CPSC 440: Machine Learning

Product of Bernoullis

Winter 2022

Last Time: Bernoulli Likelihood and Beta Prior

- We introduced the beta distribution:

$$p(\theta \mid \alpha, \beta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)} \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

  - Which is a probability over a parameter θ in the range [0,1].
- We reviewed the "∝" notation for probabilities:
  - If p(θ) ∝ g(θ) for discrete θ, then p(θ) = g(θ)/$\sum_{\theta'} g(\theta')$.
  - If p(θ) ∝ g(θ) for continuous θ, then p(θ) = g(θ)/$\int g(\theta')d\theta$.
- For Bernoulli likelihood and beta prior, we showed posterior is:

$$p(\theta \mid X, \alpha, \beta) = \frac{\theta^{\tilde{\alpha}-1}(1-\theta)^{\tilde{\beta}-1}}{B(\tilde{\alpha},\tilde{\beta})} \propto \theta^{\tilde{\alpha}-1}(1-\theta)^{\tilde{\beta}-1}$$

  - Where $\tilde{\alpha} = \alpha + n_1$ and $\tilde{\beta} = \beta + n_0$.
  - It looks like the prior, with α and β "updated" by the counts of 1s and 0s from data.

MAP Estimation for Bernoulli-Beta Model

- The posterior with a Bernoulli likelihood and beta prior is a beta:

$$p(\theta \mid X, \alpha, \beta) = \frac{\theta^{\tilde{\alpha}-1}(1-\theta)^{\tilde{\beta}-1}}{B(\tilde{\alpha}, \tilde{\beta})}$$

"beta" function (which does not depend on θ)

  - Where $\tilde{\alpha} = n_1 + \alpha$ and $\tilde{\beta} = n_0 + \beta$.
- If $\tilde{\alpha} > 1$ and $\tilde{\beta} > 1$, taking log and setting derivative to 0 gives MAP of:

$$\hat{\theta} = \frac{n_1 + \alpha - 1}{(n_1 + \alpha - 1) + (n_0 + \beta - 1)}$$

$$\boxed{\theta = (\text{sum}(X) + \alpha - 1)/(n + \alpha + \beta - 2)}$$

Cost: $O(n)$

  - If α = 1 and β = 1, we get the MLE.
  - If α = 2 and β = 2, we get Laplace smoothing (which often overfits less).
  - If $\alpha = \beta > 2$, we get a stronger bias towards $\hat{\theta} = 0.5$ than Laplace smoothing.
  - If $\alpha = \beta < 1$, we get a bias towards away from $\hat{\theta} = 0.5$ (towards 0 or 1).
  - You can also bias towards either 0 or 1; if α is large compared to β it biases towards $\hat{\theta}$=1.
  - Notice that MAP converges to MLE n → ∞, so the data eventually "takes over" estimate.
    - But we use a prior so our model does sensible things when we do not have enough data.

Review: Hyper-Parameter and [Cross]-Validation

- We call the "parameters of the prior", α and β, the hyper-parameters.
  - We usually say that hyper-parameters are "parameters affecting the complexity of the model".
  - We usually also include "parameters of the learning algorithm" as hyper-parameters.

- How can we choose hyper-parameters values?
  - Using the training likelihood does not work: it would make α and β arbitrarily small (ignoring prior).

- Usual CPSC 340 approach: use a validation set (or cross-validation).
  - Split your data 'X' into a "training" set and a "validation" set.
  - For different hyper-parameters of α and β:
    - Use the "training" examples to compute the MAP estimate.
    - Use MAP estimate to compute the likelihood of the "validation" examples.
  - Choose the hyper-parameters with the highest validation likelihood.
    - But our final goal is to **not** optimize performance on the validation set.
    - This is a surrogate for the test error (error on completely-new data), which you cannot measure.

- Take CPSC 340 to learn about many of the things that can go wrong.
  - For example, if you are not careful you can overfit to the validation set.
    - I see this all the time, even in UBC student's PhD theses!

* Blue: "Review:..." slides:
→ These are topics that covered in detail in CPSC 340, that I expect you to understand but that will not be covered in detail in this course.

Next Topic: Product of Bernoullis

Motivation: Modeling Traffic Congestion

- We want to model car "traffic congestion" in a big city.
- So we measure which intersections are busy on different days:

Inter 1	Inter 2	Inter 3	Inter 4	Inter 5	Inter 6	Inter 7	Inter 8	Inter 9
0	1	0	1	1	1	0	0	1
0	1	0	1	1	1	0	0	1
0	0	1	1	0	0	0	0	0
0	1	0	1	1	1	0	0	0
1	1	1	1	1	1	1	1	1
0	0	0	1	1	0	0	0	1
0	1	0	1	1	1	1	1	0

- We want to build a model of this data, to identify patterns/problems.
  - "Inter 4 is always busy", "Inter 1 is rarely busy".
  - "Inters 7+8 are always the same", "Inter 2 is busy when Inter 7 is busy".
  - "There is a 25% chance you will hit a busy intersection if you take Inter 1 and 8".

Problem: Multivariate Binary Density Estimation

- We can view this as multivariate binary density estimation:
  - Input: 'n' IID samples of binary vectors $x^1, x^2, x^3, \ldots, x^n$ from population.
  - Output: model that gives probability for any assignment of values $x_1, x_2, \ldots, x_d$.

X =

Inter 1	Inter 2	Inter 3	Inter 4	Inter 5	Inter 6	Inter 7	Inter 8	Inter 9
0	1	0	1	1	1	0	0	1
0	1	0	1	1	1	0	0	1
0	0	1	1	0	0	0	0	0
0	1	0	1	1	1	0	0	0
1	1	1	1	1	1	1	1	1
0	0	0	1	1	0	0	0	1
0	1	0	1	1	1	1	1	0

$\Rightarrow$ $p(x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1, x_5 = 1, x_6 = 1, x_7 = 0, x_8 = 0, x_9 = 1) = 0.11$

(Estimates probability for all 2^9 values)

- Covid example: each feature could be "are covid cases >10% in area 'j'?"
- Notation (please memorize):
  - We use 'n' for the number of examples, 'd' for the number of features.
  - Notice that x^3 is a vector with 'd' values, and x_3 is one binary value.

Product of Bernoullis Model

- There are **many** different models for binary density estimation.
  - Each one makes different assumptions, we will see lots of options!
- We will first consider the simple "product of Bernoullis" model.
  - In this model we assume that the variables are "mutually independent".
    - So if we have four variables we assume $p(x_1, x_2, x_3, x_4) = p(x_1)p(x_2)p(x_3)p(x_4)$.
  - As a picture, we treat multivariate problem as 'd' univariate problems:

X =

Inter 1	Inter 2	Inter 3	Inter 4
0	1	0	1
0	1	0	1
0	0	1	1
0	1	0	1
1	1	1	1
0	0	0	1
0	1	0	1

X_1 =

Inter 1
0
0
0
0
1
0
0

use to compute θ_1

X_2 =

Inter 2
1
1
0
1
1
0
1

use to compute θ_2

X_3 =

Inter 3
0
0
1
0
1
0
0

use to compute θ_3

X_4 =

Inter 4
1
1
1
1
1
1
1

comput θ_4

Product of Bernoullis Inference and Learning

- Key advantage of "product of Bernoullis" model: easy inference and learning.
  - For most inference tasks: do inference on each variable, then combine the results.
  - Compute joint probability.
    - $p(x_1, x_2,\dots,x_d) = p(x_1)p(x_2)\cdots p(x_d) = \theta_1\theta_2\cdots\theta_d$.
  - Compute marginal probabilities.
    - $p(x_2) = \theta_2$.
    - $p(x_2, x_3) = p(x_2)p(x_3) = \theta_2\theta_3$.
  - Compute conditional probabilities.
    - $p(x_2 \mid x_3) = p(x_2)$. $\quad p(x_2 \mid x_3) = \frac{p(x_2, x_3)}{p(x_3)} = \frac{p(x_2)p(x_3)}{p(x_3)} = p(x_2)$
    - $p(x2, x3 \mid x4) = p(x2, x3) = \theta_2\theta_3$.
  - Decoding of $p(x_1, x_2,\dots,x_d)$:
    - Set x_1 to argmax value of $p(x_1)$, set x_2 to argmax of $p(x_2)$,…, set x_d to argmax value of $p(x_d)$.
  - Sampling:
    - Sample x_1 from $p(x_1)$, sample x_2 from $p(x_2)$,…, sample x_d from $p(x_d)$.
- MLE (MAP is similar): $\hat{\theta}_1 = \frac{n_{11}}{n}$ ← number of times variable '1' is '1' $\quad \hat{\theta}_2 = \frac{n_{21}}{n} \quad \dots \quad \hat{\theta}_d = \frac{n_{d1}}{n}$

Product of Bernoullis Inference and Learning

- MLE in a product of Bernoullis:

```
θ = zeros(d)
for i in 1:n
    for j in 1:d
        if X[i,j] == 1
            θ[j] += 1
θ ./= n
```

count the number of times each $x_j^i = 1$

divide by 'n'

or

```
θ = sum(X,dims=1)./n
```

sum up columns of 'X'

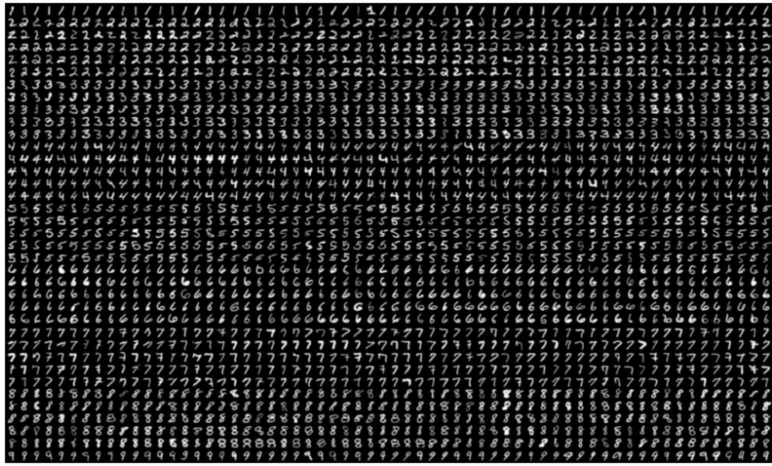
- Cost is O(nd): do an O(1) operations n*d times, then O(n) division.
  – If 'X' is stored as a "sparse" matrix, can be implemented to only cost O(z).
    - Where 'z' is the number of non-zero values ($z \leq nd$).
- Sampling code:

```
x = zeros(d)
for j in 1:d
    x[j] = sampleBinary(θ[j])
```

cost is O(d) to generate a sample

Running Example: MNIST Digits

- To illustrate density estimation, we will often use the MNIST digits:
  - 60,000 images, each a 28x28 pixel image of a number.
  - Representing as binary density estimation:
    - Each image is one training example x^i.
    - With each feature being one of the 784 pixels.
    - Threshold each pixel to make it binary.

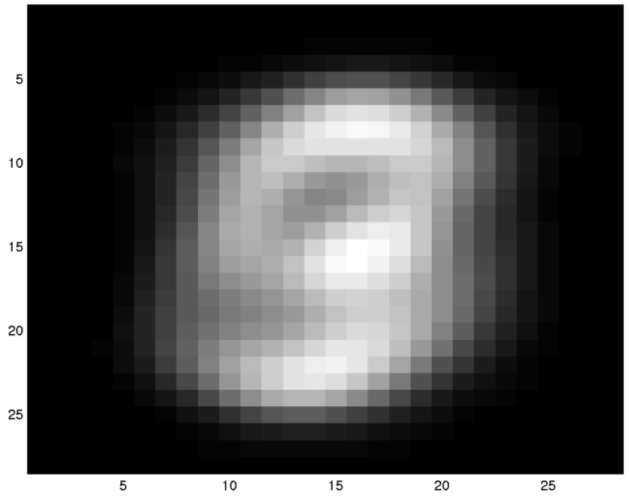


- CPSC 340 wanted to "recognize that this is a 4".
- In density estimation we want probability distribution over images.
  - Given one of the 2^{784} possible images, what is the probability it is a digit?
    - This is unsupervised, we are ignoring the class labels.
  - Sampling from the density should generate new images of digits.

https://www.kaggle.com/tarunkr/digit-recognition-tutorial-cnn-99-67-accuracy

Product of Bernoullis on MNIST Digits

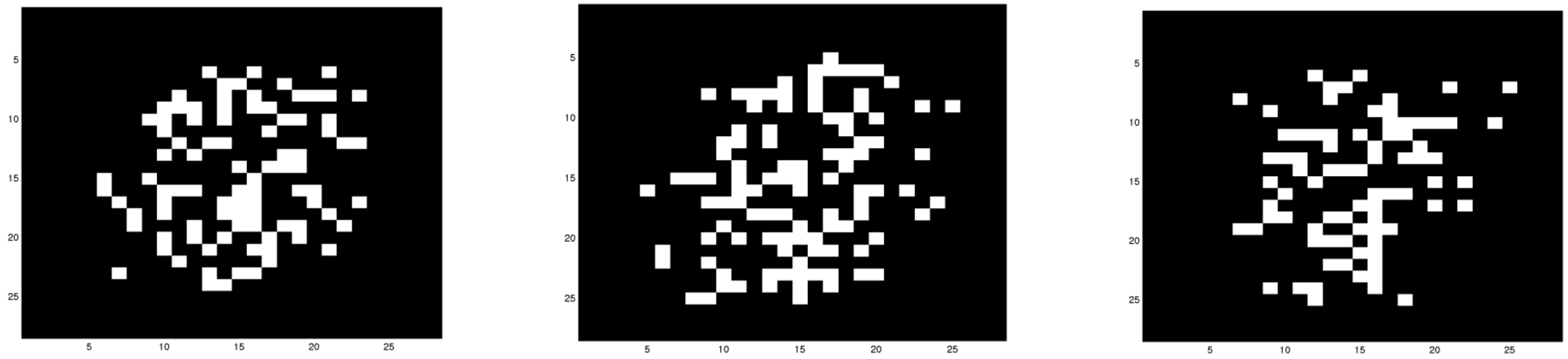
- Consider fitting the product of Bernoullis model to MNIST digits:
  - For each of the 784 pixels 'j', we will have a parameter θ_j.
    - A "position-specific Bernoulli" distribution.
  - To compute MLE for θ_j, compute fraction of times pixel 'j' was set to 1.
    - Visualizing those MLE values as an image:



    - Shows pixels near the center are more likely to be 1 than pixels near the boundary.

Product of Bernoullis on MNIST Digits

- Is product of Bernoullis a good model for the MNIST digits?
  - Samples generated from the model (independent sample from position-specific Bernoulli for each pixel):

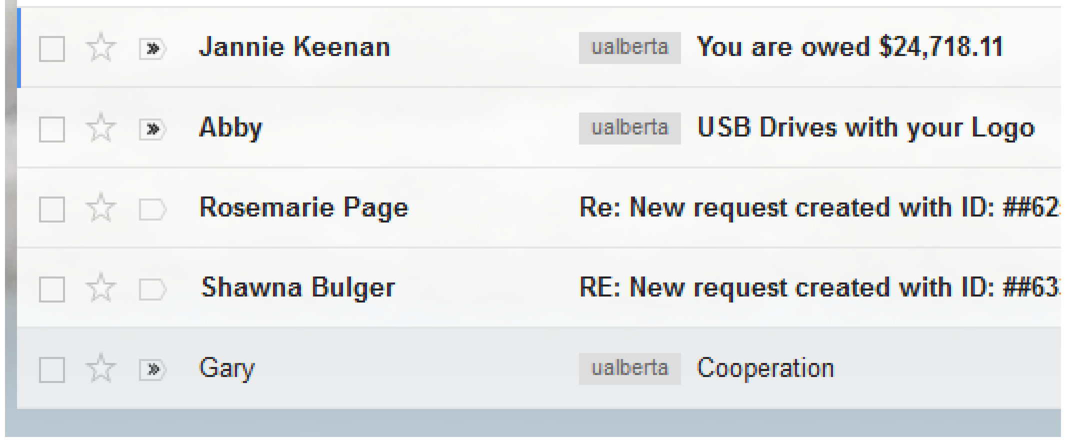


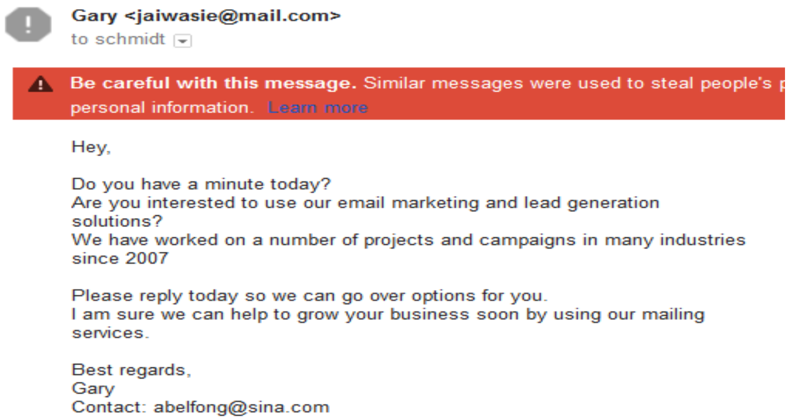
  - This is a terrible model: these samples do not look like the data.
  - Why is this a terrible model?
    - In the dataset, the pixels are not independent.
    - For example, pixels that are “next” to each other in the image are highly correlated.
  - Even it is a bad model, product of Bernoullis is often “good enough to be useful”.
    - Usually when combined with other ideas, that we will see shortly.
    - In practice, I think it is actually the most-used method for binary density estimation even though it is one of the worst.
  - Later in the course will cover several ways to relax the independence assumption.

Next Topic: Generative Classifiers

Motivation: E-mail Spam Filtering

- Want a build a system that detects spam e-mails.
  - Context: spam used to be a big problem.





- We can write this as a supervised learning problem:
  - Want to learn to map from “input” (e-mail) to “output” (spam or not).

Review: Data Collection and Feature Extraction

- Collect a large number of e-mails, gets users to label them.

$	Hi	CPSC	340	Vicodin	Offer	…
1	1	0	0	1	0	…
0	0	0	0	1	1	…
0	1	1	1	0	0	…
…	…	…	…	…	…	…

Spam?
1
1
0
…

- We can use ($y^i = 1$) if e-mail ‘i’ is spam, ($y^i = 0$) if e-mail is not spam.
- Extract features of each e-mail (like “bag of words”).
  - ($x^i_j = 1$) if word/phrase ‘j’ is in e-mail ‘i’, ($x^i_j = 0$) if it is not.
    - See CPSC 330 (or 340) for different ways to extract features from text data.

Review: Supervised Learning Notation

- Our notation for supervised learning:

$X =$

$	Hi	CPSC	340	Vicodin	Offer	…
1	1	0	0	1	0	…
0	0	0	0	1	1	…
0	1	1	1	0	0	…
…	…	…	…	…	…	…

x_6^2

x^3

$y =$

Spam?
1
1
0
…

y^i

- X is matrix of all features, y is vector of all labels.
  - We use y^i for the label of example ‘i’ (element ‘i’ of ‘y’).
  - We use x^i_j for feature ‘j’ of example ‘i‘.
  - We use x^i as the list of features of example ‘i’ (row ‘i’ of ‘X’).
    - So in the above x^3 = [0 1 1 1 0 0 …].
    - In practice, store x^i in some “sparse” format (like a list of non-zeroes, smaller memory).

Generative Classifiers

- In early 2000s, best spam filtering methods used generative classifiers.
  - Generative classifiers treat supervised learning as density estimation.

- How can we do supervised learning with density estimation?
  - Learning: use a density estimator to estimate $p(x_1, x_2, \ldots, x_d, y)$.
    - Generative classifiers model "how the features and label were generated".
  - Inference: compute conditionals $p(y \mid x_1, x_2, \ldots, x_d)$ to make predictions.
    - For example, is $p(y = 1 \mid x_1, x_2, \ldots, x_d) > p(y = 0 \mid x_1, x_2, \ldots, x_d)$?

$$\boxed{\begin{aligned} &p(y \mid x_1, x_2, \ldots, x_d) \\ &\propto p(x_1, x_2, \ldots, x_d, y) \\ &= p(x_1)p(x_2)\cdots p(x_d)p(y) \\ &= p(y) \end{aligned}}$$

- Can we use a product of Bernoullis as the density estimator?
  - You could, but it would do terrible!
  - If 'y' is independent of the features, predictions would ignore features.
  - A simple model that does assume 'y' is independent of features is naïve Bayes.

Summary

- Beta distribution:
  - Prior for Bernoulli parameter that yields a closed-form MAP estimate.
    - Laplace smoothing as a special case.
- Hyper-parameters:
  - Parameters of the prior, or other parameters affecting complexity.
  - Later we will also include "parameters of the optimization algorithm".
- Product of Bernoullis:
  - Method for multivariate binary density estimation.
  - Assumes all variables are independent.
  - Inference and learning are easy, but cannot accurate model many densities.
- Generative classifiers:
  - Classifiers that model p(x,y) and predict by doing inference.

- Next time: a bit of 340 review.

Existence of MAP Estimate under Beta Prior

- The MAP estimate for Bernoulli likelihood and beta prior:

$$\hat{\theta} = \frac{n_1 + \alpha - 1}{(n_1 + \alpha - 1) + (n_0 + \beta - 1)}$$

  - This assumes that $n_1 + \alpha > 1$ and $n_0 + \beta > 1$.

- Other cases:
  - $n_1 + \alpha > 1$ and $n_0 + \beta \leq 1$: $\hat{\theta} = 1$.
  - $n_1 + \alpha \leq 1$ and $n_0 + \beta > 1$: $\hat{\theta} = 0$.
  - $n_1 + \alpha < 1$ and $n_0 + \beta < 1$: $\hat{\theta}$ can be 0 or 1.
  - $n_1 + \alpha = 1$ and $n_0 = 1$: $\hat{\theta}$ can be anything between 0 and 1.