Tutorial 8

CPSC 340

Logistic Regression

Stochastic Gradient Descent

Logistic Regression Model

- A discriminative probabilistic model for classification e.g. spam filtering
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- what is the probabilities of $p(y = -1|x)$?
- $$p(y = -1|x) = 1 - p(y = 1|x) = \frac{1}{1 + exp(w^T x)}$$

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- Since $f(w)$ is convex function w.r.t. w we can use GD to find w^*

Learning in Logistic Regression

- Regularized loss function

$$f(w) = \frac{1}{n}\sum_{i=1}^{n} \log(1 + \exp(-y_i w^T x_i)) + \frac{\lambda}{2}\|w\|^2$$

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- Exercise: find the gradient of regularized f(w)?
- Solution

$$f_i(w) = \log(1 + \exp(-y_i w^T x_i))$$

$$\nabla f_i(w) = \frac{-y_i x_i}{1 + \exp(y_i w^T x_i)}$$

$$\nabla f(w) = \frac{1}{n}\sum_{i=1}^{n} \frac{-y_i x_i}{1 + \exp(y_i w^T x_i)} + \lambda w$$

Learning in Logistic Regression

- Coding Exercise: we want to write a function to calculate the gradient and function. The following code is given. Write the code for **loglos_subfunc**.
-

```
function [f,g] = loglos(X,Y,w,lambda)
[n,d]=size(X);
g=zeros([d,1]);
f=0;
for i=1:n
    [f_i,g_i]=loglos_subfunc(X(i,:),Y(i),w);
    g=g+g_i;
    f=f+f_i;
end
g=g/n+lambda*w;
f=f/n+lambda/2*sum(w.^2);
end
```

Learning in Logistic Regression

- Solution
-

```
function [f_i,g_i]=loglos_subfunc(x_i,y_i,w)

z_i=y_i*x_i;
f_i=1+exp(-z_i*w);

%Gradient
g_i=-z_i*(1-1/f_i);

%function value f_i
f_i=log(f_i);
end
```

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Stochastic Gradient Descent

Gradient Descent Cost for Big Data

- Assume our loss function has the following format:

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- Cost of each iteration in GD could be enormous when n is large!

Stochastic Gradient Descent(SGD) for Big Data

- SGD algorithm: in each iteration pick a f_i randomly and use its gradient

$$i \sim unif(1, n) \qquad\qquad w_{t+1} = w_t - \alpha_t \nabla f_i(w_t)$$

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- ∇f_i is an unbiased approximation of ∇f

$$\mathbb{E}\left[\nabla f_i(w)\right] = \sum_{i=1}^{n} p(i) \nabla f_i(w) = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(w) = \nabla f(w)$$

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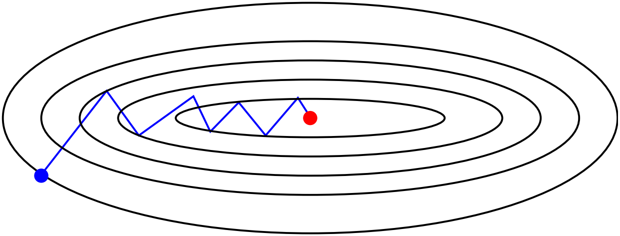
- Cost of each iteration in SGD is constant
- It does not move toward minimizer in each iteration, so is slower than GD

SGD vs GD

- GD



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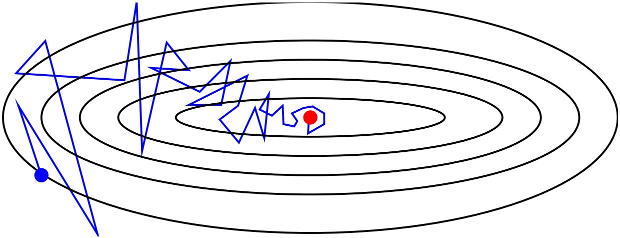

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Variance of SGD

- Variance of SGD in each iteration:

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- If variance is large, many steps jump in wrong direction!
- Variance can be controlled by decreasing step size or by variance reduction technique

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$$\sum_{t=1}^{\infty} \alpha_t = \infty \qquad \text{we can get everywhere} \tag{1}$$

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- Setting $\alpha_t = O(1/t)$ satisfies the above conditions but it is too slow
- In practice:

$$\alpha_t = \beta/(t+\gamma)$$
$$\alpha_t = O(1/\sqrt{t}) \text{ or } O(1/t^{\beta}) \text{ for } \beta \in (0, 1)$$

SGD for logistic Loss

$$\alpha_t = \beta/(t+\gamma), \quad f(w) = \frac{1}{n}\sum_{i=1}^{n} \log(1+\exp(-y_i w^T x_i)) + \frac{\lambda}{2}\|w\|^2$$

- Coding Exercise: Using the **loglos_subfunc** from previous exercise, complete the SGD code for logistic loss.

```
function w=loglosSGD(X,Y,w_0,beta,gamma,lambda, T)

w=w_0;
[n,d]=size(X);

for t=1:T

    ......

end
end
```

- T: # of iteration
- w_0: initial value

SGD for logistic Loss

- Solution:

```
function w=loglosSGD(X,Y,w_0,beta,gamma,lambda, T)

w=w_0;
[n,d]=size(X);

for t=1:T
    %generating i randomly
    i=randi(n);

    %computing the gradient of f_i
    [f_i,g_i]=loglos_subfunc(X(i,:),Y(i),w);

    %step size for this iteration
    alpha_t=beta/(t+gamma);

    %SGD update with l2 regularizer
    w=(1-alpha_t*lambda)*w-alpha_t*g_i;

end
end
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  - The memory requirement could be restrictive when n is enormous