CPSC 540: Machine Learning

Hidden Markov Models

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Last Time: Viterbi Decoding and Message Passing

- Decoding in density models: finding x with highest joint probability:

$$\operatorname*{argmax}_{x_1,x_2,\dots,x_d} p(x_1,x_2,\dots,x_d).$$

- For Markov chains, we find decoding by writing maximization as

$$\max_{x_1,x_2,x_3,x_4} p(x_1,x_2,x_3,x_4) = \max_{x_4} \underbrace{\max_{x_3} p(x_4 \mid x_3) \underbrace{\max_{x_2} p(x_3 \mid x_2) \underbrace{\max_{x_1} p(x_2 \mid x_1) \underbrace{p(x_1}_{M_1(x_1)})}_{M_2(x_2)}}_{M_3(x_3)}}_{M_4(x_4)},$$

- Viterbi decoding computes $M_1(x_1)$ for all x_1, $M_2(x_2)$ for all x_2, and so on.
  The $M_j(x_j)$ functions are called messages (summarize everything about past).

Chapman-Kolmogorov Equations as Message Passing

- We can also view Chapman Kolmogorov equations as message passing:

$$
\begin{aligned}
p(x_4) = \sum_{x_3}\sum_{x_2}\sum_{x_1} p(x_1, x_2, x_3, x_4) &= \sum_{x_3}\sum_{x_2}\sum_{x_1} p(x_4 \mid x_3)p(x_3 \mid x_2)p(x_2 \mid x_1)p(x_1) \\
&= \sum_{x3} p(x_4 \mid x_3) \sum_{x_2} p(x_3 \mid x_2) \sum_{x_1} p(x_2 \mid x_1) M_1(x_1) \\
&= \sum_{x3} p(x_4 \mid x_3) \sum_{x_2} p(x_3 \mid x_2) M_2(x_2) \\
&= \sum_{x3} p(x_4 \mid x_3) M_3(x_3) \\
&= M_4(x_4),
\end{aligned}
$$

- Messages $M_j(x_j)$ are the marginals of the Markov chain.
  - So we can view CK equations as Viterbi decoding with "max" replace by "sum".
  - These two methods are also known as "max-product" and "sum-product" algorithms.

Message-Passing Algorithms

- We've discussed several algorithms with similar structure:
  - Viterbi decoding algorithm for decoding in discrete Markov chains.
  - CK equations for marginals in discrete Markov chains.
  - Gaussian updates for marginals in Gaussian Markov chains.

- These are all special cases of message-passing algorithms:
  1. Define M_j summarizing all relevant information about the past at time j.
  2. Use Markov property to write M_j recursively in terms of M_{j-1}.
  3. Solve task by computing M_1, M_2, . . . , M_d.

- "Generalized distributive law" is a framework for describing when/why this works:
  - `https://authors.library.caltech.edu/1541/1/AJIieeetit00.pdf`

- In some cases we'll also need backwards message V_j ("cost to go"):
  - V_j summarizes all relevant information about the future at time j.

Backwards "Cost to Go" Messages

- Using backwards messages $V_j(x_j)$ to (innefficiently) compute $p(x_1)$:

$$
\begin{aligned}
p(x_1) = \sum_{x_2}\sum_{x_3}\sum_{x_4} p(x_1, x_2, x_3, x_4) &= \sum_{x_2}\sum_{x3}\sum_{x4} p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2)p(x_4 \mid x_3) \\
&= p(x_1)\sum_{x_2} p(x_2 \mid x_1)\sum_{x3} p(x_3 \mid x_2)\sum_{x4} p(x_4 \mid x_3) \\
&= p(x_1)\sum_{x_2} p(x_2 \mid x_1)\sum_{x3} p(x_3 \mid x_2)\sum_{x4} p(x_4 \mid x_3)\underbrace{V_4(x_4)}_{=1} \\
&= p(x_1)\sum_{x_2} p(x_2 \mid x_1)\sum_{x3} p(x_3 \mid x_2)\underbrace{V_3(x_3)}_{1} \\
&= p(x_1)\sum_{x_2} p(x_2 \mid x_1)\underbrace{V_2(x_2)}_{1} \\
&= p(x_1)\underbrace{V_1(x_1)}_{1}.
\end{aligned}
$$

- Observe that backwards messages $V_j(x_j)$ are not probabilities as in CK equations.
  - But they summarize everything you need to know about the future.
  - Can use this structure to condition on the future, and compute things like $p(x_1 \mid x_4)$.

Forward-Backward Algorithm

- Computing all $M_j(x_j)$ and $V_j(x_j)$ is called the forward-backward algorithm.
  - Not interesting for Markov chains since $V_j(x_j) = 1$ for all j and x_j.

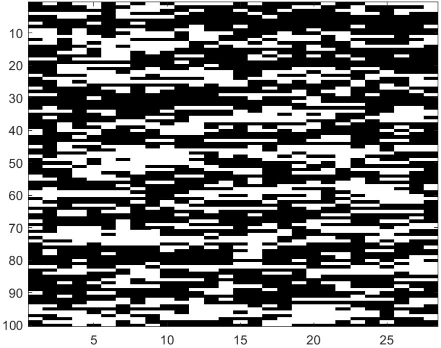
- Why do we care about backwards messages?
  - Can efficiently compute all conditionals $p(x_j = s \mid x_{10} = 3)$ (for all j and s).
    - Messages are modified when you condition (see bonus slides).
    - The modified $V_j(x_j)$ will reflect "what you need to know about the future events".

  - Can be used to compute probabilities in generalizations of Markov chains (next).
    - In this setting the forward messages may not be probabilities either.

  - In reinforcement learning, estimating the "cost to go" ("value") function is the goal.
    - We aren't covering RL, but understanding Markov chains will help you understand RL.

Back to the Rain Data

- We previously considered the “Vancouver Rain” data:



- We used homogeneous Markov chains to model between-day dependence.

Back to the Rain Data

- But doesn't it rain less in the summer?
- There are hidden clusters in the data not captured by the Markov chain.
  - But mixture of independent models are inefficient at representing direct dependency.
- Mixture of Markov chains could capture direct dependence *and* clusters,

$$p(x_1, x_2, \dots, x_d) = \sum_{c=1}^{k} p(z=c) \underbrace{p(x_1 \mid z=c)p(x_2 \mid x_1, z=c) \cdots p(x_d \mid x_{d-1}, z=c)}_{\text{Markov chain } c}.$$

- Cluster z chooses which homogeneous Markov chain parameters to use.
  - We could learn that we're more likely to have rain in winter.
  - Can modify CK equations to take into account z, and then apply EM.

Comparison of Models on Rain Data

- Independent (homogeneous) Bernoulli:
  - Average NLL: 18.9673 (1 parameter).
- Independent Bernoullis:
  - Average NLL: 18.9518, (28 parmaeters).
- Mixture of Bernoullis ($k = 10$, five random restarts of EM):
  - Average NLL: 17.0602 ($10 + 10 \times 28 = 290$ parameters)
- Homogeneous Markov chain:
  - Average NLL: 16.8065 (3 parameters)
- Mixture of Markov chains ($k = 10$, five random restarts of EM):
  - Average NLL: 16.5332 ($10 + 10 \times 3 = 40$ parameters).
  - Included what I call a “summer” cluster:

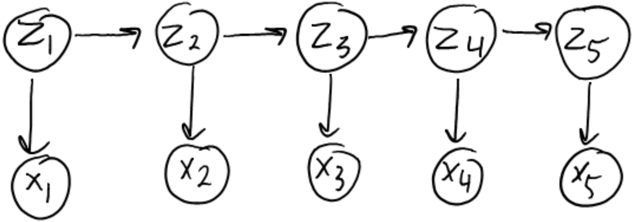
$$\begin{aligned}
&p(z = 5) = 0.14 \\
&p(x_1 = \text{“rain”} \mid z = 5) = 0.22 && \text{(instead of usual 37\%)} \\
&p(x_j = \text{“rain”} \mid x_{j-1} = \text{“rain”}, z = 5) = 0.49) && \text{(instead of usual 65\%)} \\
&p(x_j = \text{“rain”} \mid x_{j-1} = \text{“not rain”}, z = 5) = 0.11 && \text{(instead of usual 35\%)}
\end{aligned}$$

Back to the Rain Data

- The rain data is artificially divideded into months.
- We previously discussed viewing rain data as one very long sequence ($n = 1$).
- We could apply homogeneous Markov chains due to parameter tieing.
- But a mixture doesn't make sense when $n = 1$.
- What we want: different "parts" of the sequence come from different clusters.
  - We transition from "summer" cluster to "fall" cluster at some time j.
- One way to address this is with a"hidden" Markov model (HMM):
  - Instead of examples being assigned to clusters, days are assigned to clusters.
  - Have a Markov dependency between cluster values of adjacent days.

Hidden Markov Models

- Hidden Markov models have each x_j depend on hidden Markov chain.

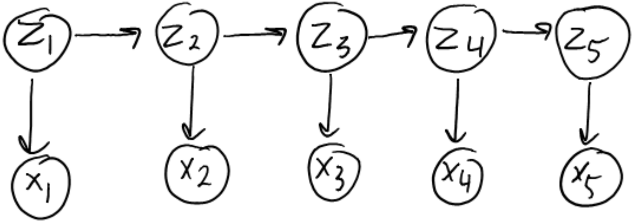


$$p(x_1, x_2, \dots, x_d, z_1, z_2, \dots z_d) = p(z_1) \prod_{j=2}^{d} p(z_j \mid z_{j-1}) \prod_{j=1}^{d} p(x_j \mid z_j).$$

- We're going to learn clusters z_j and the hidden dynamics.
  - Hidden cluster z_j could be "summer" or "winter" (we're learning the clusters).
  - Transition probability $p(z_j \mid z_{j-1})$ is probability of staying in "summer".
  - Emission probability $p(x_j \mid z_j)$ is probability of "rain" during "summer".

Hidden Markov Models

- Hidden Markov models have each x_j depend on hidden Markov chain.



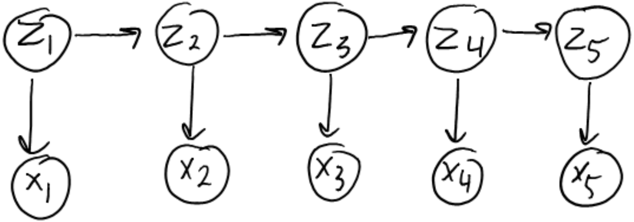
$$p(x_1, x_2, \dots, z_1, z_2, \dots z_d) = p(z_1) \prod_{j=2}^{d} p(z_j \mid z_{j-1}) \prod_{j=1}^{d} p(x_j \mid z_j).$$

- You observe the x_j values but do not see the z_j values.
    - CK equations won't work since $p(z_1 = s)$ depends on future x_j values.

- But forward-backward algorithm can be used to compute probailities.
    - And subsequently learn with EM (treating z_j values as nuissance variables).

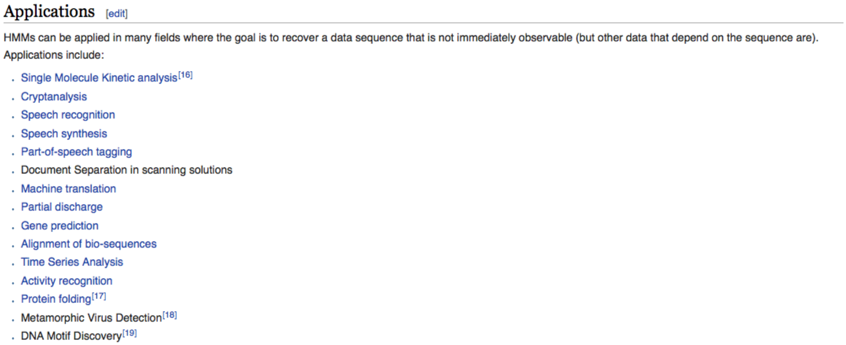
Hidden Markov Models

- Hidden Markov models have each x_j depend on hidden Markov chain.



- Note that the x_j can be continuous even with discrete clusters z_j.
  - You could have a “mixture of Gaussians” with cluster changing in time.
- If the z_j are continuous it’s often called a state-space model.
  - If everything is Gaussian, it leads to Kalman filtering.
  - Keywords for non-Gaussian: unscented Kalman filter and particle filter.
- Variants of HMMs are probably the most-used time-series model...

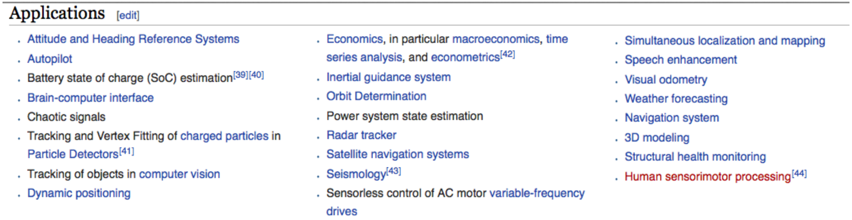
Applications of HMMs and Kalman Filters



Applications [edit]

HMMs can be applied in many fields where the goal is to recover a data sequence that is not immediately observable (but other data that depend on the sequence are). Applications include:

- Single Molecule Kinetic analysis[16]
- Cryptanalysis
- Speech recognition
- Speech synthesis
- Part-of-speech tagging
- Document Separation in scanning solutions
- Machine translation
- Partial discharge
- Gene prediction
- Alignment of bio-sequences
- Time Series Analysis
- Activity recognition
- Protein folding[17]
- Metamorphic Virus Detection[18]
- DNA Motif Discovery[19]

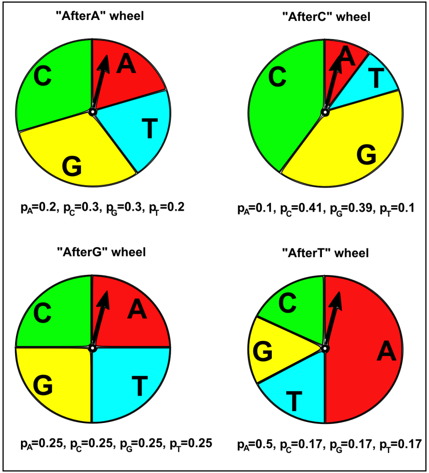


Applications [edit]

- Attitude and Heading Reference Systems
- Autopilot
- Battery state of charge (SoC) estimation[39][40]
- Brain-computer interface
- Chaotic signals
- Tracking and Vertex Fitting of charged particles in Particle Detectors[41]
- Tracking of objects in computer vision
- Dynamic positioning
- Economics, in particular macroeconomics, time series analysis, and econometrics[42]
- Inertial guidance system
- Orbit Determination
- Power system state estimation
- Radar tracker
- Satellite navigation systems
- Seismology[43]
- Sensorless control of AC motor variable-frequency drives
- Simultaneous localization and mapping
- Speech enhancement
- Visual odometry
- Weather forecasting
- Navigation system
- 3D modeling
- Structural health monitoring
- Human sensorimotor processing[44]

Example: Modeling DNA Sequences

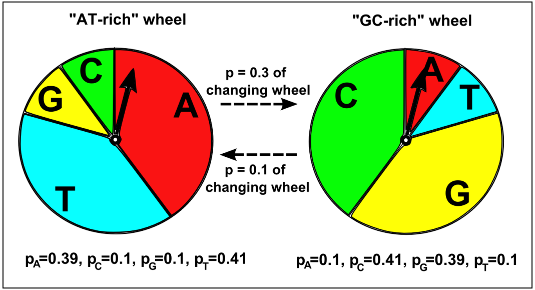
- Markov model for elements of sequence (dependence on previous symbol):


"AfterA" wheel
A
C
T
G
p_A=0.2, p_C=0.3, p_G=0.3, p_T=0.2
"AfterC" wheel
A
C
T
G
p_A=0.1, p_C=0.41, p_G=0.39, p_T=0.1
"AfterG" wheel
C
A
G
T
p_A=0.25, p_C=0.25, p_G=0.25, p_T=0.25
"AfterT" wheel
C
G
A
T
p_A=0.5, p_C=0.17, p_G=0.17, p_T=0.17

https://www.tes.com/lessons/WE5E9RncBhieAQ/dna

Example: Modeling DNA Sequences

- Hidden Markov model (HMM) for elements of sequence (two hidden clusters):


"AT-rich" wheel
"GC-rich" wheel
C
G
A
T
p = 0.3 of changing wheel
p = 0.1 of changing wheel
A
C
T
G
p_A=0.39, p_C=0.1, p_G=0.1, p_T=0.41
p_A=0.1, p_C=0.41, p_G=0.39, p_T=0.1

- This is a (hidden) state transition diagram.
  - Can reflect that probabilities are different in different regions.
  - The actual regions are not given, but instead are nuissance variables handled by EM.
- A better model might be usin a hidden and visible Markov chain.
  - With 2 hidden clusters, you would have 8 “probability wheels” (4 per cluster).
  - Would have “treewidth 2”, which we’ll show later means it’s tractable to use.

Who is Guarding Who?

- There is a lot of data on offense of NBA basketball players.
  - Every point and assist is recorded, more scoring gives more wins and $$$.
- But how do we measure defense?
  - We need to know who each player is guarding.

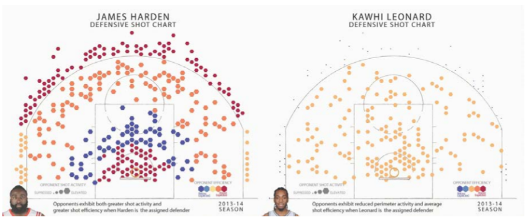


Figure 2a. Graphical depiction of a defender's volume (size) and disruption scores (color). Kawhi Leonard tends to suppress shots on the perimeter. More comparisons are provided in the Appendix.

`http://www.lukebornn.com/papers/franks_ssac_2015.pdf`

- HMMs can be used to model who is guarding who over time.
  - `https://www.youtube.com/watch?v=JvNkZdZJBt4`

Outline

1. Hidden Markov Models

2. Directed Acyclic Graphical Models

Higher-Order Markov Models

- Markov models use a density of the form

$$p(x) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2)p(x_4 \mid x_3) \cdots p(x_d \mid x_{d-1}).$$

- They support efficient computation but Markov assumption is strong.

- A more flexible model would be a second-order Markov model,

$$p(x) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2, x_1)p(x_4 \mid x_3, x_2) \cdots p(x_d \mid x_{d-1}, x_{d-2}),$$

  or even a higher-order models.

- General case is called directed acyclic graphical (DAG) models:
  - They allow dependence on any subset of previous features.

DAG Models

- As in Markov chains, DAG models use the chain rule to write

$$p(x_1, x_2, \dots, x_d) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1, x_2) \cdots p(x_d \mid x_1, x_2, \dots, x_{d-1}).$$

- We can alternately write this as:

$$p(x_1, x_2, \dots, x_d) = \prod_{j=1}^{d} p(x_j \mid x_{1:j-1}).$$

- In Markov chains, we assumed x_j only depends on previous x_{j-1} given past.

- In DAGs, x_j can depend on any subset of the past $x_1, x_2, \dots, x_{j-1}$.

DAG Models

- We often write joint probability in DAG models as

$$p(x_1, x_2, \dots, x_d) = \prod_{j=1}^{d} p(x_j \mid x_{\text{pa}(j)}),$$

where pa(j) are the "parents" of node j.
  - For Markov chains the only "parent" of j is $(j-1)$.
  - If we have k parents we only need 2^{k+1} parameters.

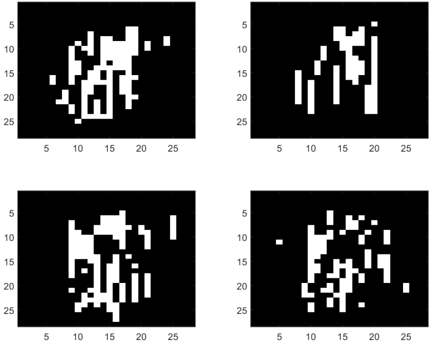
- This corresponds to a set of conditional independence assumptions,

$$p(x_j \mid x_{1:j-1}) = p(x_j \mid x_{\text{pa}(j)}),$$

that we're independent of previous non-parents given the parents.

MNIST DIgits with Markov Chains

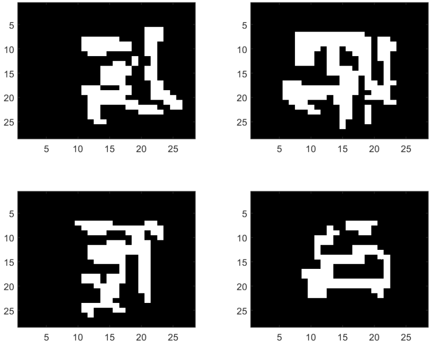
- Recall trying to model digits using an inhomogeneous Markov chain:



Only models dependence on pixel above, not on 2 pixels above nor across columns.

MNIST Digits with DAG Model (Sparse Parents)

- Samples from a DAG model with 8 parents per feature:

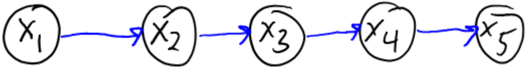


Parents of (i, j) are 8 other pixels in the neighbourhood ("up by 2, left by 2"):

$$\{(i{-}2, j{-}2), (i{-}1, j{-}2), (i, j{-}2), (i{-}2, j{-}1), (i{-}1, j{-}1), (i, j{-}1), (i{-}2, j), (i{-}1, j)\}.$$

From Probability Factorizations to Graphs

- DAG models are also known as "Bayesian networks" and "belief networks".
- "Graphical" name comes from visualizing parents/features as a graph:
  - We have a node for each feature j.
  - We place an edge into j from each of its parents.
- The DAG representation for a Markov chains is:



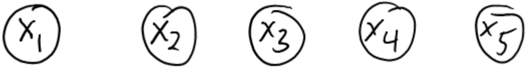
  - Different than "state transition diagrams": edges are between variables (not states).
- This graph is not just a visualization tool:
  - Can be used to test arbitrary conditional independences ("d-separation").
  - Graph structure tells us whether message passing is efficient ("treewidth").

Graph Structure Examples

With product of independent we have

$$p(x) = \prod_{j=1}^{d} p(x_j),$$

so pa$(j) = \varnothing$ and the graph is:

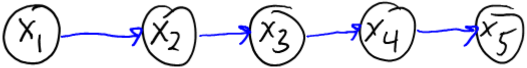


Graph Structure Examples

With Markov chain we have

$$p(x) = p(x_1)\prod_{j=2}^{d} p(x_j \mid x_{j-1}),$$

so pa$(j) = \{j-1\}$ and the graph is:

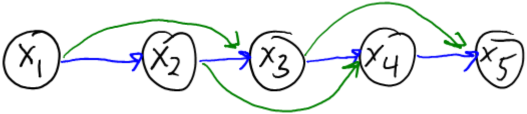


Graph Structure Examples

With second-order Markov chain we have

$$p(x) = p(x_1)p(x_2 \mid x_1)\prod_{j=3}^{d} p(x_j \mid x_{j-1}, x_{j-2}),$$

so pa$(j) = \{j-2, j-1\}$ and the graph is:

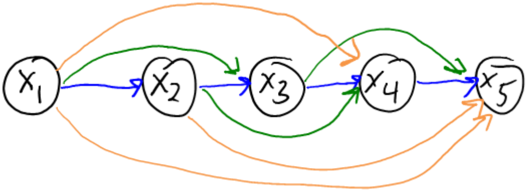


Graph Structure Examples

With general distribution we have

$$p(x) = \prod_{j=1}^{d} p(x_j \mid x_{1:j-1}).$$

so $\text{pa}(j) = \{1, 2, \dots, j-1\}$ and the graph is:

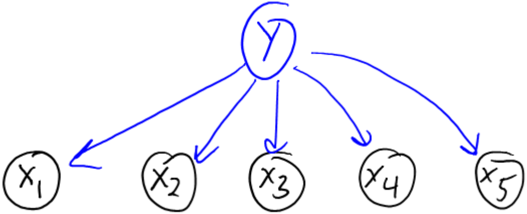


Graph Structure Examples

In naive Bayes (or GDA with diagonal Σ) we add an extra variable y and use

$$p(y,x) = p(y)\prod_{j=1}^{d} p(x_j \mid y),$$

which has $\text{pa}(y) = \emptyset$ and $\text{pa}(x_j) = y$ giving

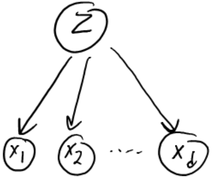


Graph Structure Examples

With mixture of independent models we have

$$p(z, x) = p(z) \prod_{j=1}^{d} p(x_j \mid z).$$

which has $\text{pa}(z) = \emptyset$ and $\text{pa}(x_j) = z$ giving same structure as naive Bayes:



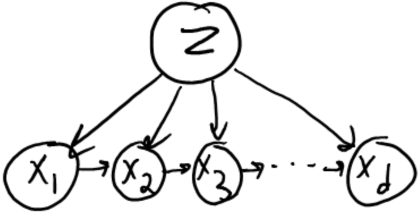
Since structure is the same, many computations will be similar.

Graph Structure Examples

With mixture of Markov chains models we have

$$p(x_1, x_2, \dots, x_d, z) = p(z)p(x_1 \mid z) \prod_{j=2}^{d} p(x_j \mid x_{j-1}, z).$$

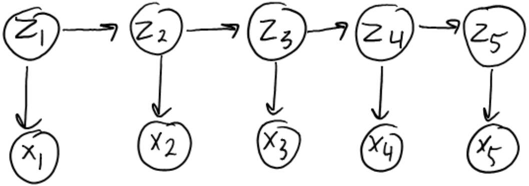
which has $\text{pa}(z) = \emptyset$ and $\text{pa}(x_j) = \{x_{j-1}, z\}$:



Graph Structure Examples

Sometimes it's easier to present a model using the graph.

In hidden Markov models we have this structure:



The graph and variable names already give you an idea of what this model does:

- We have hidden variables z_j that follow a Markov chain.
- Each feature x_j depends on corresponding hidden variable z_j.

Summary

- Message-passing allow efficient calculations with Markov chains.
- Hidden Markov models model time-series with hidden per-time cluster.
  - Tons of applications, typically more realistic than Markov models.
- DAG models factorize joint distribution into product of conditionals.
  - Assume conditionals depend on small number of "parents".
- Next time: the IID assumption as a graphical model?

Computing Conditional Probabilities

- Previously: Monte Carlo for approximating conditional probabilities
- For Gaussian/discrete Markov chains, we can do better than rejection sampling.
  1. We can generate exact samples from conditional distribution (bonus slide).
     - Rejection sampling is not needed, relies on "backwards sampling" in time.
  2. We can find conditional decoding $\max_{x \mid x_{j'}=c} p(x)$:
     - Run Viterbi decoding with $M_{j'}(c) = 1$ and $M_{j'}(c') = 0$ for $c \neq c'$.
  3. We can find univariate conditionals, $p(x_j \mid x_{j'})$.

- Example of computing $p(x_1 = c \mid x_3 = 1)$ in a length-4 discrete Markov chain:

$$\begin{aligned} p(x_1 = c \mid x_3 = 1) &\propto p(x_1 = c, x_3 = 1) \\ &= \sum_{x_4} \sum_{x_2} p(x_1 = c, x_2, x_3 = 1, x_4), \end{aligned}$$

  where the normalizing constant is the marginal $p(x_3 = 1)$.
- This is a sum over k^{d-2} possible assignments to other variables.

Distributing Sum across Product

- Fortunately, the Markov property makes the sums simplify as before:

$$
\begin{aligned}
\sum_{x_4}\sum_{x_2} p(x_1 = c, x_2, x_3 = 1, x_4) &= \sum_{x4}\sum_{x3=1}\sum_{x_2}\sum_{x_1=c} p(x_4 \mid x_3)p(x_3 \mid x_2)p(x_2 \mid x_1)p(x_1) \\
&= \sum_{x4}\sum_{x3=1}\sum_{x_2} p(x_4 \mid x_3)p(x_3 \mid x_2)\sum_{x_1=c} p(x_2 \mid x_1)p(x_1) \\
&= \sum_{x4}\sum_{x3=1} p(x_4 \mid x_3)\sum_{x_2} p(x_3 \mid x_2)\sum_{x_1=c} p(x_2 \mid x_1)M_1(x_1) \\
&= \sum_{x4}\sum_{x3=1} p(x_4 \mid x_3)\sum_{x_2} p(x_3 \mid x_2)M_2(x_2) \\
&= \sum_{x4}\sum_{x3=1} p(x_4 \mid x_3)M_3(x_3) \\
&= \sum_{x4} M_4(x_4),
\end{aligned}
$$

where $M_j(x_j)$ now sums over paths ending in x_j instead of maximizing.

- And we set $M_1(c') = 0$ if $c' \neq c$ and $M_3(c') = 0$ for $c' \neq 1$.

Conditionals via Backwards Messages

- Performing our conditional calculation using backwards messages.

$$\begin{aligned}\sum_{x_4}\sum_{x_2} p(x_1 = c, x_2, x_3 = 1, x_4) &= \sum_{x_1=c}\sum_{x_2}\sum_{x3=1}\sum_{x4} p(x_4 \mid x_3)p(x_3 \mid x_2)p(x_2 \mid x_1)p(x_1)\\ &= \sum_{x_1=c} p(x_1)\sum_{x_2} p(x_2 \mid x_1) \sum_{x3=1} p(x_3 \mid x_2)\sum_{x4} p(x_4 \mid x_3)\\ &= \sum_{x_1=c} p(x_1)\sum_{x_2} p(x_2 \mid x_1) \sum_{x3=1} p(x_3 \mid x_2)\sum_{x4} p(x_4 \mid x_3)\underbrace{V_4(x_4)}_{=1}\\ &= \sum_{x_1=c} p(x_1)\sum_{x_2} p(x_2 \mid x_1) \sum_{x3=1} p(x_3 \mid x_2)V_3(x_3)\\ &= \sum_{x_1=c} p(x_1)\sum_{x_2} p(x_2 \mid x_1)V_2(x_2)\\ &= \sum_{x_1=c} p(x_1)V_1(x_1).\end{aligned}$$

Forward-Backward Algorithm

- Generic forward and backward messages for discrete marginals have the form

$$M_j(x_j) = \sum_{x_{j-1}} p(x_j \mid x_{j-1}) M_{j-1}(x_{j-1}), \quad V_j(x_j) = \sum_{x_{j+1}} p(x_{j+1} \mid x_j) V_{j+1}(x_{j+1}).$$

- We can compute $p(x_j = c \mid x_{j'} = c')$ using only forward messages:
  - Set $M_j(c) = 1$ and $M_{j'}(c') = 1$.

- Why we would need backward messages?

Forward-Backward Algorithm

- We can compute $p(x_j = c \mid x_{j'} = c')$ for all j in $O(dk^2)$ with both messages.

- First compute all message normally with $M_{j'}(c') = 1$ and $V_{j'}(c') = 1$.

  (Set $M_{j'}(c)$ and $V_{j'}(c)$ to 0 for other values of c.)

- We then have that
  - $M_j(x_j)$ sums up all the paths that end in state x_j (with $x_{j'} = c'$).
  - $V_j(x_j)$ sums up all the paths that start in state x_j (with $x_{j'} = c'$).
  - We can combine these values to get

$$p(x_j \mid x_{j'}) \propto M_j(x_j)V_j(x_j),$$

  - Computing all M_j and V_j is called the forward-backward algorithm.

Conditional Samples from Gaussian/Discrete Markov Chain

Generating exact conditional samples from Gaussian/discrete Markov chains:

1. If we're only conditioning on first j states, $x_{1:j}$, just fix these values and start ancestral sampling from time $(j+1)$.
2. If we have the marginals $p(x_j)$, we can get the "backwards" transition probabilities using Bayes rule,

   $$p(x_j \mid x_{j+1}) = \frac{p(x_{j+1} \mid x_j)p(x_j)}{p(x_{j+1})},$$

   which lets us run ancestral sampling in reverse: sample x_d from $p(x_d)$, then x_{d-1} from $p(x_{d-1} \mid x_d)$, and so on.
3. If we're only conditioning on last j states $x_{d-j:d}$, run CK equations to get marginals and then start ancestral sampling "backwards" starting from $(d-j-1)$ to sample the earlier states.

Conditional Samples from Gaussian/Discrete Markov Chain

4. If we're conditioning on contiguous states in the middle, $x_{j:j'}$, run ancestral sampling forward starting from position $(j'+1)$ and backwards starting from position $(j-1)$.
5. If you condition on non-contiguous positions j and j' with $j < j'$, need to do (i) forward sampling starting from $(j'+1)$, (ii) backward sampling starting from $(j-1)$, and (iii) CK equations on the sequence $(j : j')$ to get marginals conditioned on value of j then backwards sampling back to j starting from $(j'-1)$.

The above are all special cases of conditioning in an undirected graphical model (UGM), followed by applying the "forward-filter backward-sampling" algorithm on each of the resulting chain-structured UGMs.