CPSC 340:
Machine Learning and Data Mining

More Linear Classifiers

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University of British Columbia, Fall 2022
https://www.students.cs.ubc.ca/~cs-340

Last Time: Classification using Regression

- Binary classification using sign of linear models.
  - Hyperplane splitting the space in half.

Fit model $y_i = w^T x_i$ and predict using $\text{sign}(w^T x_i)$

$+1$ -1

- We considered different training "error" functions:
  - Squared error: $(w^T x_i - y_i)^2$.
    - If $y_i = +1$ and $w^T x_i = +100$, then squared error $(w^T x_i - y_i)^2$ is huge.
  - 0-1 classification error: $(\text{sign}(w^T x_i) = y_i)$?
    - Ideal error but non-convex and hard to minimize in terms of 'w'.
    - Optimization only tractable if perfect classifier exists → Perceptron algorithm.
    - In the general case we need a convex approximation (today).

Degenerate Convex Approximation to 0-1 Loss

- If $y_i = +1$, we get the label right if $w^T x_i > 0$.
- If $y_i = -1$, we get the label right if $w^T x_i < 0$, or equivalently $-w^T x_i > 0$.
- So "classifying 'i' correctly" is equivalent to having $y_i w^T x_i > 0$.

- One possible convex approximation to 0-1 loss:
  - Minimize how much this constraint is violated.

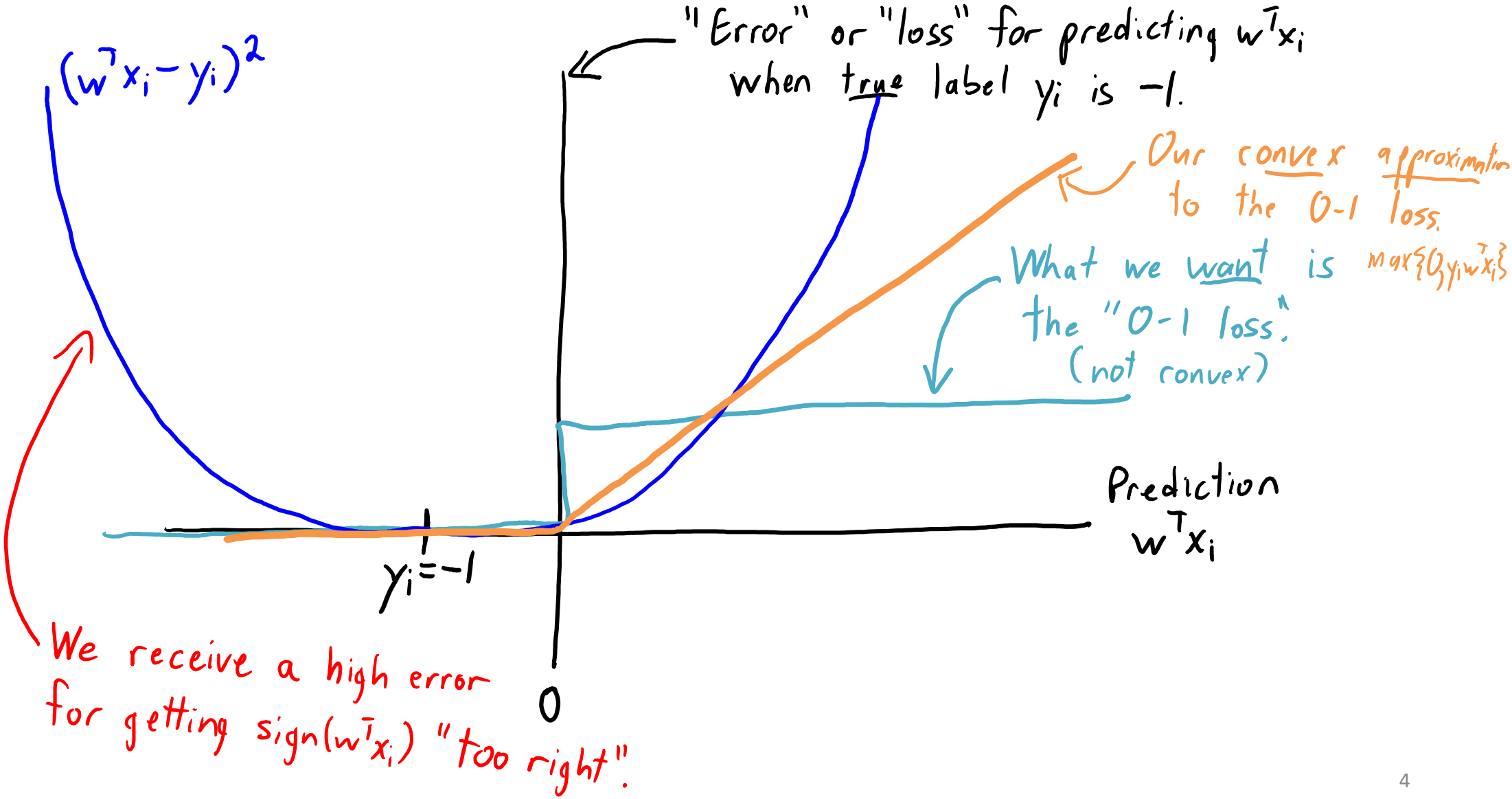
If $y_i w^T x_i > 0$ then you get an "error" of 0.

If $y_i w^T x_i < 0$ then you get an "error" of $-y_i w^T x_i$

→ So the "error" is given by $\max\{0, -y_i w^T x_i\}$

$\max\{\text{constant}, \text{linear}\} \Rightarrow$ convex

Convex Approximation to 0-1 Loss



Degenerate Convex Approximation to 0-1 Loss

- Our convex approximation of the error for one example is:

$$\max\{0, -y_i w^T x_i\}$$

- We could train by minimizing sum over all examples:

$$f(w) = \sum_{i=1}^{n} \max\{0, -y_i w^T x_i\}$$

- But this has a degenerate solution:
  - We have f(0) = 0, and this is the lowest possible value of 'f'.
- There are two standard fixes: hinge loss and logistic loss.

Hinge Loss

- We saw that we classify examples 'i' correctly if $y_i w^T x_i > 0$.
  - Our convex approximation is the amount this inequality is violated.

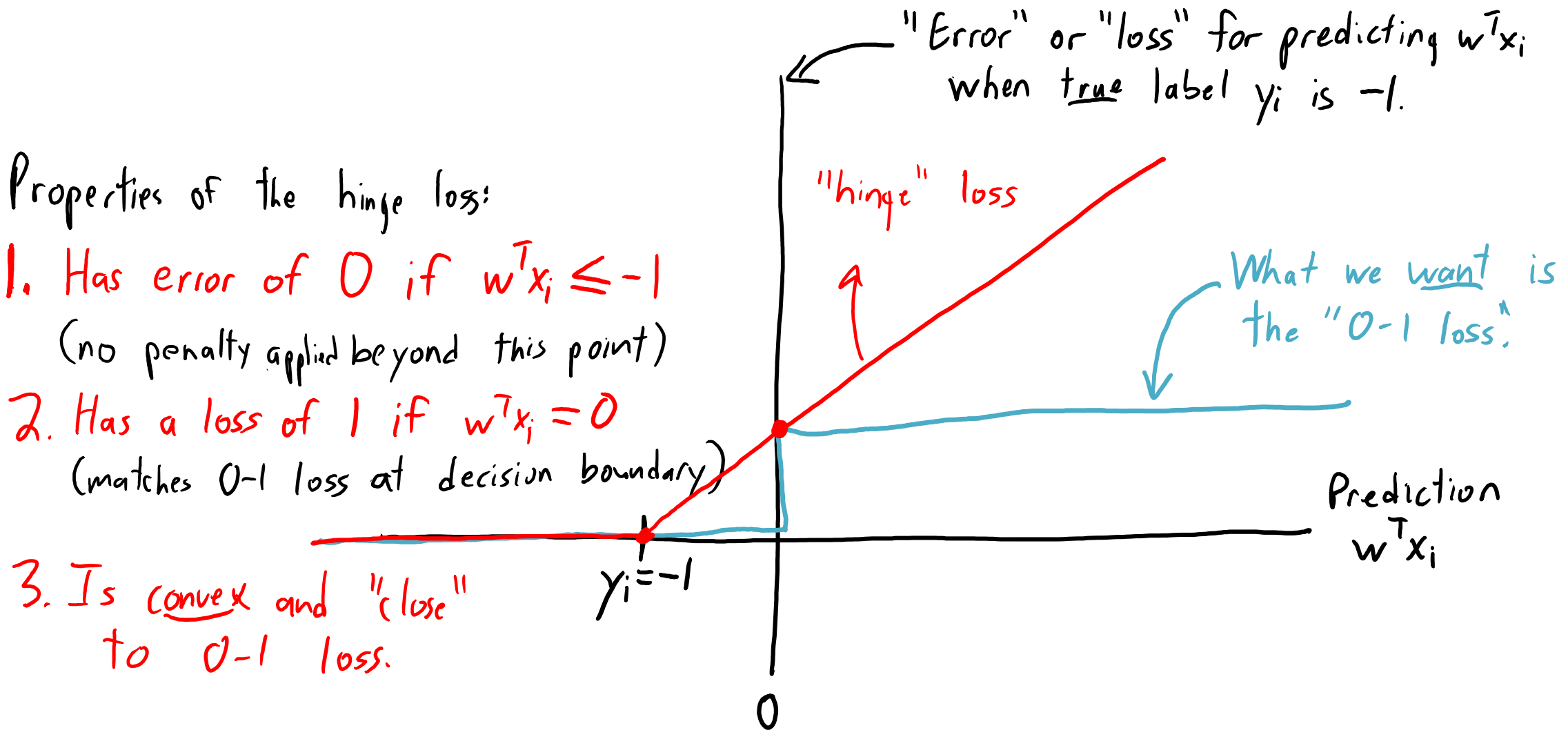
- Consider replacing $y_i w^T x_i > 0$ with $y_i w^T x_i \geq 1$.
  - (the "1" is arbitrary: we could make ||w|| bigger/smaller to use any positive constant)

- The violation of this constraint is now given by:

$$\max\{0, 1 - y_i w^T x_i\}$$

- This is the called hinge loss.
  - It's convex: max(constant, linear).
  - It's not degenerate: w=0 now gives an error of 1 instead of 0.

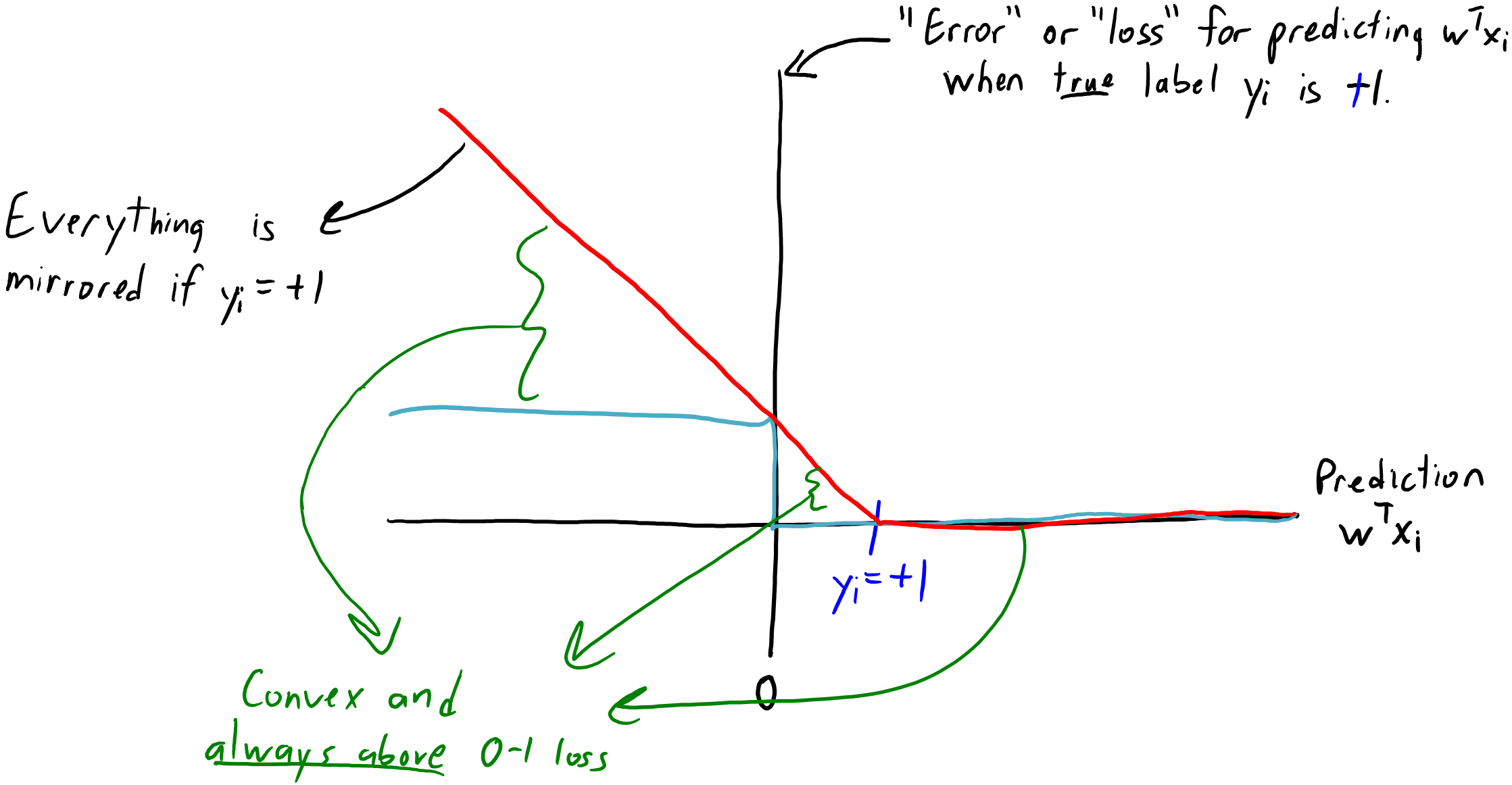
Hinge Loss: Convex Approximation to 0-1 Loss



Properties of the hinge loss:

1. Has error of 0 if $w^T x_i \leq -1$
   (no penalty applied beyond this point)
2. Has a loss of 1 if $w^T x_i = 0$
   (matches 0-1 loss at decision boundary)
3. Is convex and "close" to 0-1 loss.

Hinge Loss: Convex Approximation to 0-1 Loss



Hinge Loss

- Hinge loss for all 'n' training examples is given by:

$$f(w) = \sum_{i=1}^{n} \max\{0, 1 - y_i w^T x_i\}$$

  - Convex upper bound on 0-1 loss.
    - If the hinge loss is 18.3, then number of training errors is at most 18.
    - So minimizing hinge loss indirectly tries to minimize training error.
    - Like perceptron, finds a perfect linear classifier if one exists.
- Support vector machine (SVM) is hinge loss with L2-regularization.

$$f(w) = \sum_{i=1}^{n} \max\{0, 1 - y_i w^T x_i\} + \frac{\lambda}{2}\|w\|^2$$

  - There exist specialized optimization algorithm for this problems.
  - SVMs can also be viewed as "maximizing the margin".

‘λ’ vs ‘C’ as SVM Hyper-Parameter

- We’ve written SVM in terms of regularization parameter ‘λ’:

$$f(w) = \sum_{i=1}^{n} \max\{0, 1 - y_i w^T x_i\} + \frac{\lambda}{2}\|w\|^2$$

- Some software packages instead have regularization parameter ‘C’:

$$f(w) = C\sum_{i=1}^{n} \max\{0, 1 - y_i w^T x_i\} + \frac{1}{2}\|w\|^2$$

- In our notation, this corresponds to using $\lambda = 1/C$.
  - Equivalent to just multiplying f(w) by constant.
  - Note interpretation of ‘C’ is different: high regularization for small ‘C’.
    - You can think of ‘C’ as “how much to focus on the classification error”.

Logistic Loss

- We can smooth max the degenerate loss with log-sum-exp:

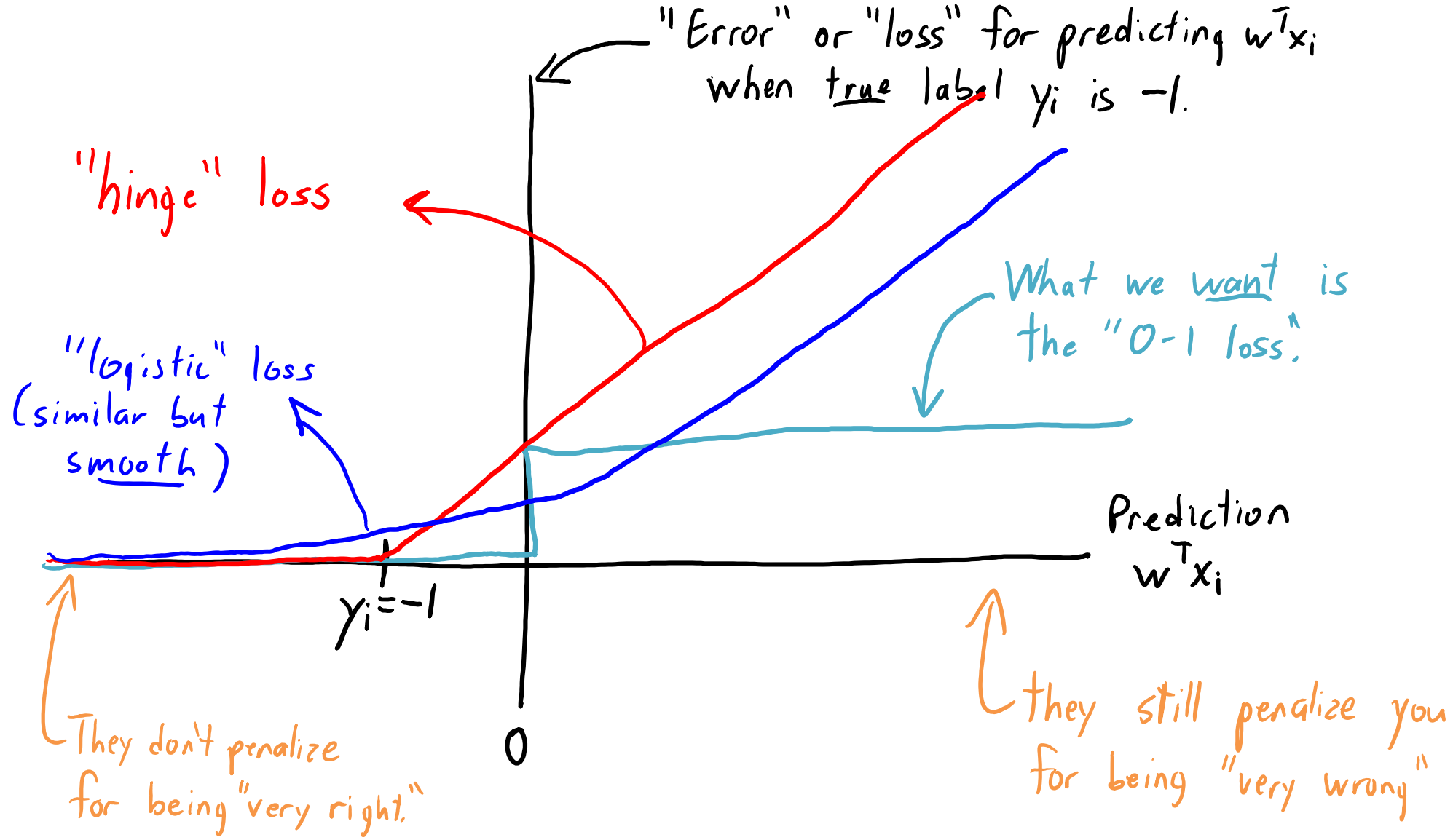
$$\max\{0, -y_i w^T x_i\} \approx \log(\exp(0) + \exp(-y_i w^T x_i))$$

- Summing over all examples gives:

$$f(w) = \sum_{i=1}^{n} \log(1 + \exp(-y_i w^T x_i))$$

- This is the "logistic loss" and model is called "logistic regression".
  - It's not degenerate: w=0 now gives an error of log(2) instead of 0.
  - Convex and differentiable: minimize this with gradient descent.
  - You should also add regularization.
  - We'll see later that it has a probabilistic interpretation.

Convex Approximations to 0-1 Loss



Logistic Regression and SVMs

- Logistic regression and SVMs are used EVERYWHERE!
  - Fast training and testing.
    - Training on huge datasets using "stochastic" gradient descent (next week).
    - Prediction is just computing $w^T x_i$.
  - Weights w_j are easy to understand.
    - It's how much w_j changes the prediction and in what direction.
  - We can often get a good test error.
    - With low-dimensional features using RBFs and regularization.
    - With high-dimensional features and regularization.
  - Smoother predictions than random forests.

Comparison of "Black Box" Classifiers

- Fernandez-Delgado et al. [2014]:
  - "Do we Need Hundreds of Classifiers to Solve Real World Classification Problems?"

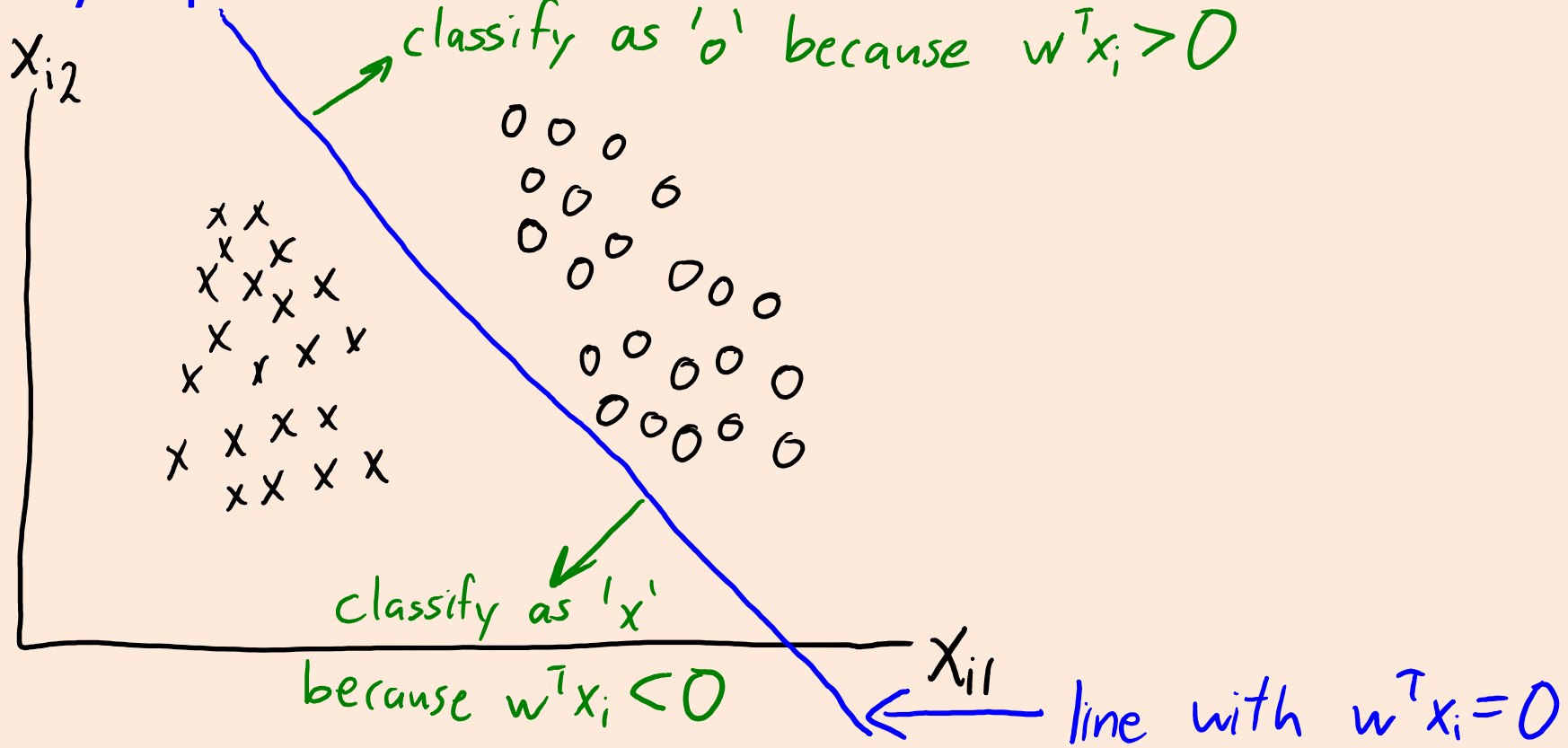
- Compared 179 classifiers on 121 datasets.
- Random forests are most likely to be the best classifier.
- Next best class of methods was SVMs (L2-regularization, RBFs).

- "Why should I care about logistic regression if I know about deep learning?"

Next Topic: Maximizing the Margin

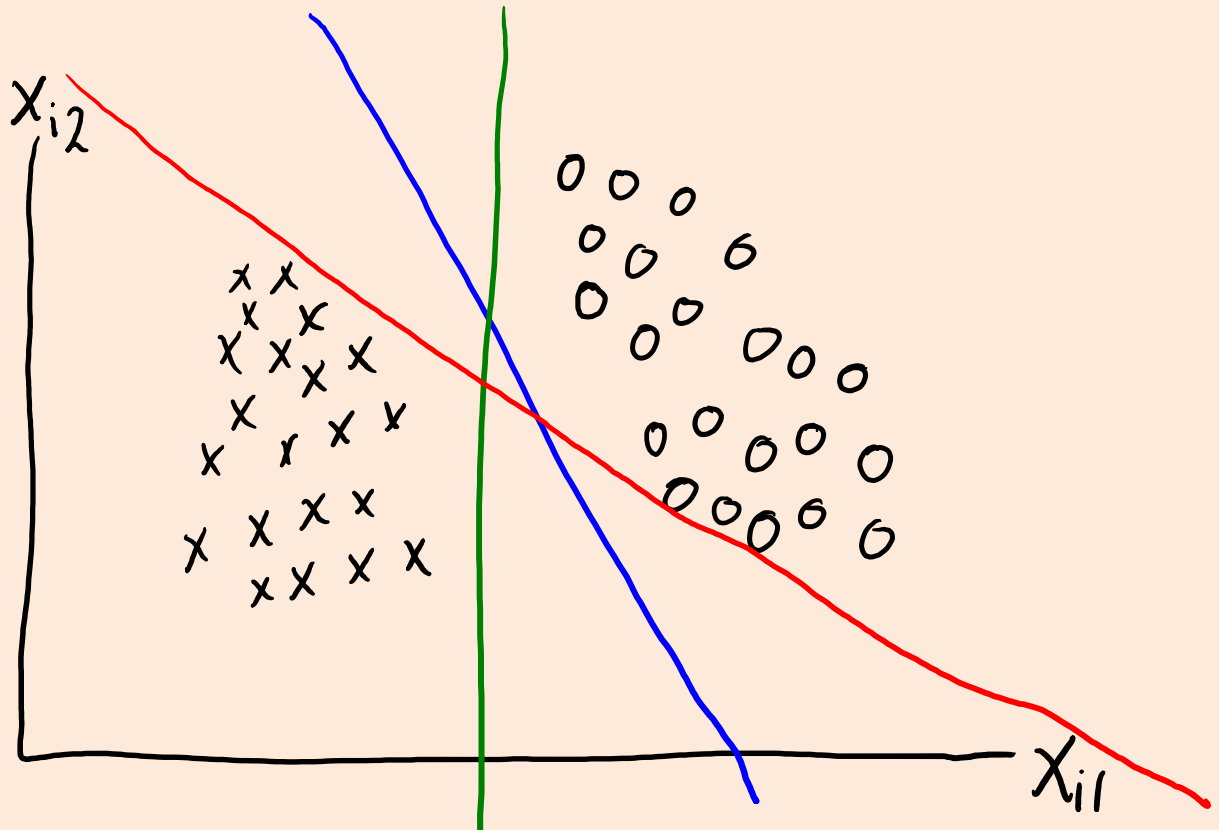
Maximum-Margin Perspective

- Consider a linearly-separable dataset.



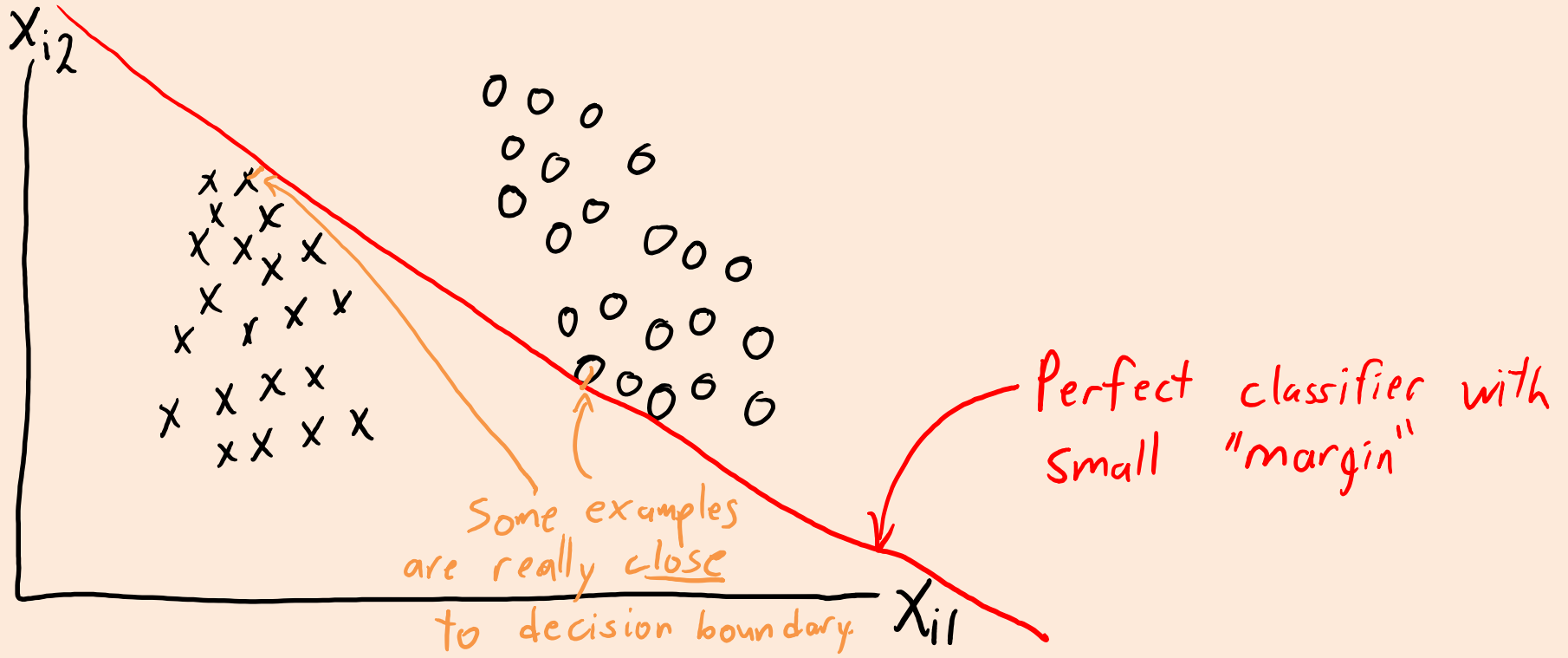
Maximum-Margin Perspective

- Consider a linearly-separable dataset.
  - Perceptron algorithm finds *some* classifier with zero error.
  - But are all zero-error classifiers equally good?



Maximum-Margin Perspective

- Consider a linearly-separable dataset.
  - Maximum-margin classifier: choose the farthest from both classes.



Maximum-Margin Perspective

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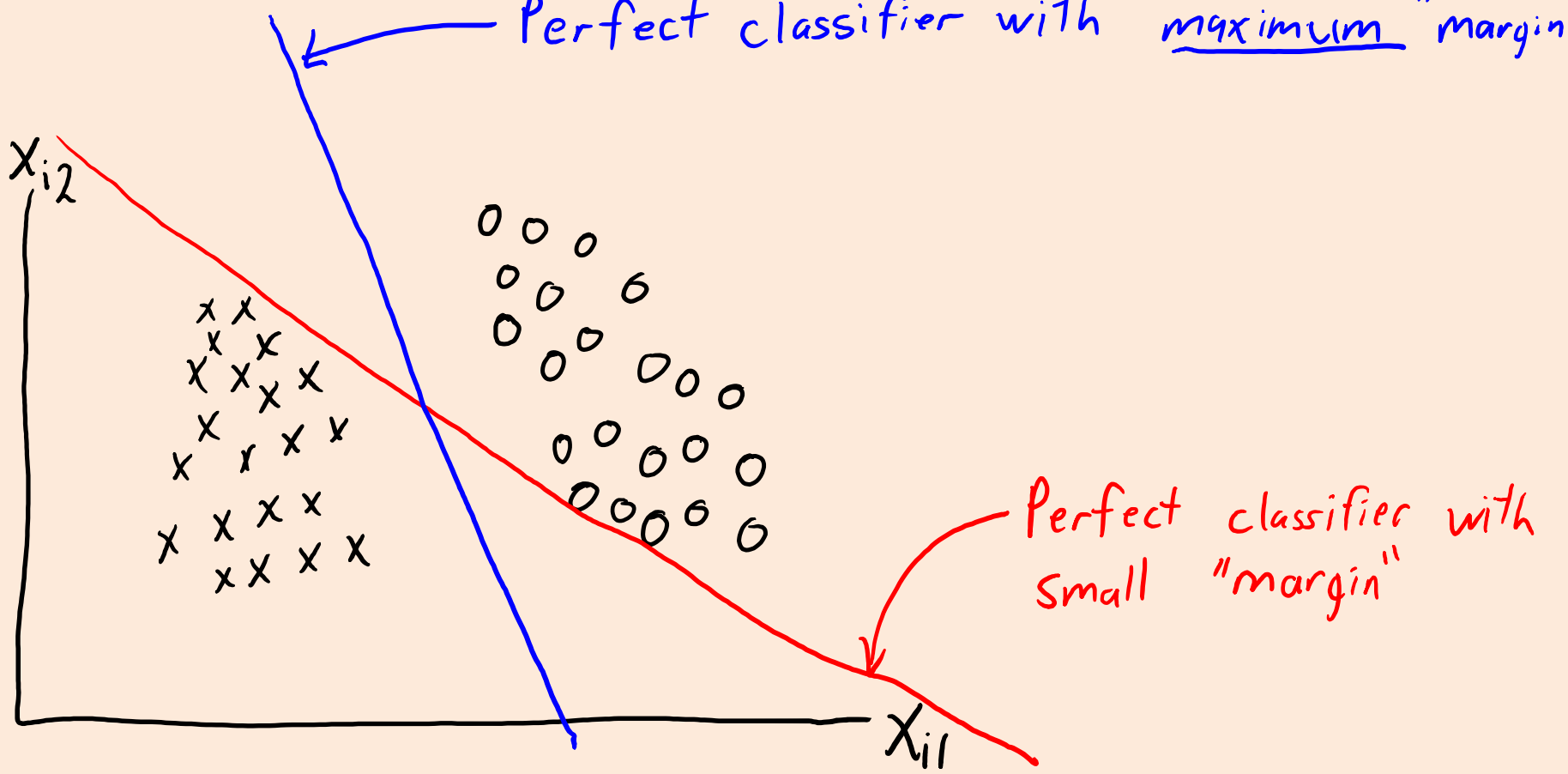
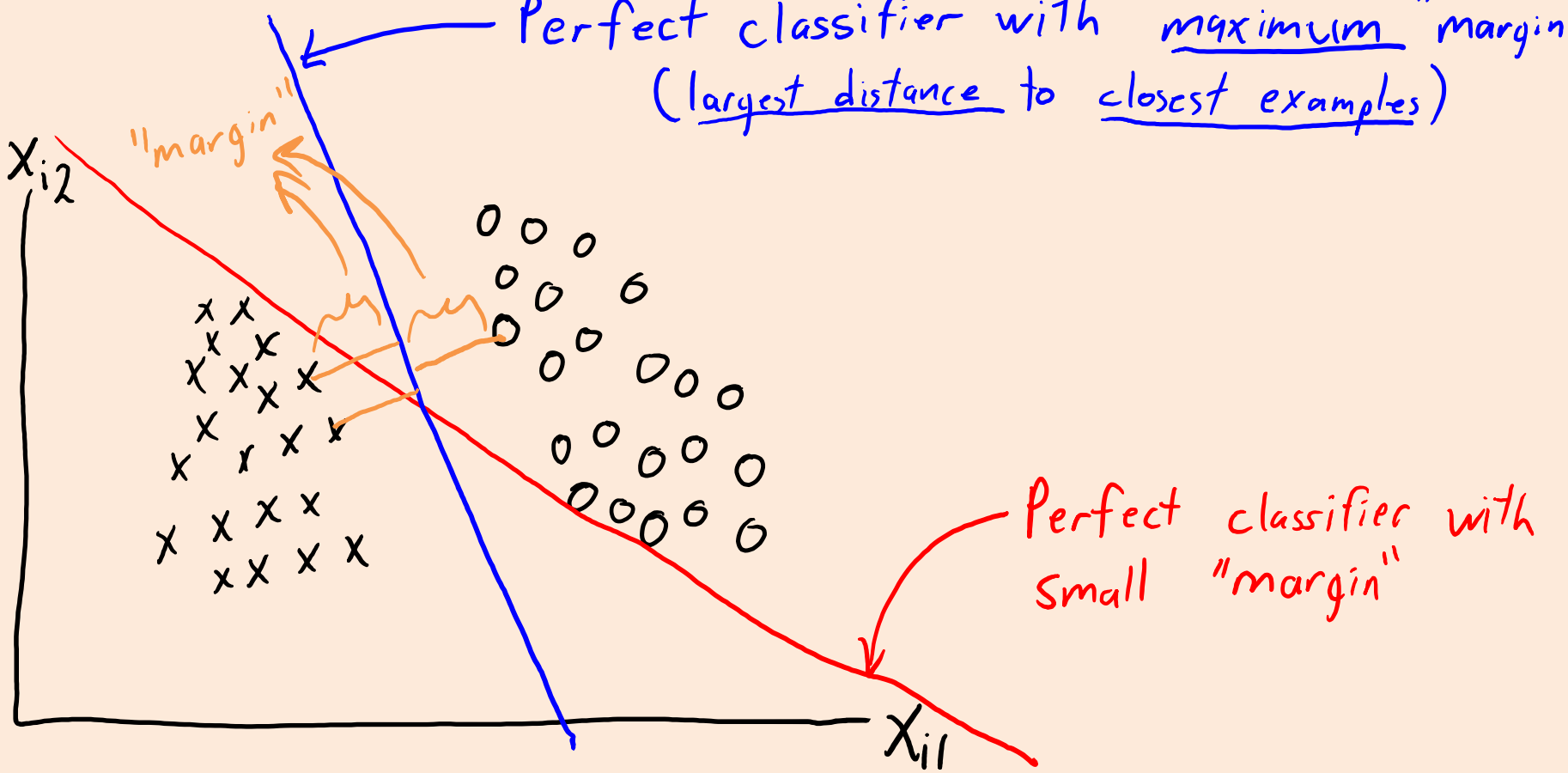


Maximum-Margin Perspective

- Consider a linearly-separable dataset.
  - Maximum-margin classifier: choose the farthest from both classes.

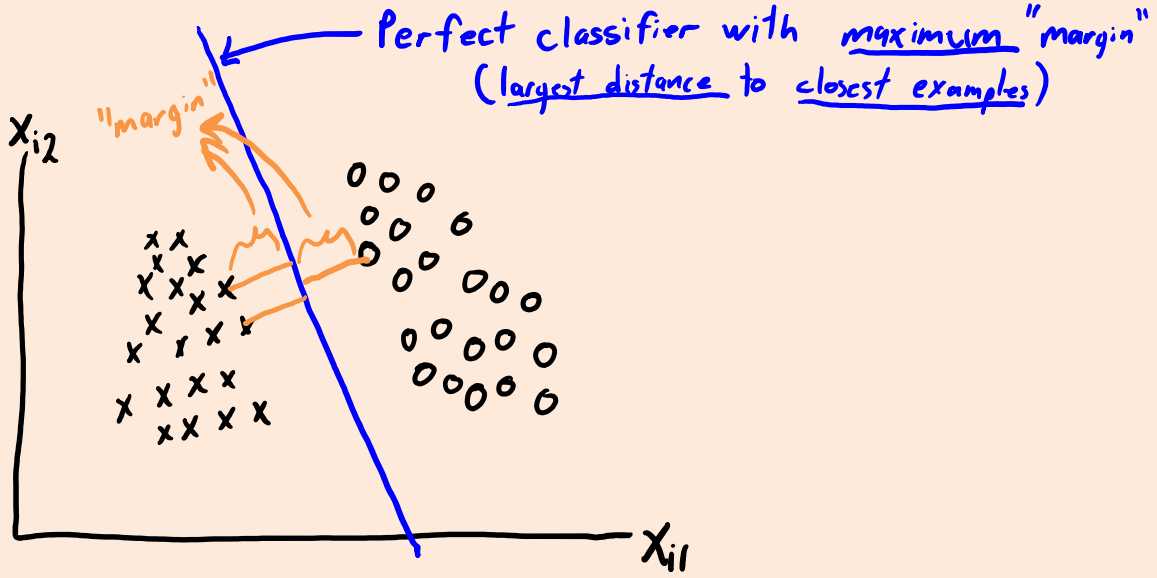
Why maximize margin?

If test data is close to training data, then max margin leaves more "room" before we make an error.



Maximum-Margin Perspective

- For linearly-separable data:



  - With small-enough $\lambda > 0$, SVMs find the maximum-margin classifier.
    - Need λ small enough that hinge loss is 0 in solution.
    - Origin of the name: the "support vectors" are the points closest to the line (see bonus).
  - Recent result: logistic regression also finds maximum-margin classifier.
    - With $\lambda=0$ and if you fit it with gradient descent (not true for many other optimizers).

Next Topic: Converting to Probabilities

Previously: Identifying Important E-mails

- Recall problem of identifying 'important' e-mails:

- We can do binary classification by taking sign of linear model:

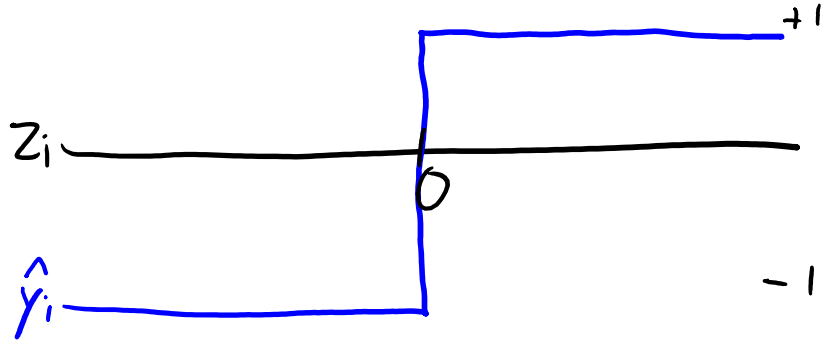
$$\hat{y}_i = \text{sign}(w^T x_i)$$

  - Convex loss functions (hinge/logistic loss) let us find an appropriate 'w'.

- But what if we want a probabilistic classifier?
  - Want a model of $p(y_i = \text{"important"} \mid x_i)$ for use in decision theory.

Predictions vs. Probabilities

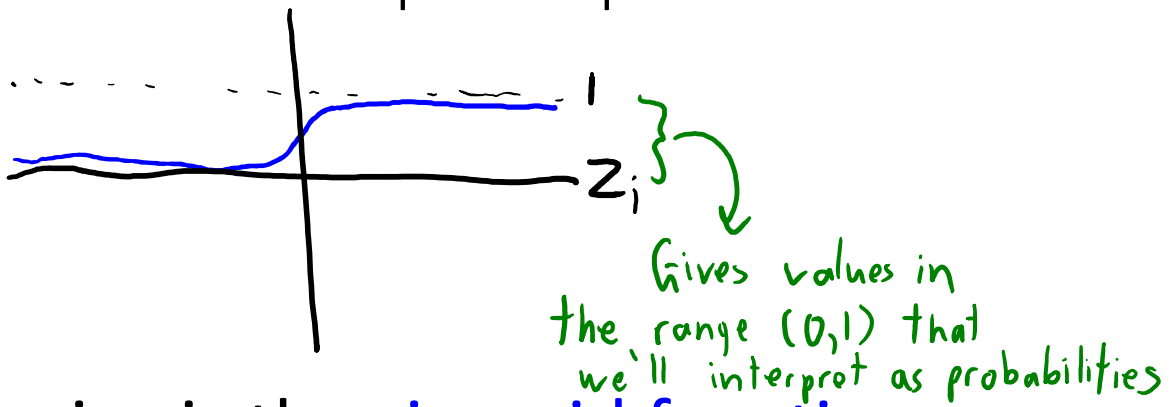
- With $z_i = w^T x_i$, linear classifiers make prediction using sign(z_i):



- For predictions, "sign" maps from $w^T x_i$ to the elements {-1,+1}.
  - If $w^T x_i$ is positive we predict +1, if it's negative we predict -1.
- For probabilities, we want to map from $w^T x_i$ to the range [0,1].
  - If $w^T x_i$ is very positive, we output a value close to +1.
  - If $w^T x_i$ is very negative, we output a value close to 0.
  - If $w^T x_i$ is close to 0, we output a value close to 0.5.

Sigmoid Function

- So we want a transformation of $z_i = w^T x_i$ that looks like this:



- The most common choice is the sigmoid function:

$$h(z_i) = \frac{1}{1 + \exp(-z_i)}$$

- Values of $h(z_i)$ match what we want:

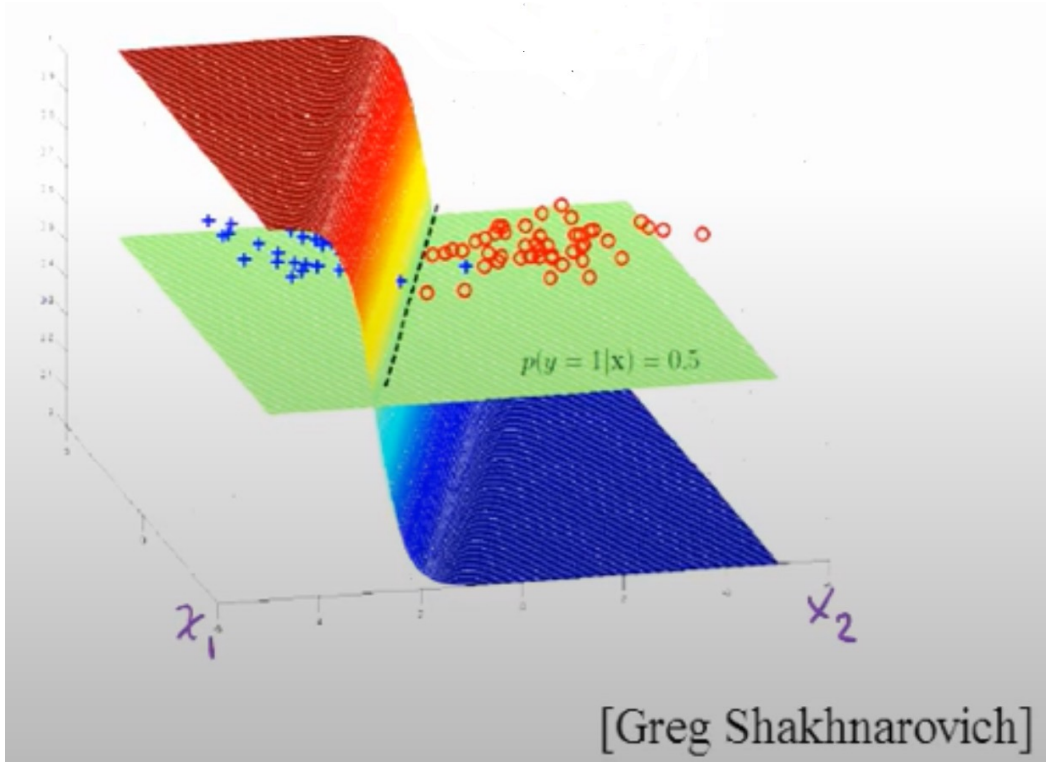
$h(-\infty) \simeq 0$ $h(-1) \simeq 0.27$ $h(0) = 0.5$ $h(0.5) \simeq 0.62$ $h(+1) \simeq 0.73$ $h(+\infty) = 1$

Probabilities for Linear Classifiers using Sigmoid

- Using sigmoid function, we output probabilities for linear models using:

$$p(y_i = 1 \mid w, x_i) = \frac{1}{1 + \exp(-w^T x_i)}$$

- Visualization for 2 features:



https://www.youtube.com/watch?v=Zc7ouSD0DEQ

Probabilities for Linear Classifiers using Sigmoid

- Using sigmoid function, we output probabilities for linear models using:

$$p(y_i = 1 \mid w, x_i) = \frac{1}{1 + \exp(-w^T x_i)}$$

- By rules of probability:

$$p(y_i = -1 \mid w, x_i) = 1 - p(y_i = 1 \mid w, x_i)$$

$$= \frac{1}{1 + \exp(w^T x_i)}$$ (with some effort)

- We then use these for "probability that e-mail is important".
- This may seem heuristic, but later we'll see that:
  - minimizing logistic loss does "maximum likelihood estimation" in this model.

Next Topic: Multi-Class Linear Classifiers

Multi-Class Linear Classification

- We've been considering linear models for binary classification:

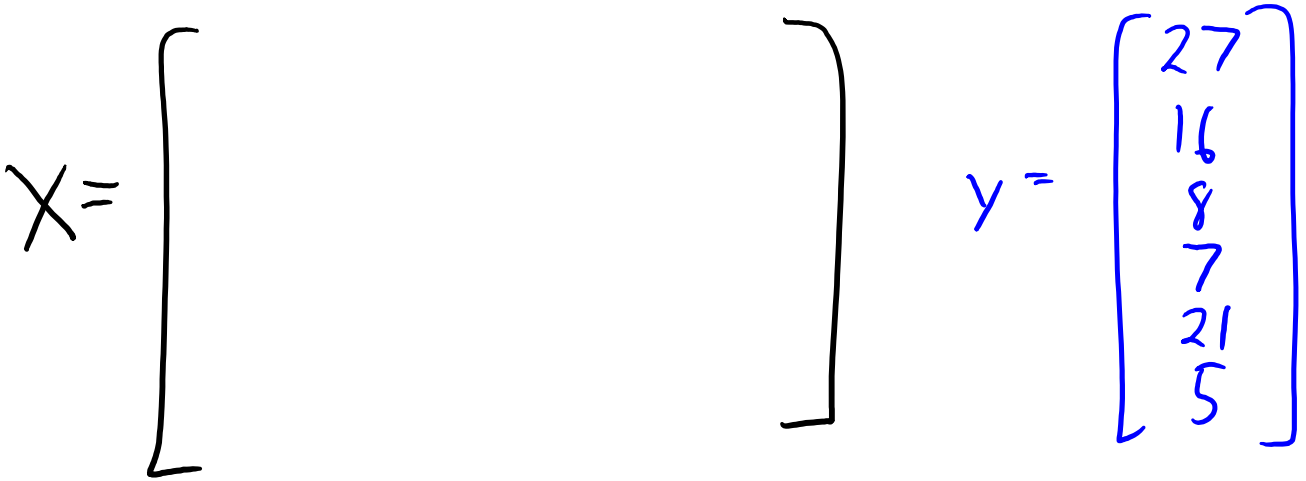
$$X = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \qquad y = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- E.g., is there a cat in this image or not?

https://www.youtube.com/watch?v=tntOCGkgt98

Multi-Class Linear Classification

- Now we'll discuss linear models for multi-class classification:



- For example, classify image as "cat", "dog", or "person".
  - This was natural for methods of Part 1 (decision trees, naïve Bayes, KNN).
  - For linear models, we need some new notation.

"One vs All" Classification

- Suppose you only know how to do binary classification:
  - "One vs all" is a way to turn a binary classifier into a multi-class method.

- Training phase:
  - For each class 'c', train binary classifier to predict whether example is a 'c'.
    - For example, train a "cat detector", a "dog detector", and a "human detector".
    - If we have 'k' classes, this gives 'k' binary classifiers .

- Prediction phase:

features x_i

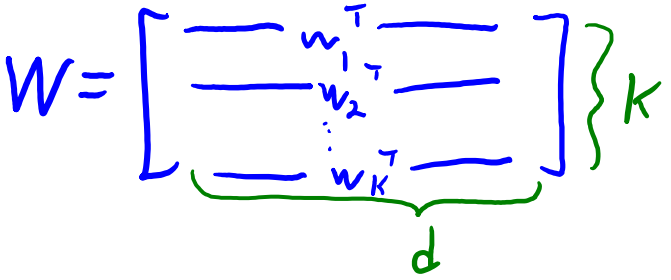
"cat detector" → 2 "cat score"
"dog detector" → 3 "dog score"
"human detector" → 10 "human score"

predict "human"

  - Apply the 'k' binary classifiers to get a "score" for each class 'c'.
  - Predict the 'c' with the highest score.

"One vs All" Linear Classification

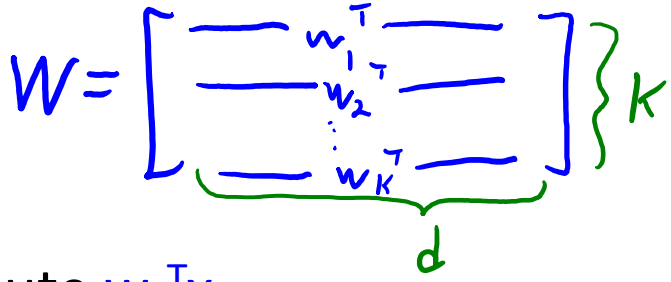
- "One vs all" logistic regression for classifying as cat/dog/person.
  - Train a separate classifier for each class.
    - Classifier 1 tries to predict +1 for "cat" images and -1 for "dog" and "person" images.
    - Classifier 2 tries to predict +1 for "dog" images and -1 for "cat" and "person" images.
    - Classifier 3 tries to predict +1 for "person" images and -1 for "cat" and "dog" images.
  - This gives us a weight vector w_c for each class 'c':
    - Weights w_c try to predict +1 for class 'c' and -1 for all others.
    - We'll use 'W' as a matrix with the w_c as rows:



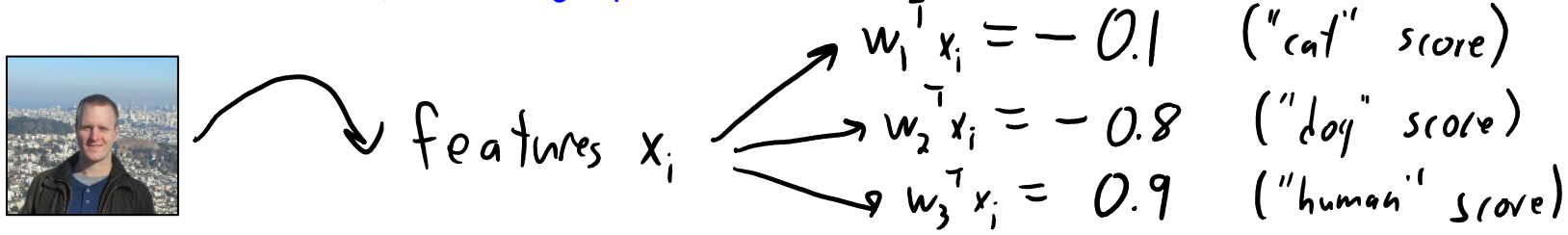
Each row 'c' gives weights w_c for a binary logistic regression model to predict class 'c'.

"One vs All" Linear Classification

- "One vs all" logistic regression for classifying as cat/dog/person.
  - Prediction on example x_i given parameters 'W' :



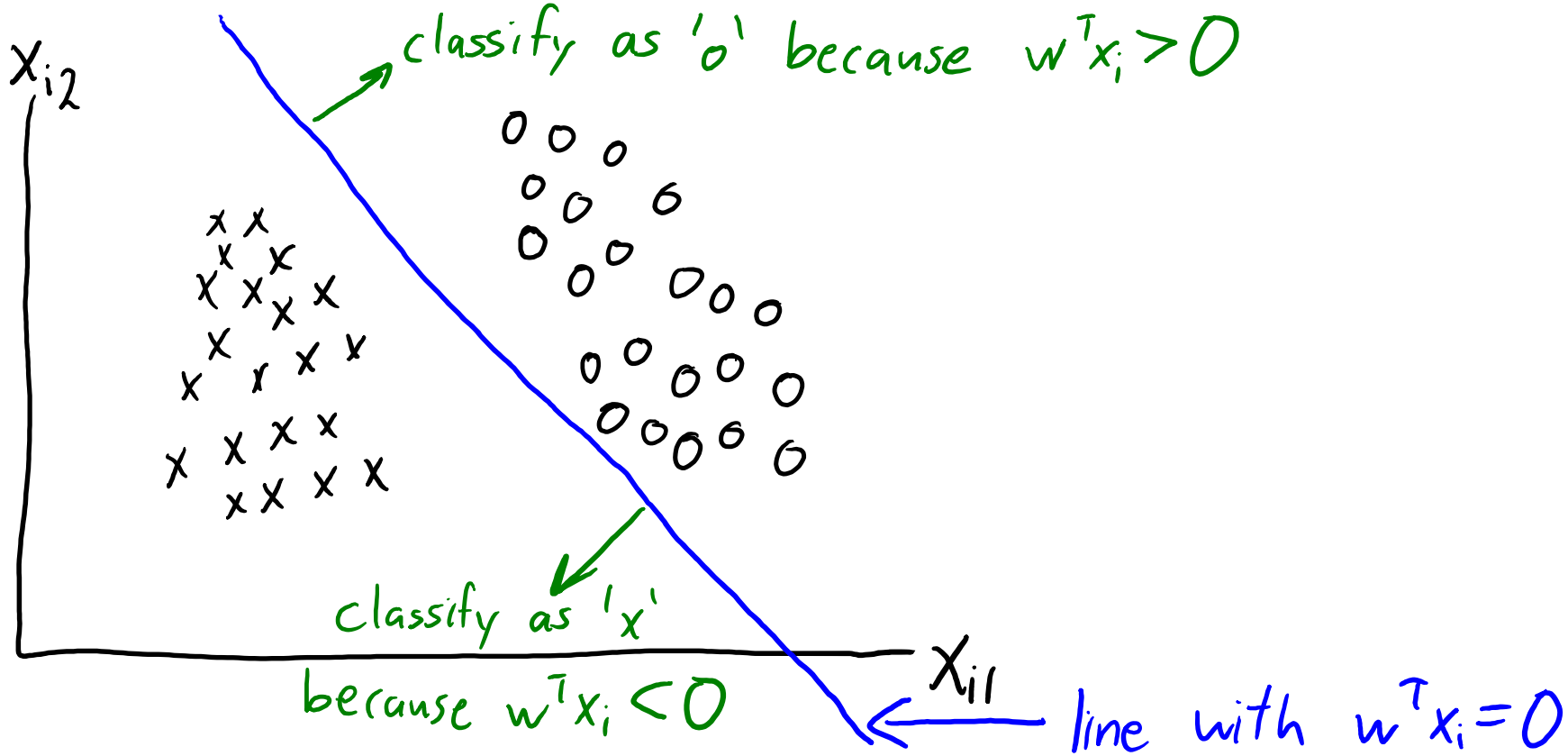
  - For each class 'c', compute $w_c^T x_i$.



    - Ideally, we'll get $\text{sign}(w_c^T x_i) = +1$ for one class and $\text{sign}(w_c^T x_i) = -1$ for all others.
    - In practice, it might be +1 for multiple classes or no class.
  - To predict class, we take maximum value of $w_c^T x_i$ ("highest score").
    - In the example above, predict "human" (0.9 is higher than -0.8 and -0.1).

Shape of Decision Boundaries

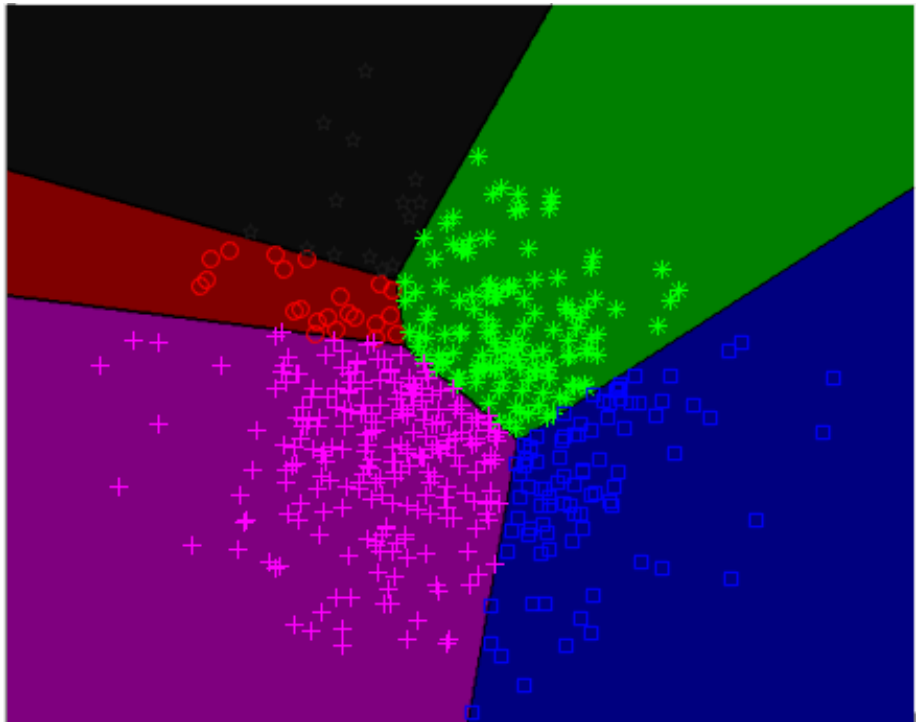
- Recall that a binary linear classifier splits space using a hyper-plane:



- Divides x_i space into 2 "half-spaces".

Shape of Decision Boundaries

- Multi-class linear classifier is intersection of these "half-spaces":
  - This divides the space into convex regions (like k-means):



"Blue" region is region where we have:

$$w_{blue}^T x_i \geq w_{green}^T x_i$$

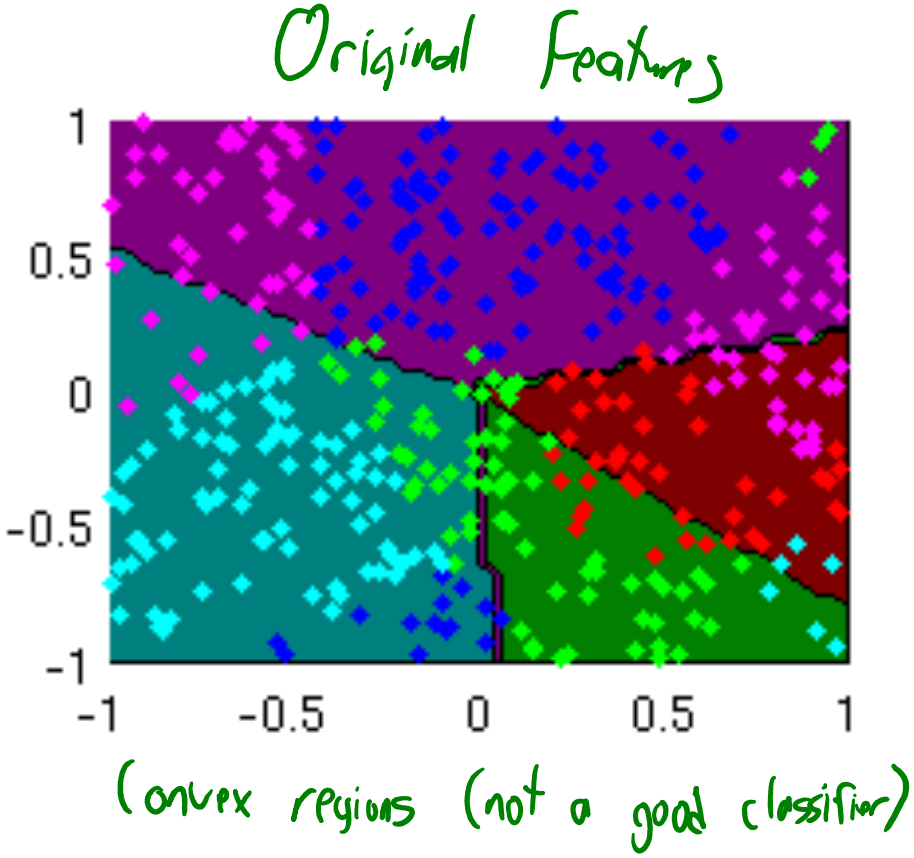
$$w_{blue}^T x_i \geq w_{magenta}^T x_i$$

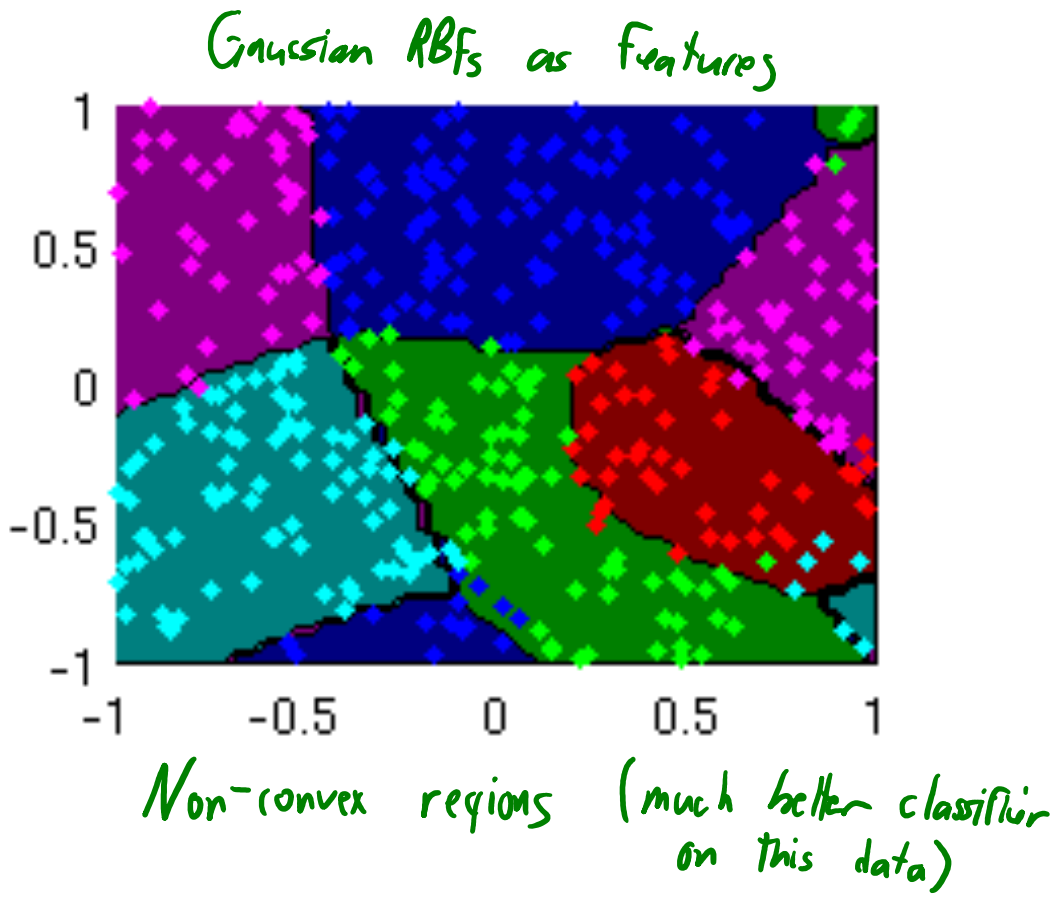
$$w_{blue}^T x_i \geq w_{red}^T x_i$$

$$w_{blue}^T x_i \geq w_{black}^T x_i$$

Shape of Decision Boundaries

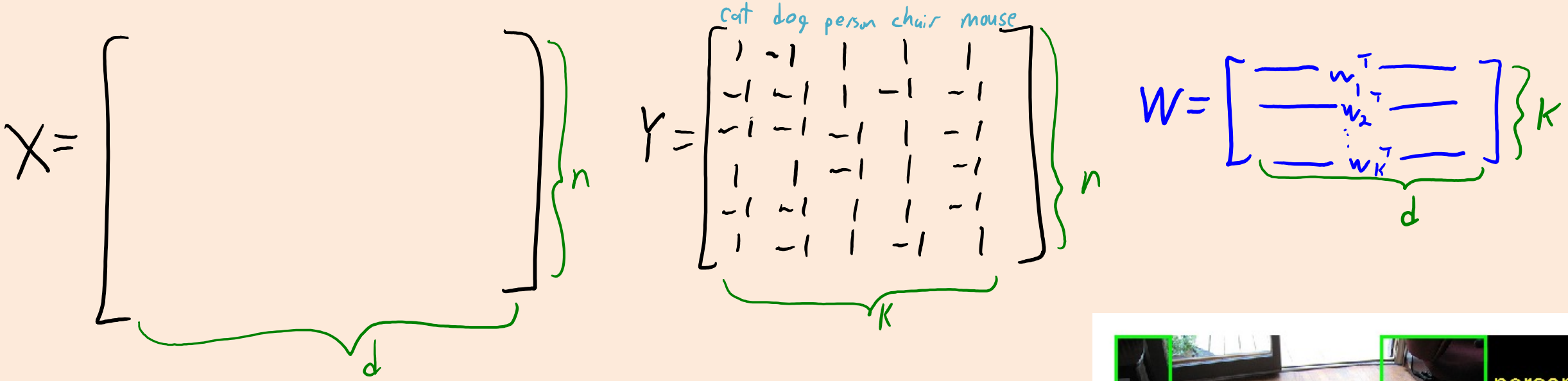
- Multi-class linear classifier is intersection of these "half-spaces":
  - Though regions could be non-convex with non-linear feature transforms:



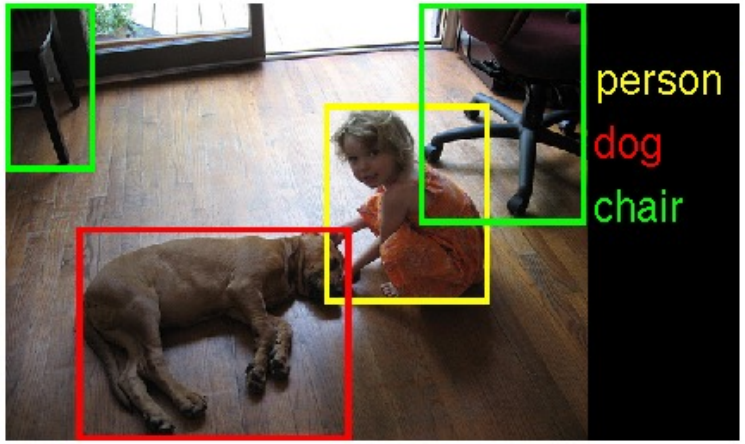


Digression: Multi-Label Classification

- A related problem is multi-label classification:



- Which of the 'k' objects are in this image?
  - There may be more than one "correct" class label.
  - Here we can also fit 'k' binary classifiers.
    - But we would take all the sign($w_c^T x_i$)=+1 as the labels.



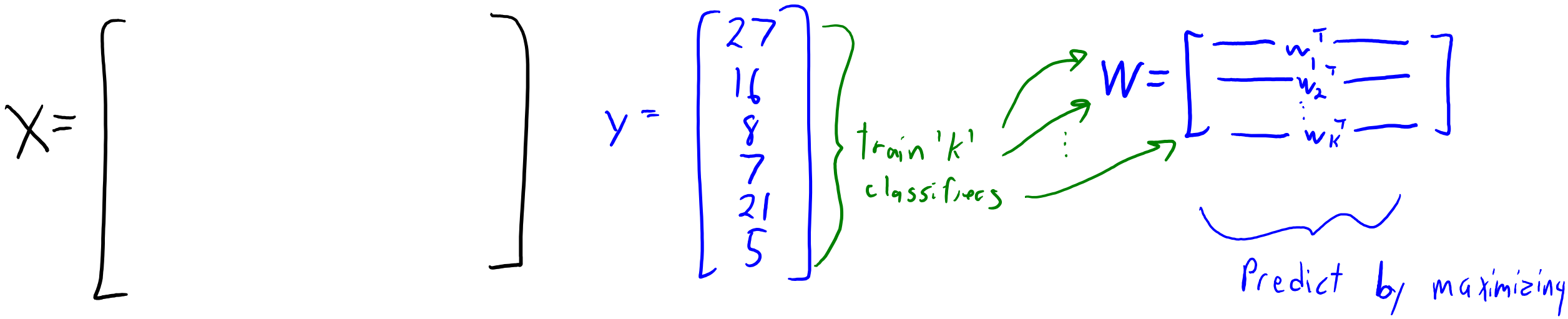
http://image-net.org/challenges/LSVRC/2013/

Summary

- Naïve convex approximation to 0-1 loss has a degenerate solution.
  - Two fixes: hinge loss and logistic loss.
- Logistic loss uses a smooth convex approximation to the 0-1 loss.
- SVMs and logistic regression are very widely-used.
  - A lot of ML consulting: "find good features, use L2-regularized logistic/SVM".
  - Under certain conditions, can be viewed as "maximizing the margin".
  - Both are just linear classifiers (a hyperplane dividing into two halfspaces).
- Sigmoid function is a way to turn linear predictions into probabilities.
- One vs all turns a binary classifier into a multi-class classifier.

Multi-Class Linear Classification (MEMORIZE)

- Back to multi-class classification where we have 1 "correct" label:



- We'll use 'w_{y_i}' as classifier where $c=y_i$ (row of correct class label).
  - So if $y_i=2$ then $w_{y_i} = w_2$.

"One vs All" Multi-Class Linear Classification

- Problem: We didn't train the w_c so that the largest $w_c^T x_i$ would be $w_{y_i}^T x_i$.
  - Each classifier is just trying to get the sign right.

features x_i

$w_1^T x_i = -5$ ("cat" score)

$w_2^T x_i = -0.1$ ("dog" score)

$w_3^T x_i = -0.2$ ("human" score)

  - Here the classifier incorrectly predicts "dog".
    - "One vs All" doesn't try to put $w_2^T x_i$ and $w_3^T x_i$ on same scale for decisions like this.
    - We should try to make $w_3^T x_i$ positive and $w_2^T x_i$ negative relative to each other.
    - The multi-class hinge losses and the multi-class logistic loss do this.

https://laughingsquid.com/pug-mask-a-latex-mask-so-you-can-look-like-a-dog/

Multi-Class SVMs

- Can we define a loss that encourages largest $w_c^T x_i$ to be $w_{y_i}^T x_i$?
  - So when we maximizing over $w_c^T x_i$, we choose correct label y_i.

- Recall our derivation of the hinge loss (SVMs):
  - We wanted $y_i w^T x_i > 0$ for all 'i' to classify correctly.
  - We avoided non-degeneracy by aiming for $y_i w^T x_i \geq 1$.
  - We used the constraint violation as our loss: $\max\{0, 1 - y_i w^T x_i\}$.

- We can derive multi-class SVMs using the same steps…

Multi-Class SVMs

- Can we define a loss that encourages largest $w_c^T x_i$ to be $w_{y_i}^T x_i$?

We want $w_{y_i}^T x_i > w_c^T x_i$ for all 'c' that are not correct label y_i

If we penalize violation of this constraint it's degenerate.

We use $w_{y_i}^T x_i \geq w_c^T x_i + 1$ for all $c \neq y_i$ to avoid strict inequality

Equivalently: $0 \geq 1 - w_{y_i}^T x_i + w_c^T x_i$

- For here, there are two ways to measure constraint violation:

"Sum"

$$\sum_{c \neq y_i} \max\{0, 1 - w_{y_i}^T x_i + w_c^T x_i\}$$

"Max"

$$\max_{c \neq y_i} \left\{ \max\{0, 1 - w_{y_i}^T x_i + w_c^T x_i\} \right\}$$

Multi-Class SVMs

- Can we define a loss that encourages largest $w_c^T x_i$ to be $w_{y_i}^T x_i$?

"Sum"

$$\sum_{c \neq y_i} \max\{0, 1 - w_{y_i}^T x_i + w_c^T x_i\}$$

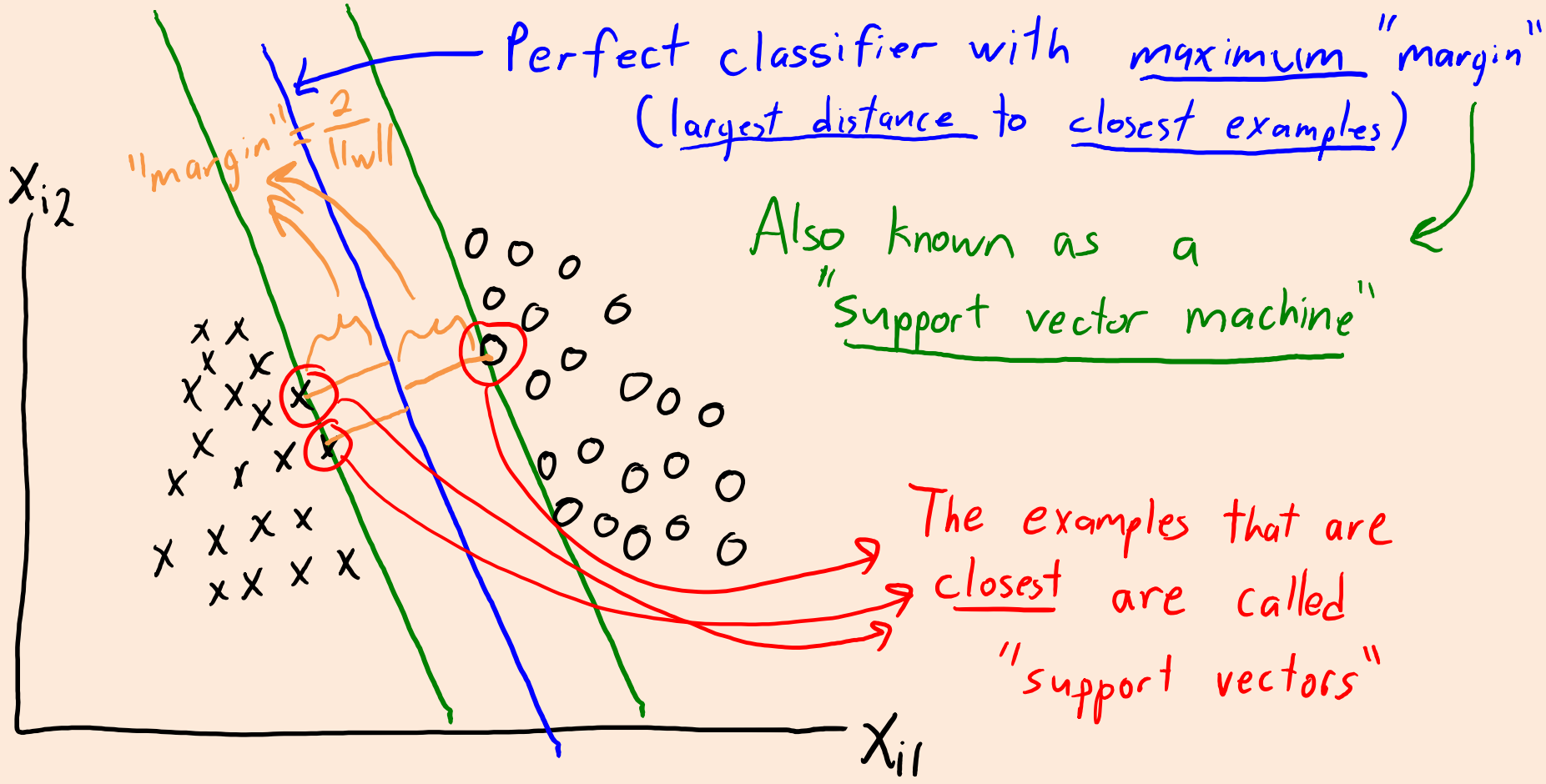
"Max"

$$\max_{c \neq y_i} \left\{ \max\{0, 1 - w_{y_i}^T x_i + w_c^T x_i\} \right\}$$

- For each training example 'i':
  - "Sum" rule penalizes for each 'c' that violates the constraint.
  - "Max" rule penalizes for one 'c' that violates the constraint the most.
    - "Sum" gives a penalty of 'k-1' for W=0, "max" gives a penalty of '1'.
- If we add L2-regularization, both are called multi-class SVMs:
  - "Max" rule is more popular, "sum" rule usually works better.
  - Both are convex upper bounds on the 0-1 loss.

Maximum-Margin Classifier

- Consider a linearly-separable dataset.
  - Maximum-margin classifier: choose the farthest from both classes.



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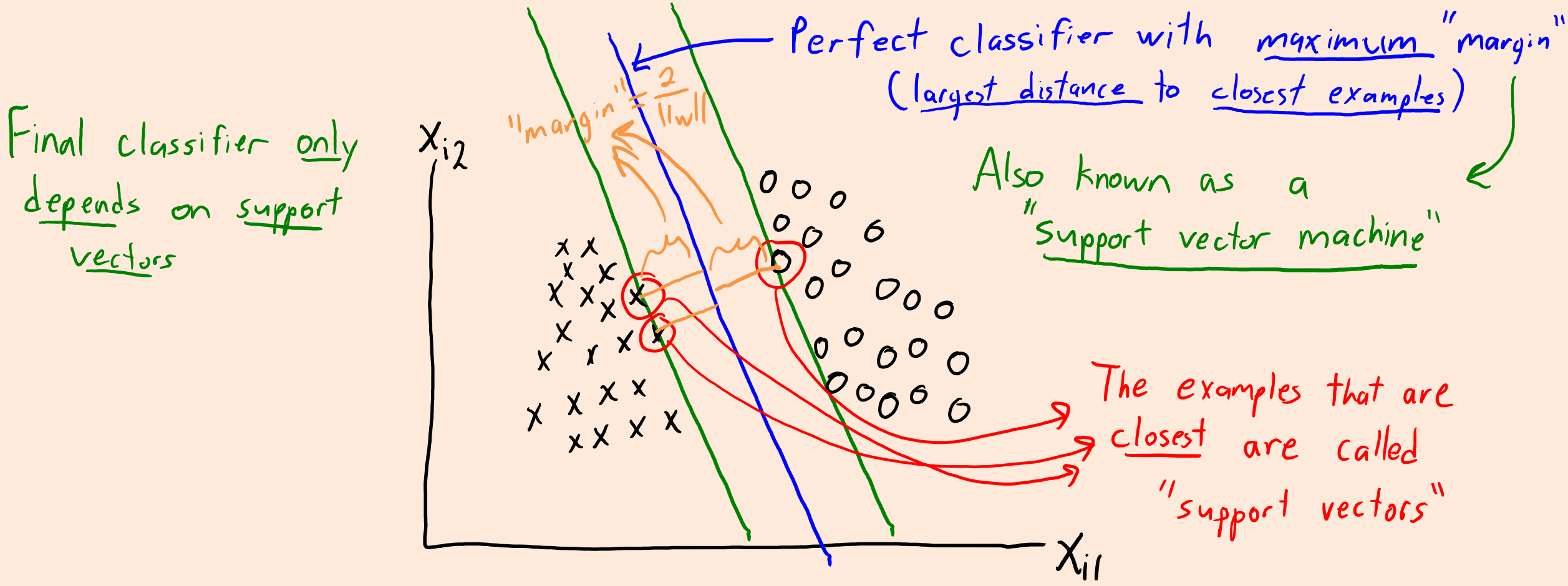

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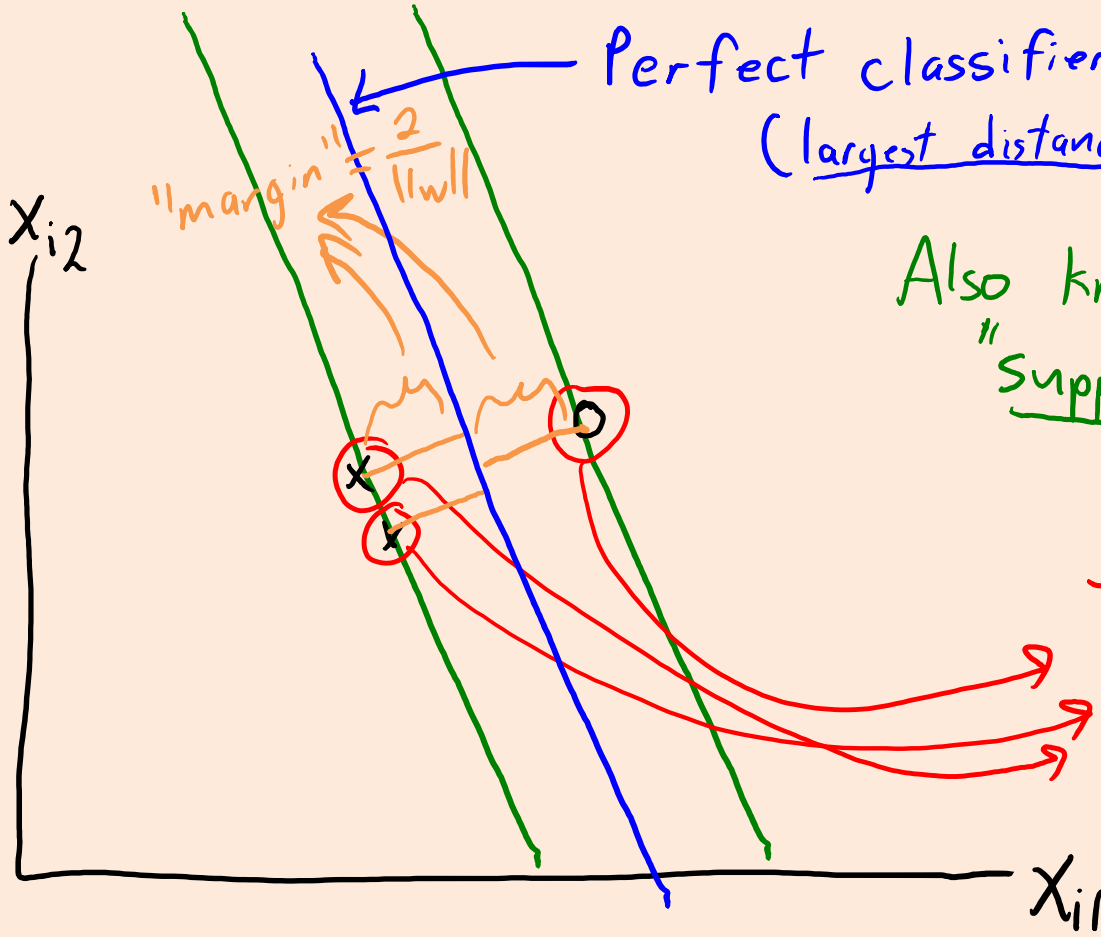


Perfect classifier with maximum "margin"
(largest distance to closest examples)

Also known as a "support vector machine"

Final classifier only depends on support vectors

You could throw away the other examples and get the same classifier.

The examples that are closest are called "support vectors"

Support Vector Machines

- For linearly-separable data, SVM minimizes:

$$f(w) = \frac{1}{2}\|w\|^2$$ (equivalent to maximizing margin $\frac{2}{\|w\|}$)

  - Subject to the constraints that: (see Wikipedia/textbooks)

$$w^T x_i \geq 1 \quad \text{for } y_i = 1$$
$$w^T x_i \leq -1 \quad \text{for } y_i = -1$$

(classify all examples correctly)

- But most data is not linearly separable.
- For non-separable data, try to minimize violation of constraints:

If $w^T x_i \leq -1$ and $y_i = -1$ then "violation" should be zero.

If $w^T x_i \geq -1$ and $y_i = -1$ then we "violate constraint" by $1 + w^T x_i$

⋮

→ Constraint violation is the hinge loss.

Support Vector Machines

- Try to maximizing margin and also minimizing constraint violation:

$$f(w) = \sum_{i=1}^{n} \max\{0, 1 - y_i w^T x_i\} + \frac{1}{2}\|w\|^2$$

Hinge loss for example 'i':

if's the amount we violate "slack" $y_i w^T x_i \geq 1$

Original SVM objective: encourages large margin.

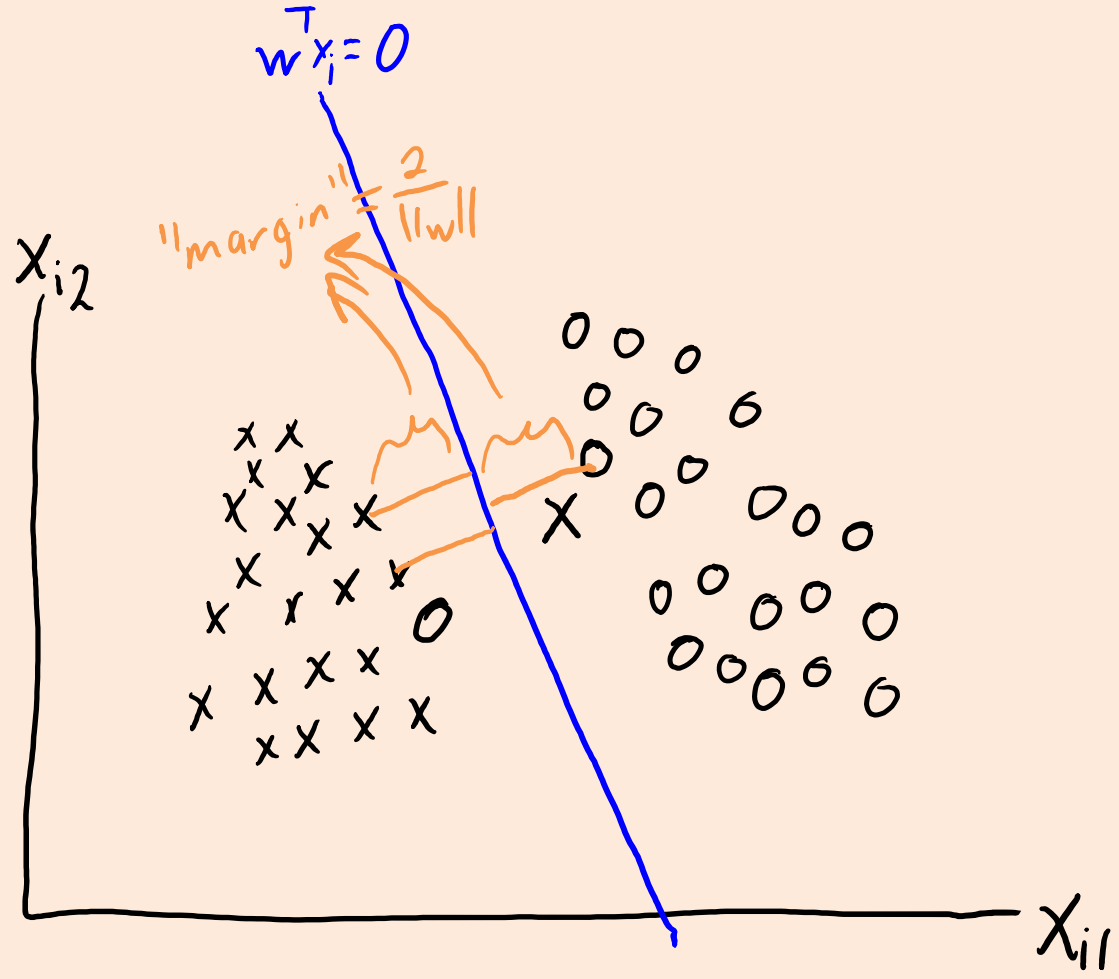
- We typically control margin/violation trade-off with parameter "λ":

$$f(w) = \sum_{i=1}^{n} \max\{0, 1 - y_i w^T x_i\} + \frac{\lambda}{2}\|w\|^2$$

- This is the standard SVM formulation (L2-regularized hinge).
  - Some formulations use $\lambda = 1$ and multiply hinge by 'C' (equivalent).

Support Vector Machines for Non-Separable

- Non-separable case:



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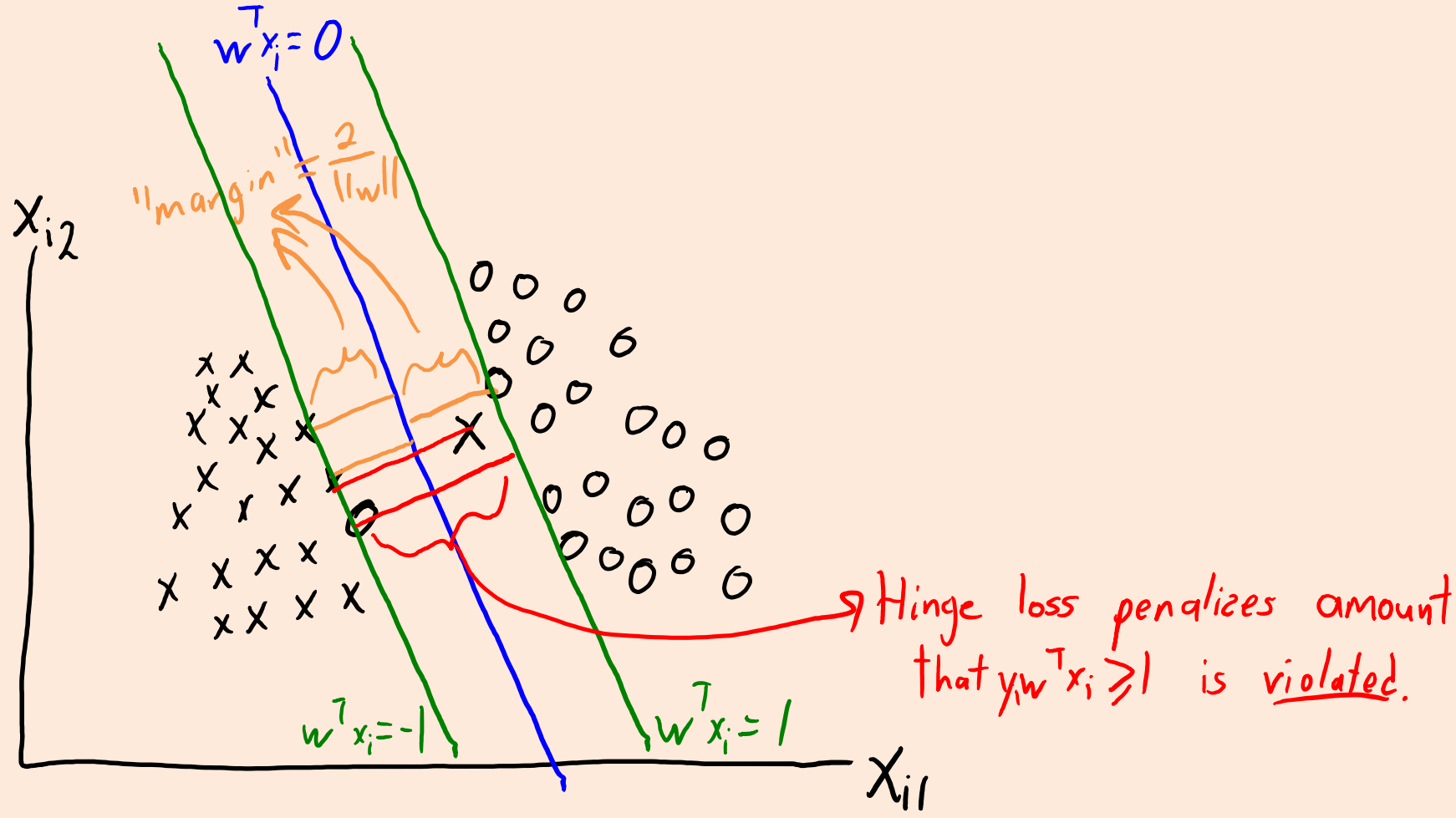

Support Vector Machines for Non-Separable

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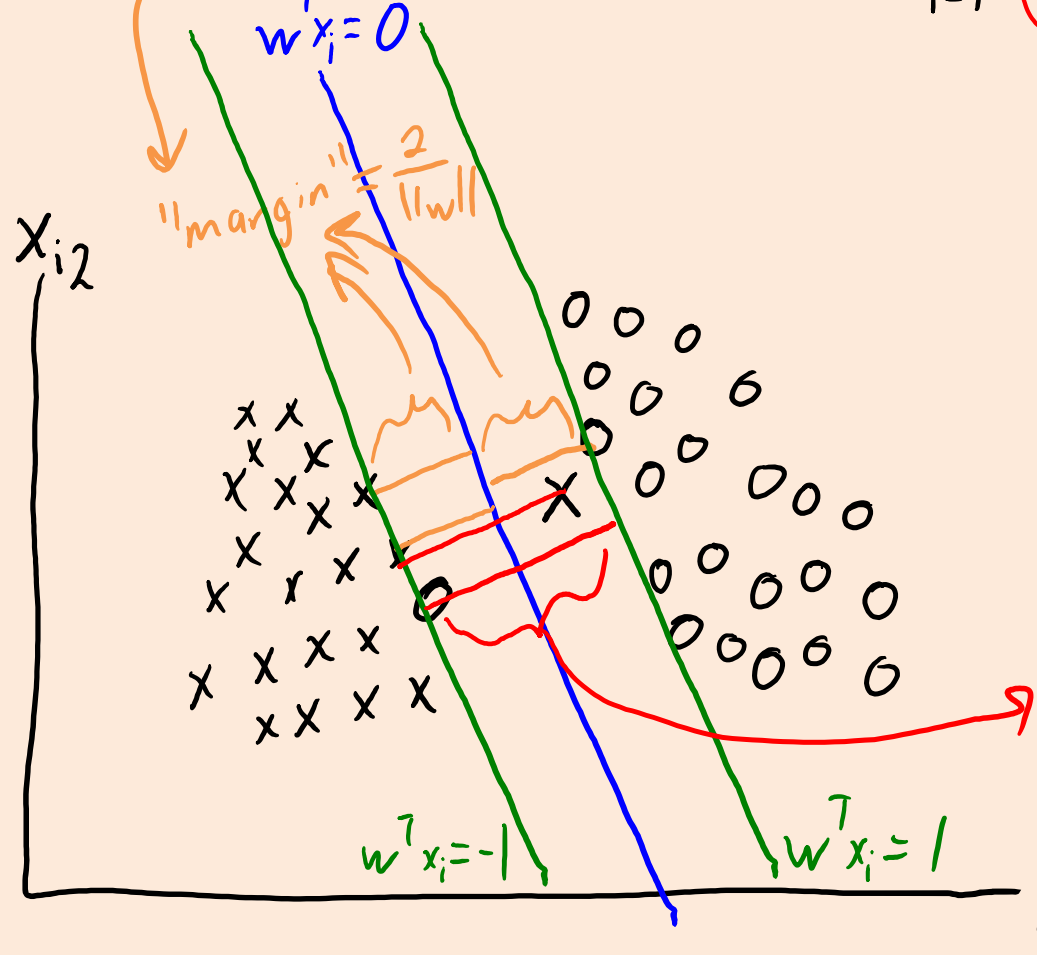

Logistic regression can be viewed as smooth approximation to SVMs.

But, no concept of "support vectors" with logistic loss.

λ controls trade-off between having large margin and classifying examples correctly.

Hinge loss penalizes amount that $y_i w^T x_i \geq 1$ is violated.

Support Vector Machines for Non-Separable

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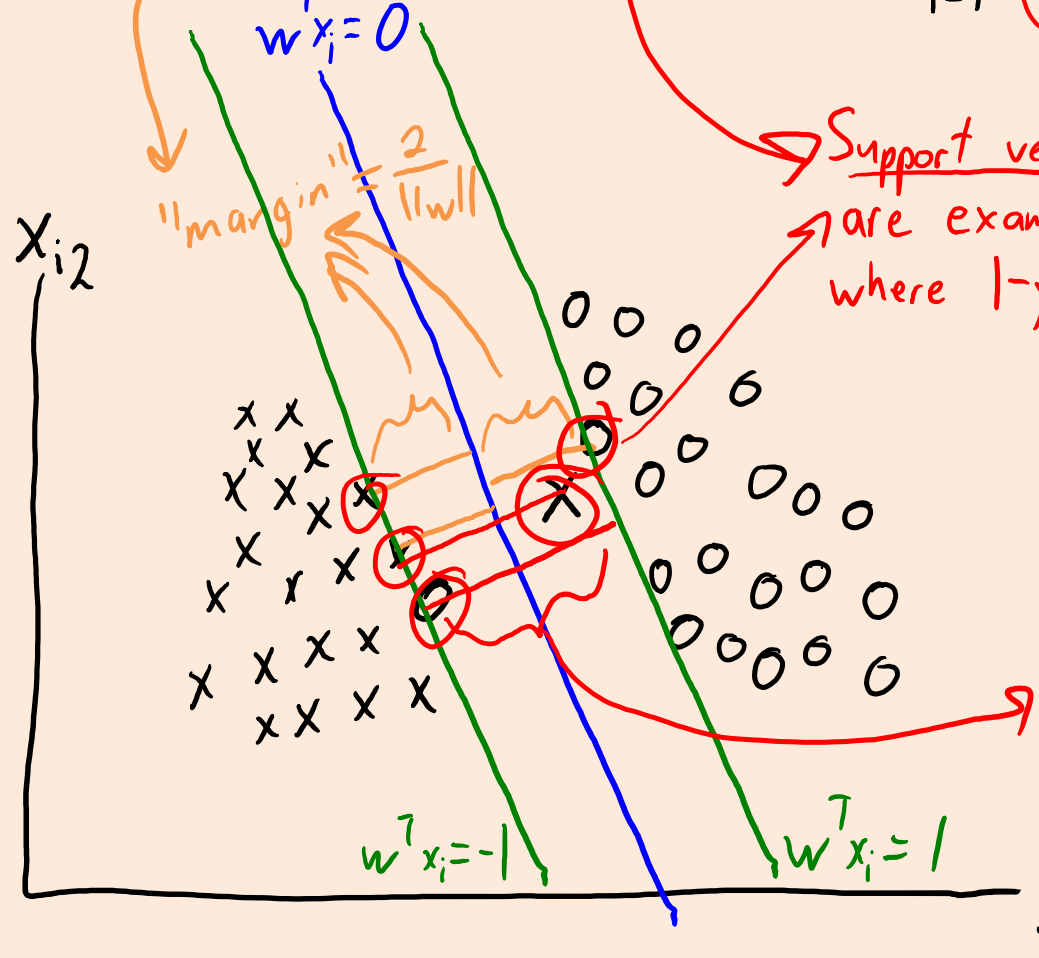


Support vectors are examples 'i' where $1 - y_i w^T x_i \geq 0$

λ controls trade-off between having large margin and classifying examples correctly.

Hinge loss penalizes amount that $y_i w^T x_i \geq 1$ is violated.

Logistic regression can be viewed as smooth approximation to SVMs.

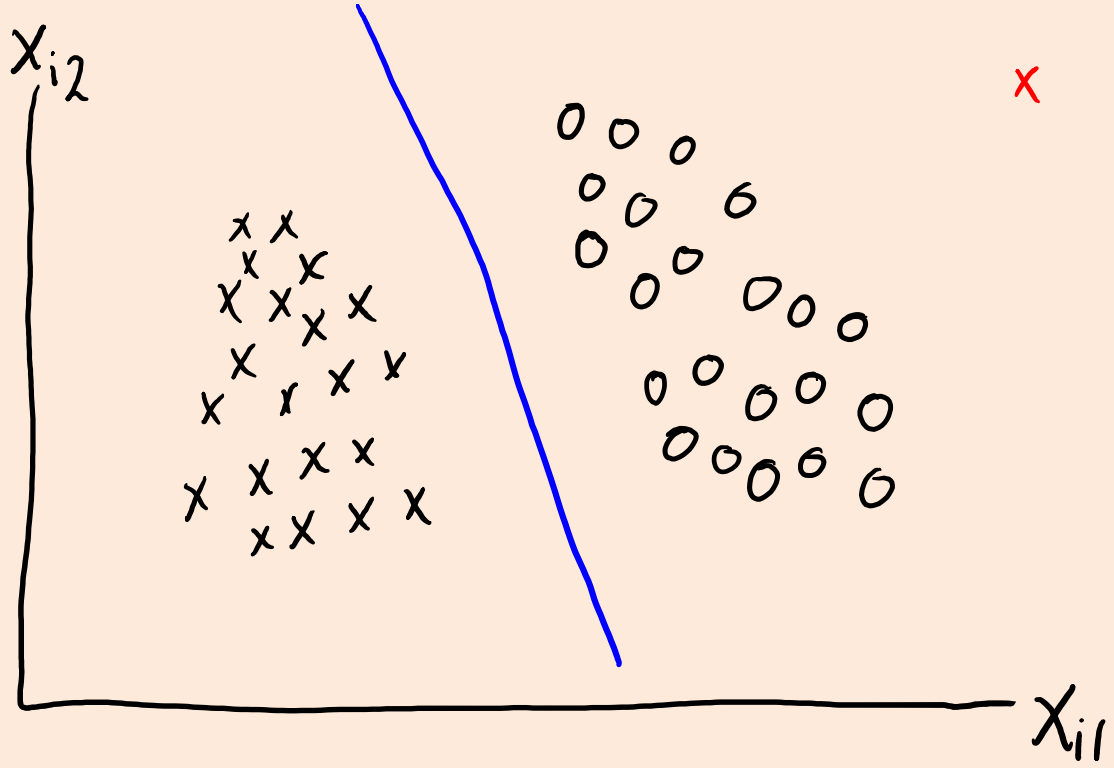
But, no concept of "support vectors" with logistic loss.

Discussion of Various Linear Classifiers

- Perceptron vs. logistic vs. SVM:
  - These linear classifiers are all extremely similar. They are basically just variations on reasonable methods to learn a classifier that uses the rule $$\hat{y}_i = \text{sign}(w^Tx_i)$$. (The online vs. offline issue is a red herring, you can train logistic/SVMs online using stochastic gradient and you can write a linear program that will give you a minimizer of the perceptron objective).
  - If you want to explore the small differences, these are some of the usual arguments:
    - The perceptron has largely been replaced by logistic/SVM, except in certain subfields like theory (it is easy to prove things about perceptrons) and natural language processing (mostly historical reasons). Perceptrons have the potential disadvantages of non-regularized models (non-uniqueness and potential non-existence of the solution, potential high sensitivity to small changes in the data, and non-robustness to irrelevant features). However, perceptrons do not interact well with regularization: if you add L2-regularization and the dataset is linearly-separable, then the solution only exists as a limit and it is actually $$w=0$$ (although it may still work in practice).
    - A usual criticism of logistic regression by people that favour SVMs is that, if the data is linearly separable, then the solution only exists as a limit as some elements $$w$$ go to plus or minus $$\infty$$. However, this argument disappears if you add regularization. A second argument traditionally made by SVM people is that you can't kernelize logistic regression, but this is now known to be incorrect (we'll cover a general kernelization strategy for L2-regularized linear classifiers in one of the next two classes).
    - The remaining differences between logistic and SVMs is that logistic regression is smooth while SVMs have support vectors. This means that the logistic regression training problem is easier from an optimization perspective (we'll get to this next class). But if you have very few support vectors, you can only take advantage of this with SVMs (or perceptrons), and this is especially important if you are using kernels.
- Regarding other linear predictors for binary classification, there are a few more:
  - Probit regression uses the Gaussian CDF in place of the logistic sigmoid function. This has very similar properties to logistic regression, but it's harder to generalize to the multi-class case (while probit regression is better if you are using a “Bayesian” estimator). You could actually use any CDF as your sigmoid function, and if there is some asymmetry between the classes using an extreme value distribution is sometimes advocated in statistics.
  - In neural networks, they sometimes use tanh in place of the logistic sigmoid function, and the reason to do this is to get values into the interval [-1,1] instead of [0,1].
  - If you want to keep support vectors but get a smooth optimization problem, you can square the hinge loss (making it once but not twice differentiable), and this is called smooth SVMs. Alternately, you could replace the non-differentiable kink with a small smooth part, and this is called Huberized SVMs.
  - Finally, some people actually just apply least squares to classification problems. If you use a flexible enough basis/kernel, then the 'bad' errors may not actually be that harmful.

Robustness and Convex Approximations

- Because the hinge/logistic grow like absolute value for mistakes, they tend not to be affected by a small number of outliers.



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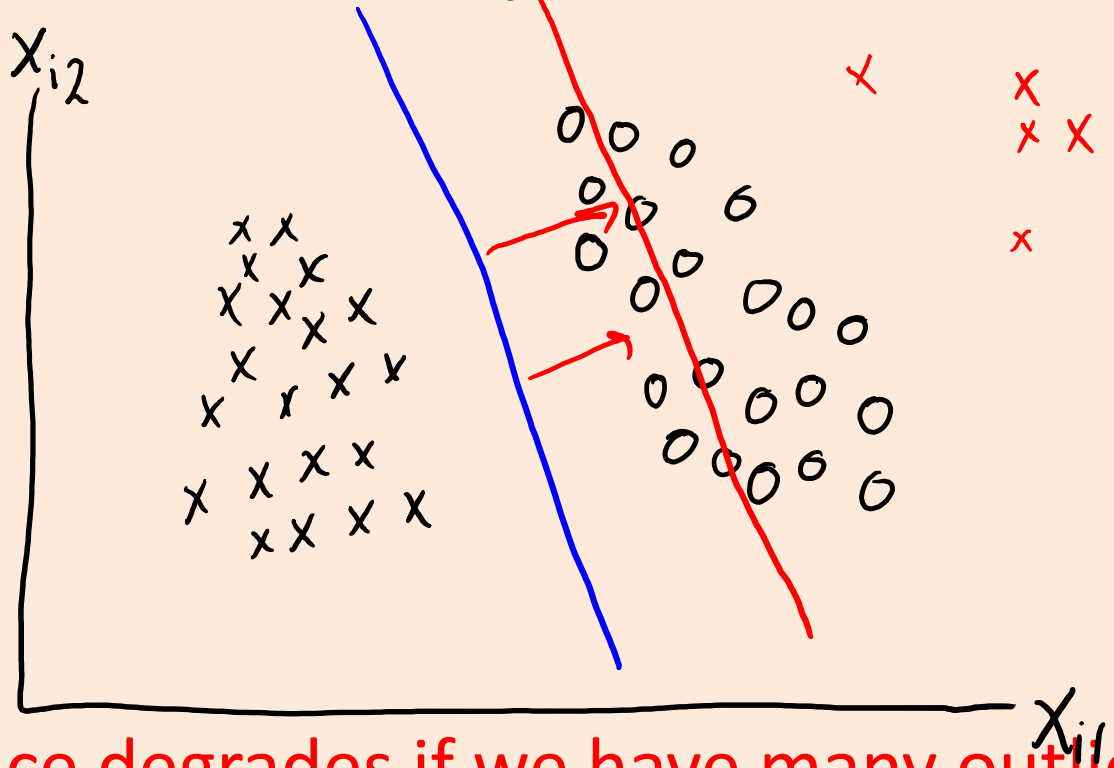
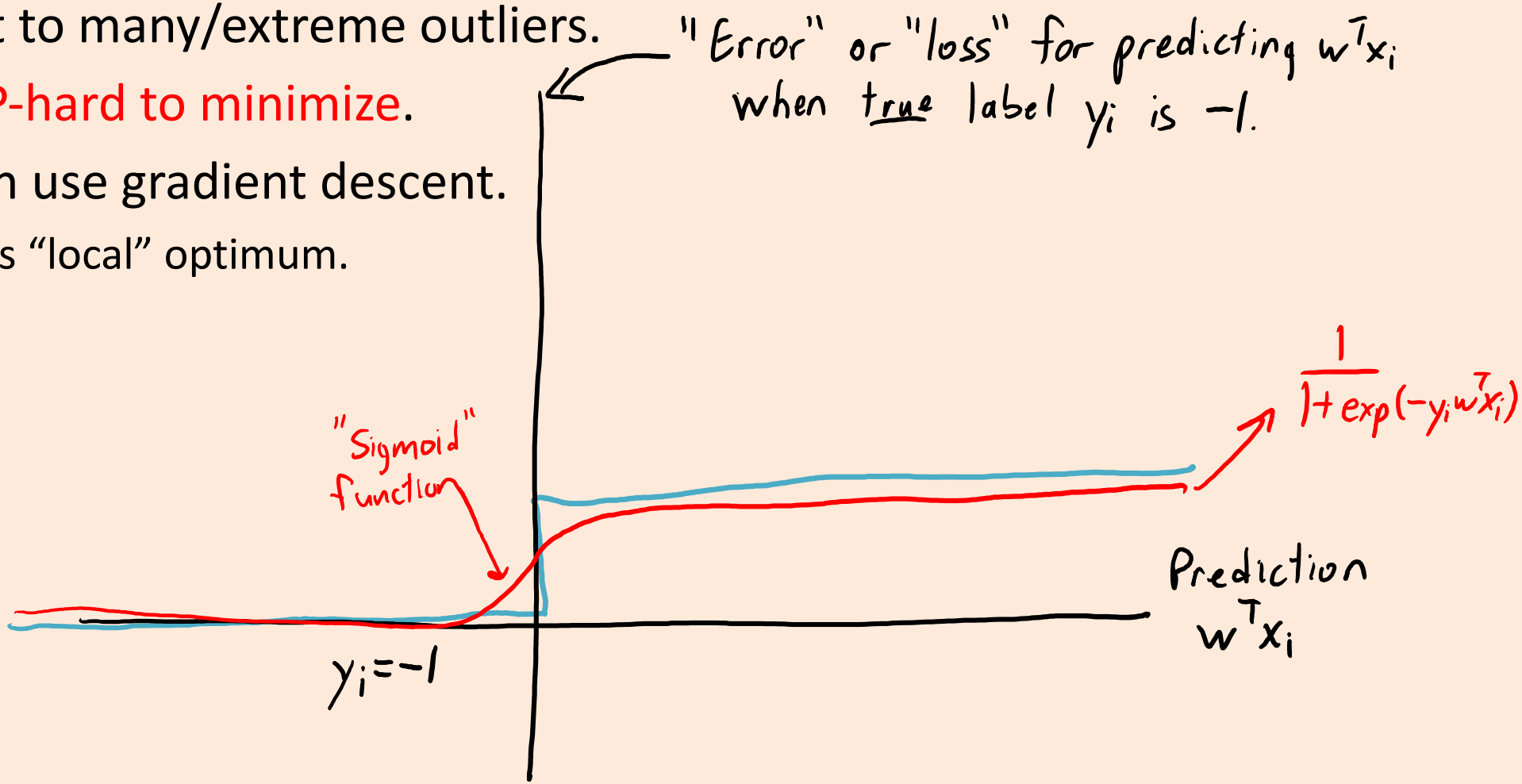


- But performance degrades if we have many outliers.

Non-Convex 0-1 Approximations

- There exists some smooth non-convex 0-1 approximations.
  - Robust to many/extreme outliers.
  - Still NP-hard to minimize.
  - But can use gradient descent.
    - Finds "local" optimum.



“Robust” Logistic Regression

- A recent idea: add a “fudge factor” v_i for each example.

$$f(w,v) = \sum_{i=1}^{n} \log\left(1 + \exp\left(-y_i w^T x_i + v_i\right)\right)$$

- If $w^T x_i$ gets the sign wrong, we can “correct” the mis-classification by modifying v_i.
  - This makes the training error lower but doesn’t directly help with test data, because we won’t have the v_i for test data.
  - But having the v_i means the ‘w’ parameters don’t need to focus as much on outliers (they can make $|v_i|$ big if $\text{sign}(w^T x_i)$ is very wrong).

"Robust" Logistic Regression

- A recent idea: add a "fudge factor" v_i for each example.

$$f(w,v) = \sum_{i=1}^{n} \log\left(1 + \exp\left(-y_i w^T x_i + v_i\right)\right)$$

- If $w^T x_i$ gets the sign wrong, we can "correct" the mis-classification by modifying v_i.
- A problem is that we can ignore the 'w' and get a tiny training error by just updating the v_i variables.
- But we want most v_i to be zero, so "robust logistic regression" puts an L1-regularizer on the v_i values:

$$f(w,v) = \sum_{i=1}^{n} \log\left(1 + \exp\left(-y_i w^T x_i + v_i\right)\right) + \lambda \|v\|_1$$

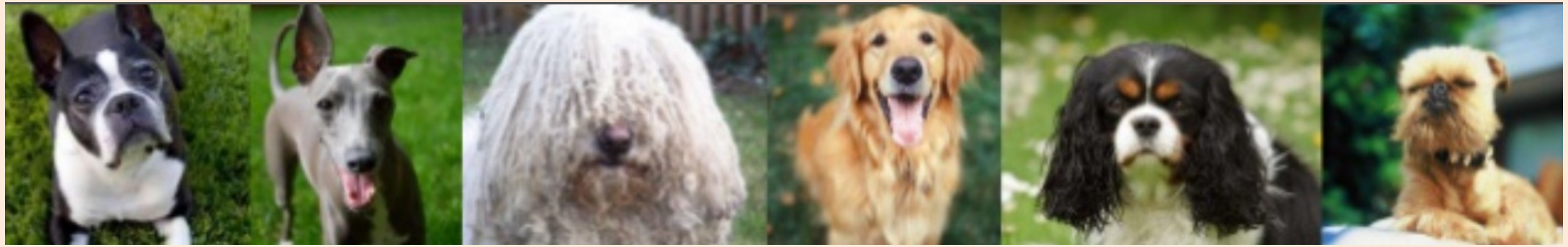
- You would probably also want to regularize the 'w' with different λ.

“All-Pairs” and ECOC Classification

- Alternative to “one vs. all” to convert binary classifier to multi-class is “all pairs”.
  - For each pair of labels ‘c’ and ‘d’, fit a classifier that predicts +1 for examples of class ‘c’ and -1 for examples of class ‘d’ (so each classifier only trains on examples from two classes).
  - To make prediction, take a vote of how many of the (k-1) classifiers for class ‘c’ predict +1.
  - Often works better than “one vs. all”, but not so fun for large ‘k’.
    - Need $O(k^2)$ classifiers.
- A variation on this is using “error correcting output codes” from information theory (see Math 342).
  - Each classifier trains to predict +1 for some of the classes and -1 for others.
  - You setup the +1/-1 code so that it has an “error correcting” property.
    - It will make the right decision even if some of the classifiers are wrong.

Motivation: Dog Image Classification

- Suppose we're classifying images of dogs into breeds:



- What if we have images where class label isn't obvious?
  - Syberian husky vs. Inuit dog?

https://www.slideshare.net/angjoo/dog-breed-classification-using-part-localization
https://ischlag.github.io/2016/04/05/important-ILSVRC-achievements

Learning with Preferences

- Do we need to throw out images where label is ambiguous?
  - We don't have the y_i.

  - We want classifier to prefer Syberian husky over bulldog, Chihuahua, etc.
    - Even though we don't know if these are Syberian huskies or Inuit dogs.

  - Can we design a loss that enforces preferences rather than "true" labels?

https://ischlag.github.io/2016/04/05/important-ILSVRC-achievements

Learning with Pairwise Preferences (Ranking)

- Instead of y_i, we're given list of (c_1,c_2) preferences for each 'i':

We want $w_{c_1}^T x_i > w_{c_2}^T x_i$ for these particular (c_1,c_2) values

- Multi-class classification is special case of choosing (y_i,c) for all 'c'.

- By following the earlier steps, we can get objectives for this setting:

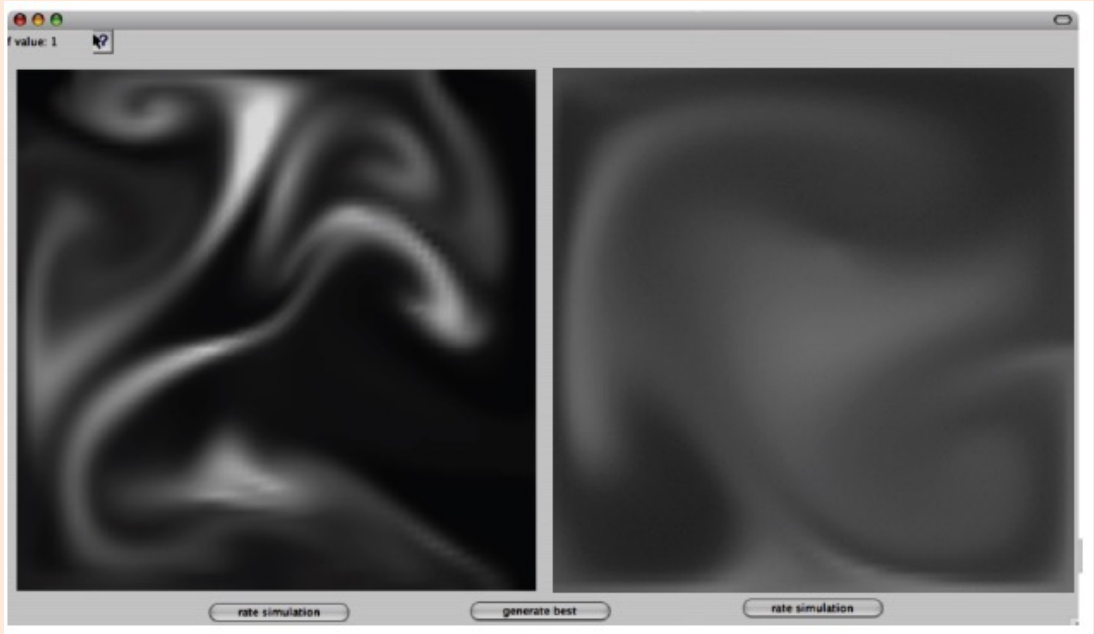
$$\sum_{i=1}^{n} \sum_{(c_1,c_2)} \max\{0, 1 - w_{c_1}^T x_i + w_{c_2}^T x_i\} + \frac{\lambda}{2} \|W\|_F^2$$

"sum" version of multi-class SVM

https://ischlag.github.io/2016/04/05/important-ILSVRC-achievements

Learning with Pairwise Preferences (Ranking)

- Pairwise preferences for computer graphics:
  - We have a smoke simulator, with several parameters:

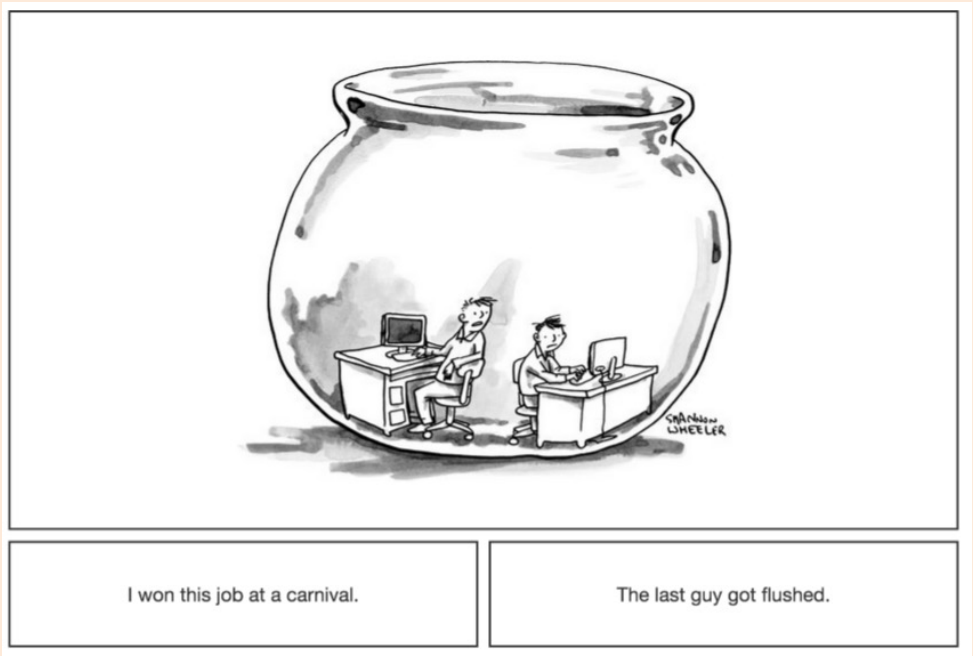


  - Don’t know what the optimal parameters are, but we can ask the artist:
    - “Which one looks more like smoke”?

https://circle.ubc.ca/bitstream/handle/2429/30519/ubc_2011_spring_brochu_eric.pdf?sequence=3

Learning with Pairwise Preferences (Ranking)

- Pairwise preferences for humour:
  - New Yorker caption contest:



  - “Which one is funnier”?

https://homes.cs.washington.edu/~jamieson/resources/next.pdf

Risk Scores

- In medicine/law/finance, risk scores are sometimes used to give probabilities:

1.	**C**ongestive Heart Failure	1 point	...
2.	**H**ypertension	1 point	+ ...
3.	**A**ge $\geq$ 75	1 point	+ ...
4.	**D**iabetes Mellitus	1 point	+ ...
5.	Prior **S**troke or Transient Ischemic Attack	**2** points	+
		SCORE	=

SCORE	0	1	2	3	4	5	6
RISK	1.9%	2.8%	4.0%	5.9%	8.5%	12.5%	18.2%

Figure 1: $CHADS_2$ risk score of Gage et al. (2001) to assess stroke risk (see www.mdcalc.com for other medical scoring systems). The variables and points of this model were determined by a panel of experts, and the risk estimates were computed empirically from data.

- Get integer-valued "points" for each "risk factor", and probability is computed from data based on people with same number of points.
- Less accurate than fancy models, but interpretable and can be done by hand.
  - Some work on trying to "learn" the whole thing (like doing feature selection then rounding).

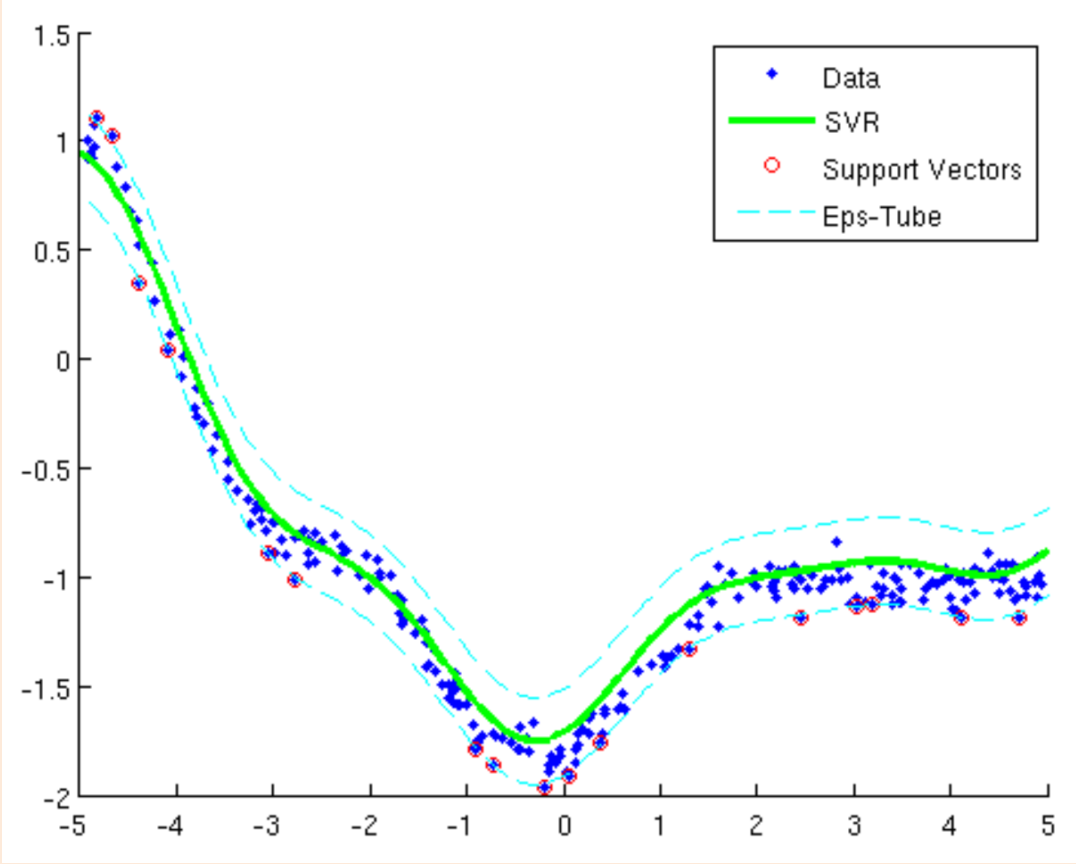
https://arxiv.org/pdf/1610.00168.pdf

Support Vector Regression

- Support vector regression objective (with hyper-parameter ϵ):

$$f(w) = \sum_{i=1}^{n} \max\{0, |w^T x_i - y_i| - \epsilon\} + \frac{\lambda}{2}\|w\|^2$$

  - Looks like L2-regularized robust regression with the L1-loss.
  - But have loss of 0 if $\hat{y}_i$ within ϵ of $\tilde{y}_i$.
    - So doesn't try to fit data exactly.
      - This can help fight overfitting.
  - Support vectors are points with loss>0.
    - Points outside the "epsilon-tube".
  - Example with Gaussian-RBFs as features:

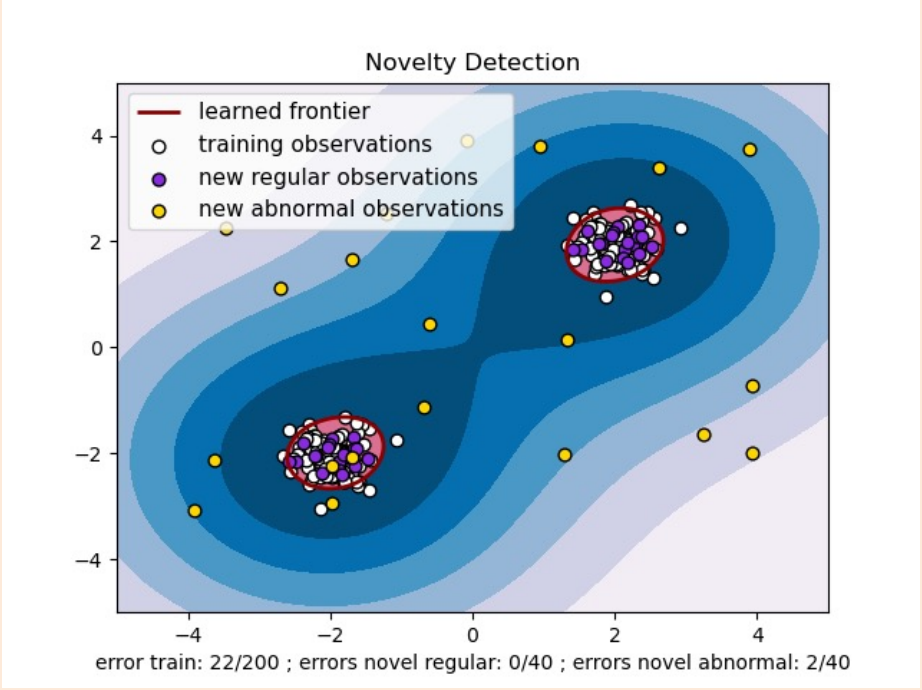


1-Class SVMs

- 1-class SVMs for outlier detection.

$$f(w, w_0) = \sum_{i=1}^{N} [\max\{0, w_0 - w^T x_i\} - w_0] + \frac{\lambda}{2}\|w\|_2^2$$

  - Variables are ‘w’ (vector) and ‘w_0’ (scalar).
  - Only trains on “inliers”.
    - Tries to make $w^T x_i$ bigger than w_0 for inliers.
    - At test time: says “outlier” if $w^T x_i < w_0$.
    - Usually used with RBFs.



  - The above is a class formulation, but there are many more.

https://scikit-learn.org/stable/auto_examples/svm/plot_oneclass.html