^{t} \alpha_k$,

$$\frac{\sum_{k=1}^{t} \alpha_{k-1} \mathbb{E}\|\nabla f(w^{k-1})\|^2}{\sum_{k=0}^{t-1} \alpha_k} \leq \frac{\sum_{k=1}^{t} [\mathbb{E} f(w^{k-1}) - \mathbb{E} f(w^k)] + \sum_{k=1}^{t} \alpha_{k-1}^2 \frac{L\sigma^2}{2}}{\sum_{k=0}^{t-1} \alpha_k}$$

- Now choose $\hat{k} \in \{0, 1, \ldots, t-1\}$ according to $p(\hat{k}) = \alpha_k / \sum_{i=0}^{t-1} \alpha_i$.
- Notice that LHS above is expectation with respect to $\hat{k}$ of $\mathbb{E}\|\nabla f(w^{\hat{k}})\|^2$,

$$\mathbb{E}\|\nabla f(w^{\hat{k}})\|^2 \leq \frac{f(w^0) - f^*}{\sum_{k=0}^{t-1} \alpha_k} + \frac{L\sigma^2}{2} \frac{\sum_{k=0}^{t-1} \alpha_k^2}{\sum_{k=0}^{t-1} \alpha_k}.$$

- So choosing an iterate in this way avoids needing to know the min.
  - Notice that RHS is the same.

Convergence Rate of Stochastic Subgradient Method

- The basic stochastic subgradient method (for random i_t and $g_{i_t} \in \partial f_{i_t}(x^t)$):

$$x^{t+1} = x^t - \alpha g_{i_t},$$

- We can't use descent lemma because f is non-differentiable.
- Since function value may not decrease, we analyze distance to x^*:

$$\begin{aligned}\|x^t - x^*\|^2 &= \|(x^{t-1} - \alpha_t g_{i_t}) - x^*\|^2 \\ &= \|(x^{t-1} - x^*) - \alpha_t g_{i_t}\|^2 \\ &= \|x^{t-1} - x^*\|^2 - 2\alpha_t g_{i_t}^\top (x^{t-1} - x^*) + \alpha_t^2 \|g_{i_t}\|^2.\end{aligned}$$

- Take expectation with respect to i_t:

$$\begin{aligned}\mathbb{E}[\|x^t - x^*\|^2] &= \mathbb{E}[\|x^{t-1} - x^*\|] - 2\alpha_t \mathbb{E}[g_{i_t}^\top (x^{t-1} - x^*)] + \alpha_t^2 \mathbb{E}[\|g_{i_t}\|^2] \\ &= \underbrace{\|x^{t-1} - x^*\|^2}_{\text{old distance}} - 2\alpha_t \underbrace{g_t^\top (x^{t-1} - x^*)}_{\text{expected progress}} + \alpha_t^2 \underbrace{\mathbb{E}[\|g_{i_t}\|^2]}_{\text{"variance"}}.\end{aligned}$$

where g_t is a subgradient of f at w^k (expected progress is positive by convexity).

Convergence Rate of Stochastic Subgradient

- Our expected distance given x^{t-1} is

$$\mathbb{E}[\|x^t - x^*\|^2] = \underbrace{\|x^{t-1} - x^*\|^2}_{\text{old distance}} - 2\alpha_t \underbrace{g_t^\top (x^{t-1} - x^*)}_{\text{expected progress}} + \alpha_t^2 \underbrace{\mathbb{E}[\|g_{i_t}\|^2]}_{\text{"variance"}}.$$

- It follows from strong-convexity that (next slide),

$$g_t^\top (x^{t-1} - x^*) \geq \mu \|x^{t-1} - x^*\|^2,$$

which gives (assuming variance is bounded by constant σ^2):

$$\begin{aligned}\mathbb{E}[\|x^t - x^*\|^2] &\leq \|x^{t-1} - x^*\|^2 - 2\alpha_t \mu \|x^{t-1} - x^*\|^2 + \alpha_t^2 \sigma^2 \\ &= (1 - 2\alpha_t \mu)\|x^{t-1} - x^*\|^2 + \alpha_t^2 \sigma^2.\end{aligned}$$

- With constant $\alpha_k = \alpha$ (with $\alpha < 2/\mu$) and applying recursively we get (with work)

$$\mathbb{E}[\|w^k - w^*\|^2] \leq (1 - 2\alpha\mu)^k \|w^0 - w^*\|^2 + \frac{\alpha\sigma^2}{2\mu},$$

where second term bounds a geometric series.

Strong-Convexity Inequalities for Non-Differentiable f

- A "first-order" relationship between subgradient and strong-convexity:
  - If f is μ-strongly convex then for all x and y we have

$$f(y) \geq f(x) + f'(y)^\top (y - x) + \frac{\mu}{2}\|y - x\|^2,$$

    for $f'(y) \in \partial f(x)$.
  - The first-order definition of strong-convexity, but with subgradient replacing gradient.
  - Reversing y and x we can write

$$f(x) \geq f(y) + f'(x)^\top (x - y) + \frac{\mu}{2}\|x - y\|^2,$$

    for $f'(x) \in \partial f(x)$.
  - Adding the above together gives

$$(f'(y) - f'(x))^\top (y - x)) \geq \mu\|y - x\|^2.$$

  - Applying this with $y = x^{t-1}$ and subgradient g_t and $x = x^*$ (which has $f'(x^*) = 0$ for some subgradient) gives

$$(g_t - 0)^\top (x^{t-1} - x^*) \geq \mu\|x^{t-1} - x^*\|^2.$$

Convergence Rate of Stochastic Subgradient

- For full details of analyzing stochastic gradient under strong convexity, see:
  - Constant α_k: `http://circle.ubc.ca/bitstream/handle/2429/50358/stochasticGradientConstant.pdf`.
  - Decreasing α_k: `http://arxiv.org/pdf/1212.2002v2.pdf`.

- For both cases under PL, see Theorem 4 here:
  - https://arxiv.org/pdf/1608.04636v2.pdf

Operations on Sparse Vectors

- Consider a vector $g \in \mathbb{R}^d$ with at most z non-zeroes:

$$g^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 0 & -0.5 & 0 & 0 & 0 \end{bmatrix}.$$

- If $z << d$, we can store the vector using $O(z)$ storage instead of $O(d)$:
  - Just store the non-zero values:

$$g_{\text{value}}^T = \begin{bmatrix} 1 & 2 & -0.5 \end{bmatrix}.$$

  - Store index of each non-zero ("pointer"):

$$g_{\text{point}}^T = \begin{bmatrix} 4 & 5 & 7 \end{bmatrix}.$$

- With this representation, we can do standard vector operations in $O(z)$:
  - Compute αg in $O(z)$ by setting $g_{\text{value}} = \alpha g_{\text{value}}$.
  - Compute $w^T g$ in $O(z)$ by multiplying g_{value} by w at positions g_{point}.

Stochastic Subgradient with Sparse Features

- Consider optimizing the hinge-loss,

$$\underset{w \in \mathbb{R}^d}{\text{argmin}} \frac{1}{n} \sum_{i=1}^{n} \max\{0, 1 - y^i(w^T x^i)\},$$

  when d is huge but each x^i has at most z non-zeroes.

- A stochastic subgradient method could use

$$w^{k+1} = w^k - \alpha_k g_{i_k}, \text{ where } g_i = \begin{cases} -y^i x^i & \text{if } 1 - y^i(w^T x^i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Calculating w^{k+1} is $O(z)$ since these are sparse vector operations.

- So stochastic subgradient is fast if z is small even if d is large.
  - This is how you "train on all e-mails": each e-mail has a limited number of words.

Stochastic Subgradient with Sparse Features

- But consider the L2-regularized hinge-loss in the same setting,

$$\underset{w\in\mathbb{R}^d}{\text{argmin}} \frac{1}{n}\sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x^i)\} + \frac{\lambda}{2}\|w\|^2,$$

  using a stochastic subgradient method,

$$w^{k+1} = w^k - \alpha_k g_{i_k} - \alpha_k \lambda w^k, \text{ where } g_{i_k} \text{ is same as before.}$$

- Problems is that w^k could have d non-zeroes:
  - So adding L2-regularization increases cost from $O(z)$ to $O(d)$?

- There are two standard ways to keep the cost at $O(z)$:
  - L2-regularization: use a $w^k = \beta^k v^k$ (scalar times vector) representation.
  - "Lazy" updates (which work for many regularizers).

Stochastic Subgradient with Sparse Features

- But consider the L2-regularized hinge-loss in the same setting,

$$\operatorname*{argmin}_{w \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x^i)\} + \frac{\lambda}{2}\|w\|^2,$$

  using a stochastic subgradient method,

$$w^{k+1} = w^k - \alpha_k g_{i_k} - \alpha_k \lambda w^k, \text{ where } g_{i_k} \text{ is same as before}$$

- Problems is that w^t could have d non-zeroes:
  - So adding L2-regularization increases cost from $O(z)$ to $O(d)$?
- To use L2-regularization and keep $O(z)$ cost, re-write iteration as

$$\begin{aligned} w^{t+1} &= w^t - \alpha_t g_{i_t} - \alpha_t \lambda w^t \\ &= \underbrace{(1 - \alpha_t \lambda) w^t}_{\text{changes scale of } w^t} - \underbrace{\alpha_t g_{i_t}}_{\text{sparse update}}. \end{aligned}$$

Stochastic Subgradient with Sparse Features

- Let's write the update as two steps

$$w^{t+\frac{1}{2}} = (1 - \alpha_t\lambda)w^t, \quad w^{t+1} = w^{t+\frac{1}{2}} - \alpha_t g_{i_t}.$$

- We can implement both steps in $O(z)$ if we re-parameterize as

$$w^t = \beta^t v^t,$$

for some scalar β^t and vector v^t.

- For the first step we can use

$$\beta^{t+\frac{1}{2}} = (1 - \alpha_t\lambda)\beta^t, \quad v^{t+\frac{1}{2}} = v^t.$$

which costs $O(1)$.

- For the second step we can use

$$\beta^{t+1} = \beta^{t+\frac{1}{2}}, \quad v^{t+1} = v^{t+\frac{1}{2}} - \frac{\alpha_t}{\beta^{t+\frac{1}{2}}} g_{i_t},$$

which costs $O(z)$.

Lazy Updates for Sparse Features with Dense Regularizers

- Consider a feature j that has been zero in the loss for 10 iterations (constant α):

$$\begin{aligned} w_j^k &= w_j^{k-1} - 0 - \alpha\lambda w_j^{k-1} \\ &= (1 - \alpha\lambda) w_j^{k-1} \\ &= (1 - \alpha\lambda)^2 w_j^{k-2} \\ &\vdots \\ &= (1 - \alpha\lambda)^{10} w_j^{k-10}. \end{aligned}$$

- So we can apply 10 regularizer gradient steps in $O(1)$.

- Lazy updates:
  - If j is zero in g_{i_k}, do nothing.
  - If j is non-zero, apply all the old regularizer updates then do the gradient step.
    - Requires keeping a "checkpoint" of the last time each variable was updated.

Lazy Updates for Sparse Features with Dense Regularizers

- Lazy updates that track cumulative effects of simple updates.

- Considern stochastic proximal-gradient for L1-regularization:
  - Soft-threshold operator with constant step-size α applies to each element,

$$w_j^{k+1} = \text{sign}(w_j^k)\max\{0, |w_j^k| - \alpha\lambda\}.$$

  - If all that happens to w_j for 10 iterations is the proximal operator, we can use

$$w_j^{k+10} = \text{sign}(w_j^k)\max\{0, |w_k^k| - 10\alpha\lambda\}.$$

- Digression: stochastic proximal-gradient methods:
  - Same convergence rates as basic stochastic gradient method (doesn't help).
  - Unlike deterministic proximal-gradient method, does not find final non-zero pattern in finite time.
    - Regularized dual averaging is a variant that has this property.