CPSC 440: Advanced Machine Learning

Markov Chains

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Example: Vancouver Rain Data

- Consider density estimation on the "Vancouver Rain" dataset:

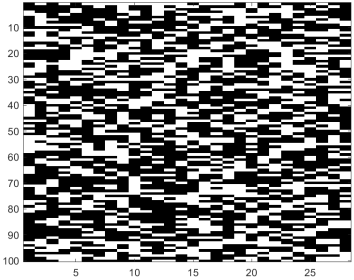
	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Day 8	Day 9	...
Month 1	0	0	0	1	1	0	0	1	1	
Month 2	1	0	0	0	0	0	1	0	0	
Month 3	1	1	1	1	1	1	1	1	1	
Month 4	1	1	1	1	0	0	1	1	1	
Month 5	0	0	0	0	1	1	0	0	0	
Month 6	0	1	1	0	0	0	0	1	1	

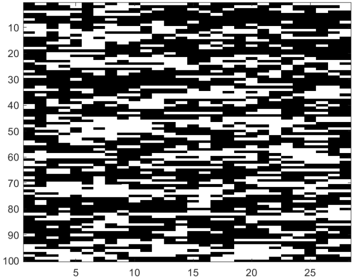
- Variable $x_j^i = 1$ if it rained on day j in month i.
  - Each row is a month, each column is a day of the month.
  - Data ranges from 1896-2004.

- The strongest signals in the data:
  - It tends to rain more in the winter than the summer.
  - If it rained yesterday, it's likely to rain today ($\approx 70\%$ chance of $(x_j^i == x_{j-1}^i)$).

Rain Data with Independent Bernoullis

- With independent Bernoullis, we get $p(x_j^i = \text{"rain"}) \approx 0.41$ (sadly).
  - Samples from product of Bernoullis model (left) vs. real data (right):

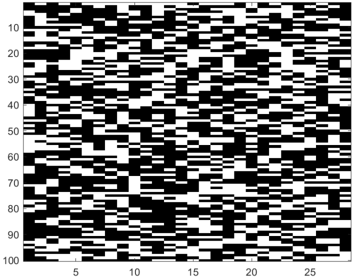


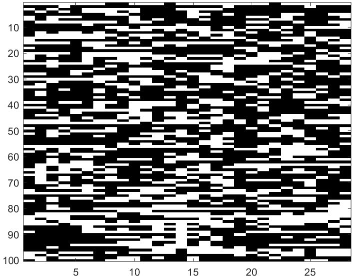


- Making days independent misses seasons and misses correlations.

Rain Data with Mixture of Bernoullis

- A better model is a mixture of Bernoullis:
  - Samples from product of Bernoullis model (left) vs. mixture of 50 Bernoullis (right):

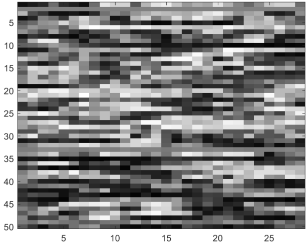




- Mixture of Bernoullis can learn that there are seasons (clusters).
- But mixture of Bernoullis can't easily learn the between-day correlations.

Rain Data with Mixture of Bernoullis

- Visualizing the mean parameters of the mixture of 50 Bernoullis:

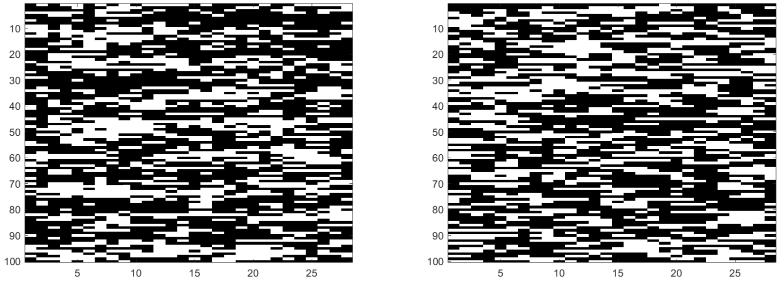


- Recall that mixture of Bernoullis assumes independence, given cluster.
- This makes it try to model between-day correlations in a weird way:
    - Uses clusters with rain for consecutuve days, during different parts of month.

- So you would need a lot of clusters to model all between-day correlations.
    - Need cluster that correlate that day 1 and 2, that correlate day 2 and 3, and so on.
    - Doesn't account for "position independence" of the correlation.

Markov Chains

- A better model for the between-day correlations is a Markov chain.
  - Models $p(x_j^i \mid x_{j-1}^i)$: probability of rain today given yesterday's value.
    - Captures dependency between adjacent days.



- It can perfectly capture the "position-independent" between-day correlation.
  - With only a few parameters and a closed-form MLE (no EM or non-convexity).

Markov Chain for Rain

- Markov chain ingredients and MLE for rain data:
  - State space:
    - At time j, we can be in the "rain" state or the "not rain" state.
  - Initial probabilities:

c	$p(x_1 = c)$
Rain	0.37
Not Rain	0.63

  - Transition probabilities (assumed to the same for all times j):

c'	c	$p(x_j = c \mid x_{j-1} = c')$
Rain	Rain	0.65
Rain	Not Rain	0.35
Not Rain	Rain	0.25
Not Rain	Not Rain	0.75

- Becuase of "sum to 1" constraints, there are only 3 parameters in this model.

Markov Chain Ingredients

- Markov chain ingredients and MLE for rain data:
  - State space:
    - Set of possible states (indexed by c) we can be in at time j ("rain" or "not rain").
  - Initial probabilities:
    - $p(x_1 = c)$: probability that we start in state c at time $j = 1$ (p("rain") on day 1).
  - Transition probabilities:
    - $p(x_j = c \mid x_{j-1} = c')$: probability that we move from state c' to state c at time j.
    - Probability that it rains today, given what happened yesterday.

- Notation alert: I'm going to start using "x_j" as short for "x_j^i" for a generic i.

- We're assuming that the order of features is meaningful.
  - We're modeling dependency of each feature on the previous feature.

Chain Rule of Probability

- By using the product rule, $p(a, b) = p(a)p(b \mid a)$, we can write any density as

$$\begin{aligned} p(x_1, x_2, \ldots, x_d) &= p(x_1)p(x_2, x_3, \ldots, x_d \mid x_1) \\ &= p(x_1)p(x_2 \mid x_1)p(x_3, x_4, \ldots, x_d \mid x_1, x_2) \\ &= p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2, x_1)p(x_4, x_5, \ldots, x_d \mid x_1, x_2, x_3), \end{aligned}$$

  and so on until we get

$$p(x_1, x_2, \ldots, x_d) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1, x_2) \cdots p(x_d \mid x_1, x_2, \ldots x_{d-1}).$$

- This factorization of a density is called the chain rule of probability.
- But it leads to complicated conditionals:
  - For binary x_j, we need 2^d parameters for $p(x_d \mid x_1, x_2, \ldots, x_{d-1})$ alone.

Markov Chains

- Markov chains we simplify the distribution by assuming the Markov property:

$$p(x_j \mid x_{j-1}, x_{j-2}, \ldots, x_1) = p(x_j \mid x_{j-1}),$$

that x_j is independent of the past given x_{j-1}.

  - "Don't care what happened 2 days ago if you know what happened yesterday".

- The probability for a sequence $x_1, x_2, \cdots, x_d$ in a Markov chain simplifies to

$$\begin{aligned} p(x_1, x_2, \ldots, x_d) &= p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2, x_1) \cdots p(x_d \mid x_{d-1}, x_{d-2}, \ldots, x_1) \\ &= p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2) \cdots p(x_d \mid x_{d-1}) \end{aligned}$$

- Another way to write this joint probability is

$$p(x_1, x_2, \ldots, x_d) = \underbrace{p(x_1)}_{\text{initial prob.}} \prod_{j=2}^{d} \underbrace{p(x_j \mid x_{j-1})}_{\text{transition prob.}}.$$

Markov Chains

- Markov chains are ubiquitous in sequence/time-series models:

9 Applications
- 9.1 Physics
- 9.2 Chemistry
- 9.3 Testing
- 9.4 Speech Recognition
- 9.5 Information sciences
- 9.6 Queueing theory
- 9.7 Internet applications
- 9.8 Statistics
- 9.9 Economics and finance
- 9.10 Social sciences
- 9.11 Mathematical biology
- 9.12 Genetics
- 9.13 Games
- 9.14 Music
- 9.15 Baseball
- 9.16 Markov text generators

Outline

Homogenous Markov Chains

- For rain data it makes sense to use a homogeneous Markov chain:
  - Transition probabilities $p(x_j \mid x_{j-1})$ are the same for all times j.

- With discrete states, we could parameterize transition probabilities by

$$p(x_j = c \mid x_{j-1} = c') = \theta_{c,c'},$$

where $\theta_{c,c'} \geq 0$ and $\sum_{c=1}^{k} \theta_{c,c'} = 1$ (and we use the same $\theta_{c,c'}$ for all j).
  - So we have a categorical distribution over c values for each c' value.

- MLE for homogeneous Markov chain with discrete x_j is:

$$\theta_{c,c'} = \frac{(\text{number of transitions from } c' \text{ to } c)}{(\text{number of times we went from } c' \text{ to anything})},$$

so learning is just counting.

Parameter Tieing

- Using same parameters $\theta_{c,c'}$ for different j is called parameter tieing.
  - "Making different parts of the model use the same parameters."

- Key advantages to parameter tieing:
  1. You have more data available to estimate each parameter.
     - Don't need to independently learn $p(x_j \mid x_{j-1})$ for days 3 and 24.
  2. You can have training examples of different sizes.
     - Same model can be used for any number of days.
     - We could even treat the data as one long Markov chain $(n = 1)$.

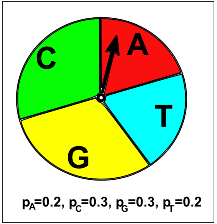
- We've seen parameter tieing before:
  - In 340 we discussed convolutional neural networks, which repeat same filters.
  - Throughout 340/540, we've assumed tied parameters across training examples.
    - That you use the same parameter for x^i and x^j.
    - Can think of mixtures models as relaxing this (same parameters only within cluster).

Example: Modeling DNA Sequences

- A nice demo of independent vs. Markov (and HMMs) for DNA sequences:
  - http://a-little-book-of-r-for-bioinformatics.readthedocs.io/en/latest/src/chapter10.html

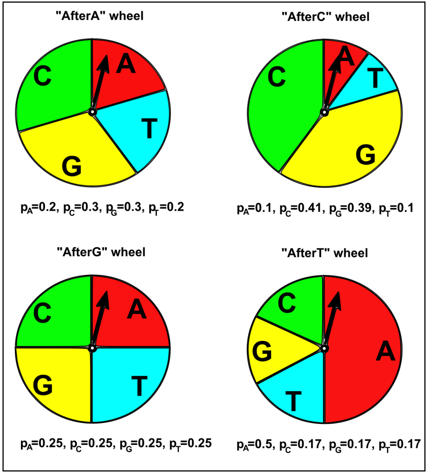
https://www.tes.com/lessons/WE5E9RncBhieAQ/dna

- Independent model for elements of sequence:



Example: Modeling DNA Sequences

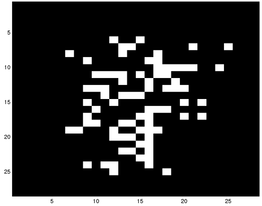
- Transition probabilities in a Markov chain model for elements of sequence:

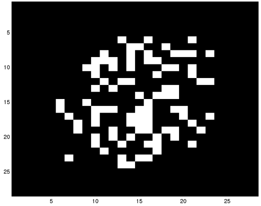


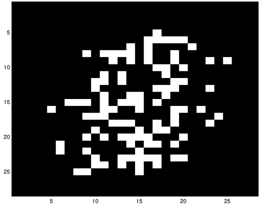
(visualizing transition probabilities based on previous symbol):

Density Estimation for MNIST Digits

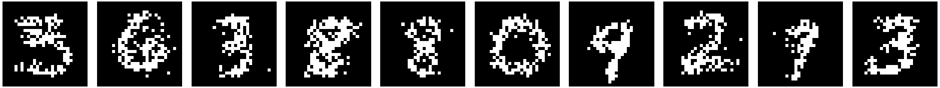
- We've previously considered density estimation for MNIST images of digits.
- We saw that independent Bernoullis do terrible







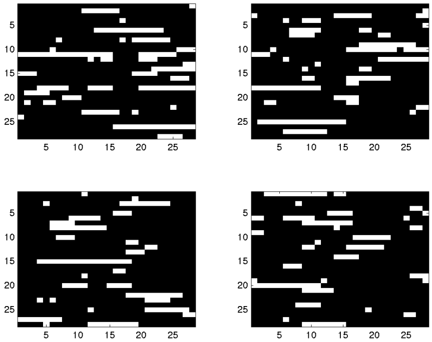
- We saw that a mixture of Bernoullis does better:



- The shape is looking better, but it's missing correlation between adjacent pixels.
  - Could we capture this with a Markov chain?

Density Estimation for MNIST Digits

- Samples from a homogeneous Markov chain (putting rows into one long vector):



- Captures correlations between adjacent pixels in the same row.
  - But misses long-range dependencies in row and dependencies between rows.
  - Also, "position independence" of homogeneity means it loses position information.

Inhomogeneous Markov Chains

- Markov chains could allow a different $p(x_j \mid x_{j-1})$ for each j.
  - This makes sense for digits data, but probably not for the rain data.

- For discrete x_j we could use

$$p(x_j = c \mid x_{j=1} = c') = \theta^{j}_{c,c'}.$$

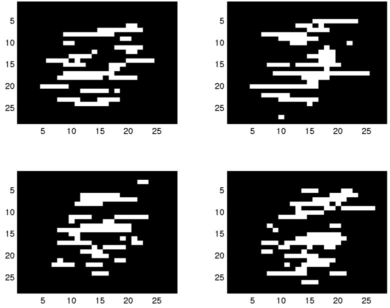
- MLE for discrete x_j values is given by

$$\theta^{j}_{c,c'} = \frac{(\text{number of transitions from } c' \text{ to } c \text{ starting at } (j-1))}{(\text{number of times we saw } c' \text{ at position } (j-1))},$$

- Such inhomogeneous Markov chains include independent models as special case:
  - If we set $p(x_j \mid x_{j-1}) = p(x_j)$ for all j we get independent model.

Density Estimation for MNIST Digits

- Samples from an inhomogeneous Markov chain fit to digits:



- We have correlations between adjacent pixels in rows and position information.
  - But isn't capturing long-range dependencies or dependency between rows.
  - Later we'll discuss graphical models which address this.
  - You could alternately consider a mixture of Markov chains.

Summary

- Markov chains model dependencies between adjacent features.
- Parameter tieing uses same parameters in different parts of a model.
  - Example of "homogeneous" Markov chain.
  - Allows to use more data to estimate each parameter.
  - Allows models of different sizes.
  - But the wrong assumption for some datasets.
    - May prefer "inhomogeneous" Markov chain.
- Next time: the other "MC" in MCMC.