CPSC 540: Machine Learning

SGD Convergence Rate

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Last time: Stochastic sub-gradient

- We discussed minimizing finite sums,

$$f(w) = \frac{1}{n}\sum_{i=1}^{n} f_i(w),$$

  when n is very large.

- For non-smooth f_i, we discussed stochastic subgradient method,

$$w^{k+1} = w^k - \alpha_k g_{i_k},$$

  for some $g_{i_k} \in \partial f_{i_k}(w^k)$ for some random $i_k \in \{1, 2, \dots, n\}$.
    - May increase f, but moves closer to w^* for small α_k in expectation.

- Same rate as deterministic subgradient method but n times faster.

Stochastic Gradient Convergence Assumptions

- We're going to analyze stochastic gradient rate under these assumptions:
  - f is bounded below (not necessarily convex).
  - ∇f is L-Lipschitz continuous.
  - $\mathbb{E}[\|\nabla f_i(w)\|^2] \leq \sigma^2$ for some constant σ^2 and all w ("variance" is bounded).
    - This bounds the worst-case effect of the "bad term".
- Possible to relax noise bound to more-realistic $\mathbb{E}[\|\nabla f_i(w^k) - \nabla f(w^k)\|^2] \leq \sigma^2$.
  - Just get some extra terms in the result.
- Possible to show similar results for non-smooth functions.
  - Need something stronger than "bounded below" ("weak convexity" or "tame").
  - 2018: first result that applied to ReLU neural networks.

Convergence Rate of Stochastic Gradient Method

- Let's use the "variance" bound inside previous bound,

$$\begin{aligned}\mathbb{E}[f(w^{k+1})] &\leq f(w^k) - \alpha_k \|\nabla f(w^k)\|^2 + \alpha_k^2 \frac{L}{2}\mathbb{E}[\|\nabla f_{i_k}(w^k)\|^2] \\ &\leq f(w^k) - \alpha_k \|\nabla f(w^k)\|^2 + \alpha_k^2 \frac{L\sigma^2}{2}\end{aligned}$$

- As before, re-arrange to get the gradient norm on the left side,

$$\alpha_k \|\nabla f(w^k)\|^2 \leq f(w^k) - \mathbb{E}[f(w^{k+1})] + \alpha_k^2 \frac{L\sigma^2}{2}.$$

- Sum this up (and use iterated expectation) to get

$$\sum_{k=1}^{t} \alpha_{k-1}\mathbb{E}\|\nabla f(w^{k-1})\|^2 \leq \sum_{k=1}^{t}[\mathbb{E}f(w^{k-1}) - \mathbb{E}f(w^k)] + \sum_{k=1}^{t} \alpha_{k-1}^2 \frac{L\sigma^2}{2}.$$

Convergence Rate of Stochastic Gradient Method

- The bound from the previous slide:

$$\sum_{k=1}^{t} \alpha_{k-1} \mathbb{E} \underbrace{\|\nabla f(w^{k-1})\|^2}_{\text{bound by min}} \leq \sum_{k=1}^{t} \underbrace{[\mathbb{E} f(w^{k-1}) - \mathbb{E} f(w^k)]}_{\text{telescope}} + \sum_{k=1}^{t} \alpha_{k-1}^2 \underbrace{\frac{L\sigma^2}{2}}_{\text{no } k}.$$

- Applying the above operations gives

$$\min_{k=0,1,\dots,t-1} \{\mathbb{E}\|\nabla f(w^k)\|^2\} \sum_{k=0}^{t-1} \alpha_k \leq f(w^0) - \mathbb{E} f(w^t) + \frac{L\sigma^2}{2} \sum_{k=0}^{t-1} \alpha_k^2.$$

- Using $\mathbb{E} f(w^k) \geq f^*$ and dividing both sides by $\sum_k \alpha_{k-1}$ gives

$$\min_{k=0,1,\dots,t-1} \{\mathbb{E}\|\nabla f(w^k)\|^2\} \leq \frac{f(w^0) - f^*}{\sum_{k=0}^{t-1} \alpha_k} + \frac{L\sigma^2}{2} \frac{\sum_{k=0}^{t-1} \alpha_k^2}{\sum_{k=0}^{t-1} \alpha_k}.$$

Convergence Rate of Stochastic Gradient Method

- The final bound:

$$\min_{k=0,1,\dots,t-1}\{\mathbb{E}\|\nabla f(w^k)\|^2\} \leq \frac{f(w^0)-f^*}{\sum_{k=0}^{t-1}\alpha_k} + \frac{L\sigma^2}{2}\frac{\sum_{k=0}^{t-1}\alpha_k^2}{\sum_{k=0}^{t-1}\alpha_k}.$$

- If $\sigma^2 = 0$, then we could use a constant step-size and would get a $O(1/t)$ rate.
  - Same as regular gradient descent.
- But due to stochasticity, convergence rate is determined by $\sum_k \alpha_k^2/\sum_k \alpha_k$.

- Classic decreasing step-sizes: set $\alpha_k = \alpha/k$ for some α.
  - Gives $\sum_k \alpha_k = O(\log(t))$ and $\sum_k \alpha_k^2 = O(1)$, so error at t is $O(1/\log(t))$.

- Bigger decreasing step-sizes: set $\alpha_k = \alpha/\sqrt{k}$ for some α.
  - Gives $\sum_k \alpha_k = O(\sqrt{k})$ and $\sum_k \alpha_k^2 = O(\log(k))$, so error at t is $O(\log(t)/\sqrt{t})$.

- Constant step-sizes: set $\alpha_k = \alpha$ for some α.
  - Gives $\sum_k \alpha_k = k\alpha$ and $\sum_k \alpha_k^2 = k\alpha^2$, so error at t is $O(1/t) + O(\alpha)$

Convergence of Stochastic [Sub]Gradient under Strong Convexity

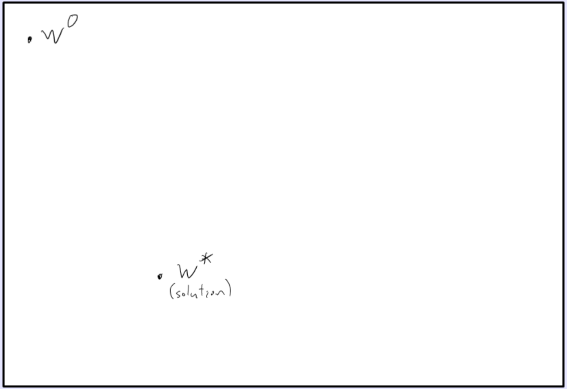
- You can get faster rates if f is strongly-convex:
  - With decreasing $\alpha_k = 1/\mu k$ you get $O(1/t)$ for t iterations (but not linear).
    - But be careful, if you over-estimate μ rate can be much worse.
    - Also, initial steps are huge (this approach only seems to work for binary SVMs).
  - With constant $\alpha_k = \alpha < 1/2\mu$ you get $O(\rho^k) + O(\alpha)$ for t iterations.
    - Linear convergence up to some accuracy proportional to α for sufficiently small α.
- For non-smooth strongly-convex f you get similar results:
  - Setting $\alpha_k = 1/\mu k$ gives $O(\log(t)/t)$.
    - Can improve to $O(1/t)$ by using averaging of the last $t/2$ values of w^k.
  - Setting $\alpha_k = \alpha < 1/2\mu$ still gives $\mathbb{E}[\|w^k - w^*\|^2] = O(\rho^k) + O(\alpha)$.
    - Looks like linear convergence if far from solution (or gradients are similar).
    - No progress if close to solution or have high variance in gradients.

Stochastic Subradient with Constant Step Size

- Expected distance with constant step-size and strong convexity (see bonus):

$$\mathbb{E}[\|w^k - w^*\|^2] \leq (1-2\alpha\mu)^k\|w^0 - w^*\|^2 + \frac{\alpha\sigma^2}{2\mu},$$

- First term looks like linear convergence, but second term does not go to zero.



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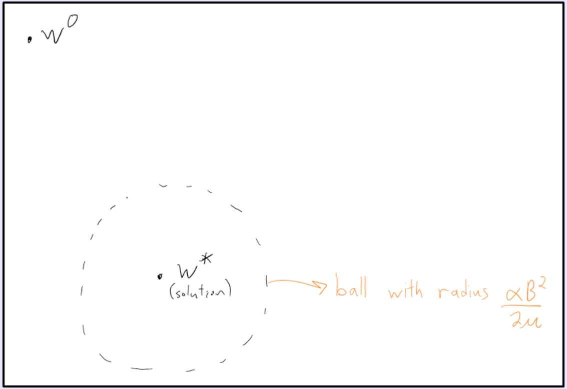

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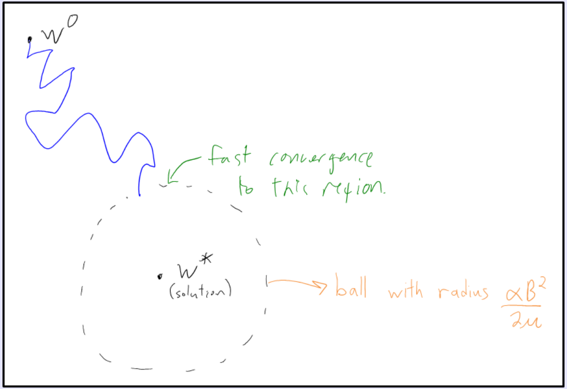
w^0
fast convergence to this region.
w^* (solution)
ball with radius $\frac{\alpha B^2}{2\mu}$

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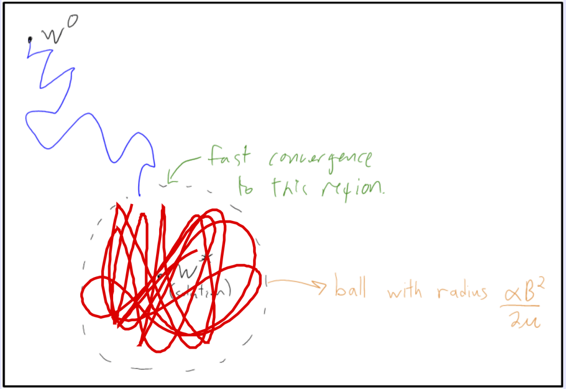
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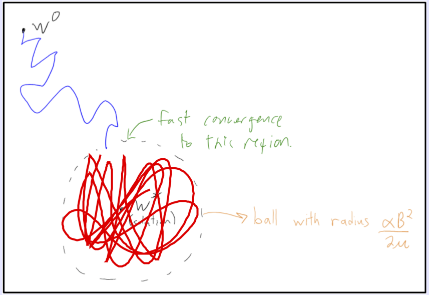
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w^* (solution)
ball with radius $\frac{\alpha B^2}{2\mu}$

- Theory justifies "divide the step-size in half if it looks like it's stalled" heuristic.
  - Halving α divides radius of the ball around w^* in half (similar for non-convex).

Digression: Sparse Features

- For many datasets, our feature vectors x^i are very sparse:

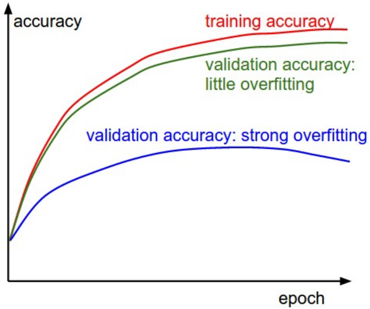
"CPSC"	"Expedia"	"vicodin"	<recipient name>	...
1	0	0	0	...
0	1	0	0	...
0	0	1	0	...
0	1	0	1	...
1	0	1	1	...

- Consider case where d is huge but each row x^i has at most z non-zeroes:
  - The $O(d)$ cost of stochastic subgradient might be too high.
  - We can often modify stochastic subgradient to have $O(z)$ cost.

- See bonus slides for details on this issue, and how to handle regularization.
  - Various forms of "lazy updates" to deal with non-sparse gradient of regularizer.

Early Stopping

- It's hard to decide when to stop stochastic gradient.
- Common heuristic is "early stopping":
  - Every m iterations, stop and compute the validation error.
  - Stop if the validation error starts increasing.

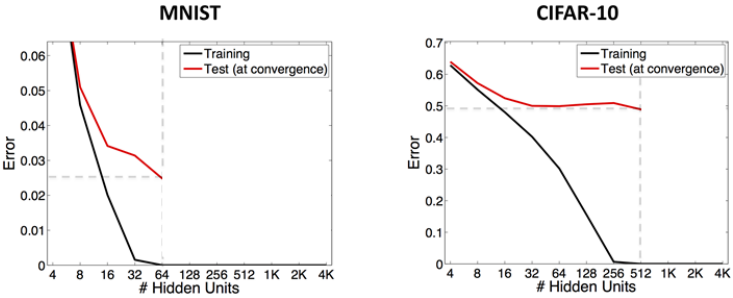


http://cs231n.github.io/neural-networks-3

- This can be viewed as a form regularization ("stop overfitting before it happens").

“Hidden” Regularization in Neural Networks

- Fitting single-layer neural network with SGD and no regularization:



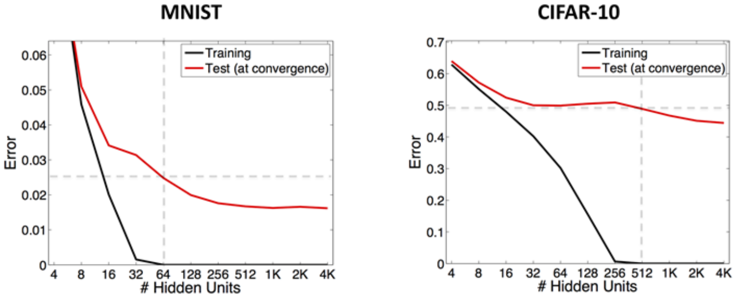
https://www.neyshabur.net/papers/inductive_bias_poster.pdf

- Training error goes to 0 with enough units: we’re finding a global min.

- What should happen to training and test error for larger number of hidden?

"Hidden" Regularization in Neural Networks

- Fitting single-layer neural network with SGD and no regularization:



https://www.neyshabur.net/papers/inductive_bias_poster.pdf

- Test error continues to go down!?! Where is fundamental trade-off??
- There exists global mins where large number of units leads to accuracy 0.
    - But among the global minima, SGD is somehow converging to "good" ones.

Implicit Regularization of SGD

- There is growing evidence that using SGD regularizes parameters.
- Beyond empirical evidence, we know this happens in simpler cases.
- Example:
    - Consider a least squares problem where there exists a w with $Xw = y$.
        - Residuals are all zero, we fit the data exactly.
    - You run [stochastic] gradient descent starting from $w = 0$.
    - Converges to solution w^* of $Xw = y$ that has minimum L2-norm.
        - So using SGD is equivalent to L2-regularization here, but regularization is "implicit".

Implicit Regularization of SGD

- There is growing evidence that using SGD regularizes parameters.
- Beyond empirical evidence, we know this happens in simpler cases.
- Example:
  - Consider a logistic regression problem where data is linearly separable.
    - We can fit the data exactly.
  - You run [stochastic] gradient descent starting from $w = 0$.
  - Converges to max-margin solution w^* of the problem.
    - So using SGD is equivalent to encouraging large margin (and to SVM solutions).

Stochastic Nesterov/Newton Methods?

- Should we use Nesterov/Newton-like stochastic methods?
  - These do not improve the $O(1/\epsilon)$ convergence rate.
- In fact, there is a negative result due to Polyak and Ruppert:
  - Classic result is that scaling by $\nabla^2 f(w^*)$ gives optimal asymptotic rate.
  - You can get same rate without Hessian, by just averaging the later iterations:

$$\bar{w}^t = \frac{1}{t-k}\sum_{k'=k}^{t} w^{k'},$$

- Practical averaging strategies:
  - Could weight all iterations equally.
  - Could ignore first half of the iterations then weight equally.
  - Could weight proportional to k.

Stochastic Nesterov/Newton Methods?

- Some positive results regarding stochastic Nesterov/Newton:
  - Nesterov/Newton can improve dependence on L and μ.
  - Two-phase Newton-like method achieves $O(1/\epsilon)$ without strong-convexity.
  - AdaGrad method,

$$w^{k+1} = w^k + \alpha D^{-1} g_{i_k}, \quad \text{with diagonal } D_{jj} = \sqrt{\delta + \sum_{k'=0}^{k} (\nabla_j f_{i_{k'}}(w^{k'}))^2},$$

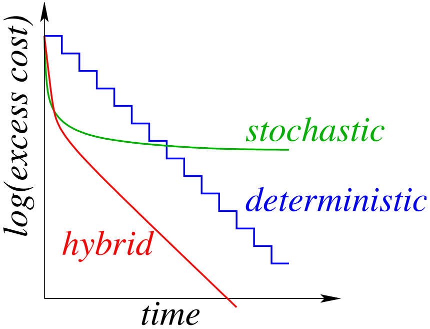
    improves "regret" but not optimization error.
  - Some heuristic extensions of AdaGrad:
    - RMSprop: variant of AdaGrad where step-size does not go to zero.
    - Adam: variant where momentum is added.
    - Recent work: these AdaGrad variants have worse implicit regularization.

Outline

1 More Stochastic Subgradient

2 Mini-Batches and Batching

Better Methods for Smooth Objectives and Finite Datasets?



- Stochastic methods:
  - $O(1/\epsilon)$ iterations but requires 1 gradient per iterations.
  - Rates are unimprovable for general stochastic objectives.
- Deterministic methods:
  - $O(\log(1/\epsilon))$ iterations but requires n gradients per iteration.
  - The faster rate is possible because n is finite.
- For finite n, can we design a better method?

Hybrid Deterministic-Stochastic

- Approach 1: control the sample size.
- Deterministic method uses all n gradients,

$$\nabla f(w^k) = \frac{1}{n}\sum_{i=1}^{n} \nabla f_i(w^k).$$

- Stochastic method approximates it with 1 sample,

$$\nabla f_{i_k}(w^k) \approx \frac{1}{n}\sum_{i=1}^{n} \nabla f_i(w^k).$$

- A common variant is to use larger sample $\mathcal{B}^k$ ("mini-batch"),

$$\frac{1}{|\mathcal{B}^k|}\sum_{i\in\mathcal{B}^k} \nabla f_i(w^k) \approx \frac{1}{n}\sum_{i=1}^{n} \nabla f_i(w^k),$$

particularly useful for vectorization/parallelization.

  - For example, with 16 cores set $|\mathcal{B}^k| = 16$ and compute 16 gradients at once.

Mini-Batching as Gradient Descent with Error

- The SG method with a sample $\mathcal{B}^k$ ("mini-batch") uses iterations

$$w^{k+1} = w^k - \frac{\alpha_k}{|\mathcal{B}^k|} \sum_{i \in \mathcal{B}^k} \nabla f_i(w^k).$$

- Let's view this as a "gradient method with error",

$$w^{k+1} = w^k - \alpha_k(\nabla f(w^k) + e^k),$$

where e^k is the difference between approximate and true gradient.

- If you use $\alpha_k = 1/L$, then using descent lemma this algorithm has

$$f(w^{k+1}) \leq f(w^k) - \underbrace{\frac{1}{2L}\|\nabla f(w^k)\|^2}_{\text{good}} + \underbrace{\frac{1}{2L}\|e^k\|^2}_{\text{bad}},$$

for any error e^k.

Effect of Error on Convergence Rate

- Our progress bound with $\alpha_k = 1/L$ and error in the gradient of e^k is

$$f(w^{k+1}) \leq f(w^k) - \underbrace{\frac{1}{2L}\|\nabla f(w^k)\|^2}_{\text{good}} + \underbrace{\frac{1}{2L}\|e^k\|^2}_{\text{bad}}.$$

- Connection between "error-free" rate and "with error" rate:
  - If "error-free" rate is $O(1/k)$, you maintain this rate if $\|e^k\|^2 = O(1/k)$.
  - If "error-free" rate is $O(\rho^k)$, you maintain this rate if $\|e^k\|^2 = O(\rho^k)$.
  - If error goes to zero more slowly, then rate that it goes to zero becomes bottleneck.

- So to understanding effect of batch-size, need to know how $|\mathcal{B}^k|$ affects $\|e^k\|^2$.

Effect of Batch Size on Error

- Effect of batch size $|\mathcal{B}^k|$ control error size e^k.
  - If we sample with replacement we get

    $$\mathbb{E}[\|e^k\|^2] = \frac{1}{|\mathcal{B}^k|}\sigma^2,$$

    where σ^2 is the variance of the gradient norms.
    - "Doubling the batch size cuts the error in half".
  - If we sample without replacement from a training set of size n we get

    $$\mathbb{E}[\|e^k\|^2] = \frac{n - |\mathcal{B}^k|}{n}\frac{1}{|\mathcal{B}^k|}\sigma^2,$$

    which drives error to zero as batch size approaches n.
- For $O(\rho^k)$ linear convergence, need a schedule like $|\mathcal{B}^{k+1}| = |\mathcal{B}^k|/\rho$.
- For $O(1/k)$ sublinear convergence, need a schedule like $|\mathcal{B}^{k+1}| = |\mathcal{B}^k| + \text{const}$.

Batching: Growing-Batch-Size Methods

- The SG method with a sample $\mathcal{B}^k$ uses iterations

$$w^{k+1} = w^k - \frac{\alpha_k}{|\mathcal{B}^k|} \sum_{i \in \mathcal{B}^k} \nabla f_i(w^k).$$

- For a fixed sample size $|\mathcal{B}^k|$, the rate is sublinear.
  - Still need step-size to go to zero to get convergence.

- But we can grow $|\mathcal{B}^k|$ to achieve a faster rate:
  - Early iterations are cheap like SG iterations.
  - Later iterations can use a sophisticated gradient method.
    - No need to set a magical step-size: use a line-search.
    - Can incorporate linear-time approximations to Newton.

- Another approach: at some point switch from stochastic to deterministic:
  - Often after a small number of passes (but hard to know when to switch).

Summary

- Stochastic subgadient convergence rate:
  - Constant step-size: subgradient quickly converges to approximate solution.
  - Decreasing step-size: subgradient slowly converges to exact solution.
- Practical aspects of stochastic gradient methods:
  - Sparse datasets, early stopping, "hidden regularization".
  - Negative and positive results regarding second-order methods.
- Increasing batch sizes:
  - Leads to faster rate in terms of iterations.
  - Makes setting the step-size easier
- Next time: new stochastic methods with fast rates batch size of 1.

Convergence Rate of Stochastic Subgradient Method

- The basic stochastic subgradient method (for random i_t and $g_{i_t} \in \partial f_{i_t}(x^t)$):

$$x^{t+1} = x^t - \alpha g_{i_t},$$

- We can't use descent lemma because f is non-differentiable.
- Since function value may not decrease, we analyze distance to x^*:

$$\begin{aligned}\|x^t - x^*\|^2 &= \|(x^{t-1} - \alpha_t g_{i_t}) - x^*\|^2\\ &= \|(x^{t-1} - x^*) - \alpha_t g_{i_t}\|^2\\ &= \|x^{t-1} - x^*\|^2 - 2\alpha_t g_{i_t}^\top (x^{t-1} - x^*) + \alpha_t^2 \|g_{i_t}\|^2.\end{aligned}$$

- Take expectation with respect to i_t:

$$\begin{aligned}\mathbb{E}[\|x^t - x^*\|^2] &= \mathbb{E}[\|x^{t-1} - x^*\|] - 2\alpha_t \mathbb{E}[g_{i_t}^\top (x^{t-1} - x^*)] + \alpha_t^2 \mathbb{E}[\|g_{i_t}\|^2]\\ &= \underbrace{\|x^{t-1} - x^*\|^2}_{\text{old distance}} - 2\alpha_t \underbrace{g_t^\top (x^{t-1} - x^*)}_{\text{expected progress}} + \alpha_t^2 \underbrace{\mathbb{E}[\|g_{i_t}\|^2]}_{\text{“variance”}}.\end{aligned}$$

where g_t is a subgradient of f at w^k (expected progress is positive by convexity).

Convergence Rate of Stochastic Subgradient

- Our expected distance given x^{t-1} is

$$\mathbb{E}[\|x^t - x^*\|^2] = \underbrace{\|x^{t-1} - x^*\|^2}_{\text{old distance}} - 2\alpha_t \underbrace{g_t^\top (x^{t-1} - x^*)}_{\text{expected progress}} + \alpha_t^2 \underbrace{\mathbb{E}[\|g_{i_t}\|^2]}_{\text{“variance”}}.$$

- It follows from strong-convexity that (next slide),

$$g_t^\top (x^{t-1} - x^*) \geq \mu\|x^{t-1} - x^*\|^2,$$

which gives (assuming variance is bounded by constant σ^2):

$$\begin{aligned}\mathbb{E}[\|x^t - x^*\|^2] &\leq \|x^{t-1} - x^*\|^2 - 2\alpha_t\mu\|x^{t-1} - x^*\|^2 + \alpha_t^2\sigma^2 \\ &= (1 - 2\alpha_t\mu)\|x^{t-1} - x^*\|^2 + \alpha_t^2\sigma^2.\end{aligned}$$

- With constant $\alpha_k = \alpha$ (with $\alpha < 2/\mu$) and applying recursively we get (with work)

$$\mathbb{E}[\|w^k - w^*\|^2] \leq (1 - 2\alpha\mu)^k\|w^0 - w^*\|^2 + \frac{\alpha\sigma^2}{2\mu},$$

where second term bounds a geometric series.

Strong-Convexity Inequalities for Non-Differentiable f

- A "first-order" relationship between subgradient and strong-convexity:
  - If f is μ-strongly convex then for all x and y we have
    $$f(y) \geq f(x) + f'(y)^\top(y-x) + \frac{\mu}{2}\|y-x\|^2,$$
    for $f'(y) \in \partial f(x)$.
  - The first-order definition of strong-convexity, but with subgradient replacing gradient.
  - Reversing y and x we can write
    $$f(x) \geq f(y) + f'(x)^\top(x-y) + \frac{\mu}{2}\|x-y\|^2,$$
    for $f'(x) \in \partial f(x)$.
  - Adding the above together gives
    $$(f'(y) - f'(x))^\top(y-x)) \geq \mu\|y-x\|^2.$$
  - Applying this with $y = x^{t-1}$ and subgradient g_t and $x = x^*$ (which has $f'(x^*) = 0$ for some subgradient) gives
    $$(g_t - 0)^\top(x^{t-1} - x^*) \geq \mu\|x^{t-1} - x^*\|^2.$$

Convergence Rate of Stochastic Subgradient

- For full details of analyzing stochastic gradient under strong convexity, see:
  - Constant α_k: `http://circle.ubc.ca/bitstream/handle/2429/50358/stochasticGradientConstant.pdf`.
  - Decreasing α_k: `http://arxiv.org/pdf/1212.2002v2.pdf`.
- For both cases under PL, see Theorem 4 here:
  - https://arxiv.org/pdf/1608.04636v2.pdf

Operations on Sparse Vectors

- Consider a vector $g \in \mathbb{R}^d$ with at most z non-zeroes:

$$g^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 0 & -0.5 & 0 & 0 & 0 \end{bmatrix}.$$

- If $z << d$, we can store the vector using $O(z)$ storage instead of $O(d)$:
  - Just store the non-zero values:

$$g_{\text{value}}^T = \begin{bmatrix} 1 & 2 & -0.5 \end{bmatrix}.$$

  - Store index of each non-zero ("pointer"):

$$g_{\text{point}}^T = \begin{bmatrix} 4 & 5 & 7 \end{bmatrix}.$$

- With this representation, we can do standard vector operations in $O(z)$:
  - Compute αg in $O(z)$ by setting $g_{\text{value}} = \alpha g_{\text{value}}$.
  - Compute $w^T g$ in $O(z)$ by multiplying g_{value} by w at positions g_{point}.

Stochastic Subgradient with Sparse Features

- Consider optimizing the hinge-loss,

$$\operatorname*{argmin}_{w\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n}\max\{0, 1 - y^i(w^Tx^i)\},$$

  when d is huge but each x^i has at most z non-zeroes.

- A stochastic subgradient method could use

$$w^{k+1} = w^k - \alpha_k g_{i_k}, \text{ where } g_i = \begin{cases} -y^ix^i & \text{if } 1 - y^i(w^Tx^i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Calculating w^{k+1} is $O(z)$ since these are sparse vector operations.

- So stochastic subgradient is fast if z is small even if d is large.
  - This is how you "train on all e-mails": each e-mail has a limited number of words.

Stochastic Subgradient with Sparse Features

- But consider the L2-regularized hinge-loss in the same setting,

$$\operatorname*{argmin}_{w\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^n \max\{0, 1 - y_i(w^Tx^i)\} + \frac{\lambda}{2}\|w\|^2,$$

  using a stochastic subgradient method,

$$w^{k+1} = w^k - \alpha_k g_{i_k} - \alpha_k \lambda w^k, \text{ where } g_{i_k} \text{ is same as before.}$$

- Problems is that w^k could have d non-zeroes:
  - So adding L2-regularization increases cost from $O(z)$ to $O(d)$?

- There are two standard ways to keep the cost at $O(z)$:
  - L2-regularization: use a $w^k = \beta^k v^k$ (scalar times vector) representation.
  - "Lazy" updates (which work for many regularizers).

Stochastic Subgradient with Sparse Features

- But consider the L2-regularized hinge-loss in the same setting,

$$\operatorname*{argmin}_{w \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \max\{0, 1 - y_i(w^T x^i)\} + \frac{\lambda}{2}\|w\|^2,$$

  using a stochastic subgradient method,

$$w^{k+1} = w^k - \alpha_k g_{i_k} - \alpha_k \lambda w^k, \text{ where } g_{i_k} \text{ is same as before}$$

- Problems is that w^t could have d non-zeroes:
  - So adding L2-regularization increases cost from $O(z)$ to $O(d)$?
- To use L2-regularization and keep $O(z)$ cost, re-write iteration as

$$\begin{aligned} w^{t+1} &= w^t - \alpha_t g_{i_t} - \alpha_t \lambda w^t \\ &= \underbrace{(1 - \alpha_t \lambda) w^t}_{\text{changes scale of } w^t} - \underbrace{\alpha_t g_{i_t}}_{\text{sparse update}}. \end{aligned}$$

Stochastic Subgradient with Sparse Features

- Let's write the update as two steps
$$w^{t+\frac{1}{2}} = (1 - \alpha_t \lambda) w^t, \quad w^{t+1} = w^{t+\frac{1}{2}} - \alpha_t g_{i_t}.$$
- We can implement both steps in $O(z)$ if we re-parameterize as
$$w^t = \beta^t v^t,$$
for some scalar β^t and vector v^t.
- For the first step we can use
$$\beta^{t+\frac{1}{2}} = (1 - \alpha_t \lambda)\beta^t, \quad v^{t+\frac{1}{2}} = v^t.$$
which costs $O(1)$.
- For the second step we can use
$$\beta^{t+1} = \beta^{t+\frac{1}{2}}, \quad v^{t+1} = v^{t+\frac{1}{2}} - \frac{\alpha_t}{\beta^{t+\frac{1}{2}}} g_{i_t},$$
which costs $O(z)$.

Lazy Updates for Sparse Features with Dense Regularizers

- Consider a feature j that has been zero in the loss for 10 iterations (constant α):

$$\begin{aligned} w_j^k &= w_j^{k-1} - 0 - \alpha\lambda w_j^{k-1} \\ &= (1-\alpha\lambda)w_j^{k-1} \\ &= (1-\alpha\lambda)^2 w_j^{k-2} \\ &\vdots \\ &= (1-\alpha\lambda)^{10} w_j^{k-10}. \end{aligned}$$

- So we can apply 10 regularizer gradient steps in $O(1)$.

- Lazy updates:
  - If j is zero in g_{i_k}, do nothing.
  - If j is non-zero, apply all the old regularizer updates then do the gradient step.
    - Requires keeping a "checkpoint" of the last time each variable was updated.

Lazy Updates for Sparse Features with Dense Regularizers

- Lazy updates that track cumulative effects of simple updates.

- Considern stochastic proximal-gradient for L1-regularization:
  - Soft-threshold operator with constant step-size α applies to each element,

$$w_j^{k+1} = \text{sign}(w_j^k)\max\{0, |w_j^k| - \alpha\lambda\}.$$

  - If all that happens to w_j for 10 iterations is the proximal operator, we can use

$$w_j^{k+10} = \text{sign}(w_j^k)\max\{0, |w_k^k| - 10\alpha\lambda\}.$$

- Digression: stochastic proximal-gradient methods:
  - Same convergence rates as basic stochastic gradient method (doesn't help).
  - Unlike deterministic proximal-gradient method, does not find final non-zero pattern in finite time.
    - Regularized dual averaging is a variant that has this property.