Assignment 6

Question 1.1

$$y_i \in \{-1, 1\}$$

Odds ratio

$$\frac{p(y_i|w^T x_i)}{p(-y_i|w^T x_i)}$$

Linear model

$$\log\left(\frac{p(y_i|w^T x_i)}{p(-y_i|w^T x_i)}\right) = w^T x.$$

Objective function

$$\underset{w \in \mathbb{R}^d}{\operatorname{argmin}} \sum_{i=1}^{n} -\log(p(y_i|w^T x_i)).$$

Question 1.1

Odds ratio

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Linear model

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Objective function

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Starting from equation 1

First step

replace $p(-y_i|w^T x_i)$ with $p(y_i|w^T x_i)$ using the fact that,

$$p(y_i|w^T x_i) + p(-y_i|w^T x_i) = 1$$

Second step

Apply “exp” on both sides to get rid of the log

Third step

Solve for $p(y_i|w^T x_i)$ and plug it into the objective function

Question 1.2 One-vs-all Logistic Regression

```
% Classification using one-vs-all least squares

% Compute sizes
[n,d] = size(X);
k = max(y);

W = zeros(d,k); % Each column is a classifier
for c = 1:k
    yc = ones(n,1); % Treat class 'c' as (+1)
    yc(y ~= c) = -1; % Treat other classes as (-1)
    W(:,c) = (X'*X)\(X'*yc);
end

model.W = W;
model.predict = @predict;
end

function [yhat] = predict(model,X)
W = model.W;
    [~,yhat] = max(X*W,[],2);
end
```

Question 1.2 One-vs-all Logistic Regression

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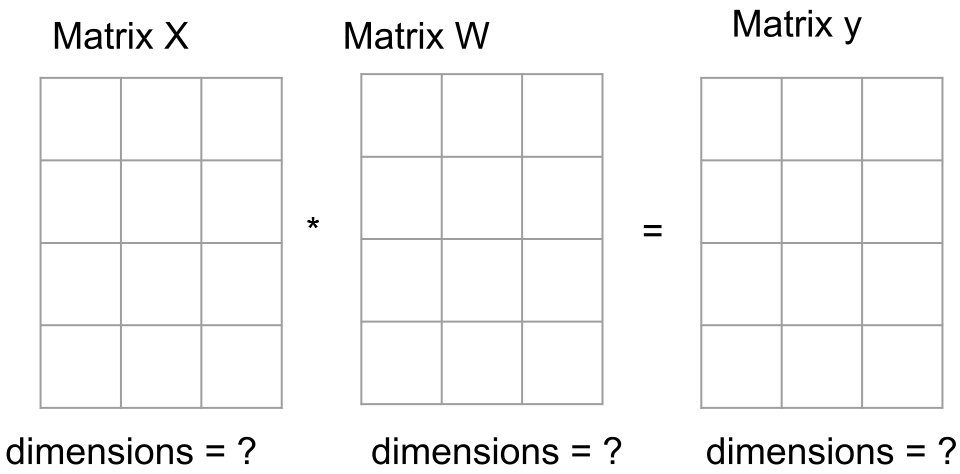

n samples, k classes, p features

use *findMin* with *LogisticLoss* instead (see assignment 4 for the *LogisticLoss* function)

Question 1.2 One-vs-all Logistic Regression

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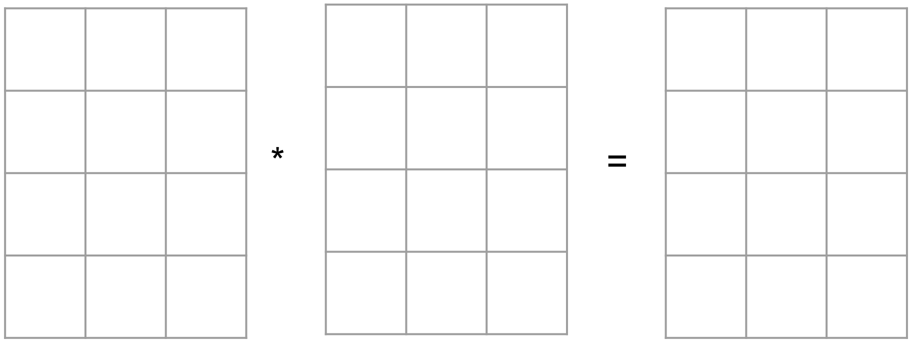
function [yhat] = predict(model,X)
W = model.W;
    [~,yhat] = max(X*W,[],2);
end
```

Matrix X   Matrix W   Matrix y



dimensions = n x p   **dimensions = p x k**   **dimensions = n x k**

n samples, k classes, p features

use *findMin* with *LogisticLoss* instead (see assignment 4 for the *LogisticLoss* function)

Question 1.3 Softmax Loss and derivative

- The softmax probability function is given as,

$$p(y_i|W, x_i) = \frac{\exp(w_{y_i}^T x_i)}{\sum_{c'=1}^{k} \exp(w_c'^T x_i)}.$$

- Assume $y_i \in \{1,2,3\}$

- Convert ***y*** to binary form; i.e.

$$\bar{y_i} = \begin{cases} [1,0,0] & \text{if } y_i = 1 \\ [0,1,0] & \text{if } y_i = 2 \\ [0,0,1] & \text{if } y_i = 3 \end{cases}$$

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y_{i1} y_{i2}. y_{i3}

- Therefore,

$$P(y_i|W,x) = p(\bar{y}_i|W,x)$$

$$= \bar{y}_{i1}\frac{\exp(x_i W^1)}{\sum_{c=1}^{k}\exp(x_i W^c)} + \bar{y}_{i2}\frac{\exp(x_i W^2)}{\sum_{c=1}^{k}\exp(x_i W^c)} + \dots + \bar{y}_{iC}\frac{\exp(x_i W^C)}{\sum_{c=1}^{k}\exp(x_i W^c)}$$

- C is the number of classes
- W^j refers to column j of W
- y_{ic} is the binary predicted target value for class *‘c’* of sample *‘i’*

Question 1.3 Softmax Loss and derivative

- The softmax probability function is now formulated as,

$$P(y_i|W,x) = p(\bar{y}_i|W,x) = \bar{y}_{i1}\frac{\exp(x_i W^1)}{\sum_{c=1}^{k}\exp(x_i W^c)} + \bar{y}_{i2}\frac{\exp(x_i W^2)}{\sum_{c=1}^{k}\exp(x_i W^c)} + \ldots + \bar{y}_{iC}\frac{\exp(x_i W^C)}{\sum_{c=1}^{k}\exp(x_i W^c)}$$

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Only one of these terms is non-zero for any training example ‘i’

Question 1.3 Softmax Loss and derivative

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- The negative logarithm of the probability:

$\log(p(\bar{y}_i|W,x))$ = ? (apply log)

$-\log(p(\bar{y}_i|W,x))$ = ? (multiply by -1)

- The derivative of the negative log probability with respect to W_j^c can be broken into two cases :

$$\frac{\partial -\log(p(\bar{y}_i|W,x))}{\partial W_j^c} = \begin{cases} ? & \text{if } y_i = c; \text{ i.e. } \bar{y}_{ic} = 1 \\ ? & \text{if } y_i \neq c; \text{ i.e. } \bar{y}_{ic} = 0 \end{cases}$$

Question 1.3 Softmax Loss and derivative

- The softmax probability function is now formulated as,

$$P(y_i|W,x) = p(\bar{y}_i|W,x) = \bar{y}_{i1}\frac{\exp(x_i W^1)}{\sum_{c=1}^{k}\exp(x_i W^c)} + \bar{y}_{i2}\frac{\exp(x_i W^2)}{\sum_{c=1}^{k}\exp(x_i W^c)} + ... + \bar{y}_{iC}\frac{\exp(x_i W^C)}{\sum_{c=1}^{k}\exp(x_i W^c)}$$

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W_j^c is column *c* of row *j* of *W* which corresponds to the coefficient of feature *j* of class *c*

Question 1.3 Softmax Loss and derivative

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$$P(y_i|W,x) = p(\bar{y}_i|W,x) = \bar{y}_{i1}\frac{\exp(x_i W^1)}{\sum_{c=1}^{k}\exp(x_i W^c)} + \bar{y}_{i2}\frac{\exp(x_i W^2)}{\sum_{c=1}^{k}\exp(x_i W^c)} + \ldots + \bar{y}_{iC}\frac{\exp(x_i W^C)}{\sum_{c=1}^{k}\exp(x_i W^c)}$$

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Hint: Use the indicator function to distinguish between the two cases

Question 1.4 - Softmax Classifier

```
function [model] = leastSquaresClassifier(X,y)
% Classification using one-vs-all least squares

% Compute sizes
[n,d] = size(X);
k = max(y);

W = zeros(d,k); % Each column is a classifier
for c = 1:k
    yc = ones(n,1); % Treat class 'c' as (+1)
    yc(y ~= c) = -1; % Treat other classes as (-1)
    W(:,c) = (X'*X)\(X'*yc);
end

model.W = W;
model.predict = @predict;
end
```

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end

model.W = W;
model.predict = @predict;
end
```

Use findMin instead with the softmax loss grad function

Question 1.4 - Softmax Classifier

```matlab
function [model] = leastSquaresClassifier(X,y)
% Classification using one-vs-all least squares

% Compute sizes
[n,d] = size(X);
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    W(:,c) = (X'*X)\(X'*yc);
end

model.W = W;
model.predict = @predict;
end
```

```matlab

W = zeros(d,k); % Each column is a classifier
% findMin expects 1-dimensional parameter vector
% Therefore use W(:) to get W's 1-dimensional form
W(:) = findMin(@yourSoftmaxLossFunction, W(:), ....)

model.W = W;
model.predict = @predict;
end

function [loss, grad] = yourSoftmaxLossFunction(w, X, y, k)
    % reshape w's dimensions to "p x k"
    % p is the number of features, k is the number of classes
    W = reshape(w, [p k]);

    % Compute loss
    loss = the softmax loss function you derived for Q1.3

    % Compute gradient
    grad = the softmax gradient function you derived for Q1.3

    % reshape grad's dimensions to "1 x (p * k)"
    % i.e. convert the grad matrix to a 1-dimensional vector
    grad = reshape(grad, [p*k 1]);
end
```

Change the contents of the green box on the left using that of the green boxes on the right

Question 1.4 - Softmax Classifier

```matlab
function [model] = leastSquaresClassifier(X,y)
% Classification using one-vs-all least squares

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[n,d] = size(X);
k = max(y);

W = zeros(d,k); % Each column is a classifier
for c = 1:k
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W = zeros(d,k); % Each column is a classifier
% findMin expects 1-dimensional parameter vector
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model.W = W;
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function [loss, grad] = yourSoftmaxLossFunction(w, X, y, k)
    % reshape w's dimensions to "p x k"
    % p is the number of features, k is the number of classes
    W = reshape(w, [p k]);

    % Compute loss
    loss = the softmax loss function you derived for Q1.3

    % Compute gradient
    grad = the softmax gradient function you derived for Q1.3

    % reshape grad's dimensions to "1 x (p * k)"
    % i.e. convert the grad matrix to a 1-dimensional vector
    grad = reshape(grad, [p*k 1]);
end
```

Change the contents of the green box on the left using that of the green boxes on the right

Question 1.5 - Cost of Multinomial Logistic Regression

Time complexity for processing one example = ?

Time complexity for predicting one example = ?

```
# Training
# Run for T iterations
for t = 1 to T
  # Loop over training examples
  for i = 1 to n
      for k = 1 to K
        softmax_value(i,k) = compute softmax for class k for training example i over the 'd' features
  for j = 1 to d
      for k = 1 to K
        softmax_gradient(j, k) = compute the gradient for the cofficient of feature j of class k using softmax_value
  for j = 1 to d
    for k = 1 to K
      update w(j,k) using softmax_gradient(j, k)
# Testing
# Loop over test examples
for i = 1 to n_test
  for k = 1 to K
    softmax_value(i,k) = compute softmax for class k for test example i  over the 'd' features
end for

for i = 1 to n_test
  yhat(i) = argmax of softmax_value(i,k) over 'k'
end for
```

Question 1.5 - Cost of Multinomial Logistic Regression

```
# Training
# Run for T iterations
for t = 1 to T
    # Loop over training examples
    for i = 1 to n
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for i = 1 to n_test
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for i = 1 to n_test
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```
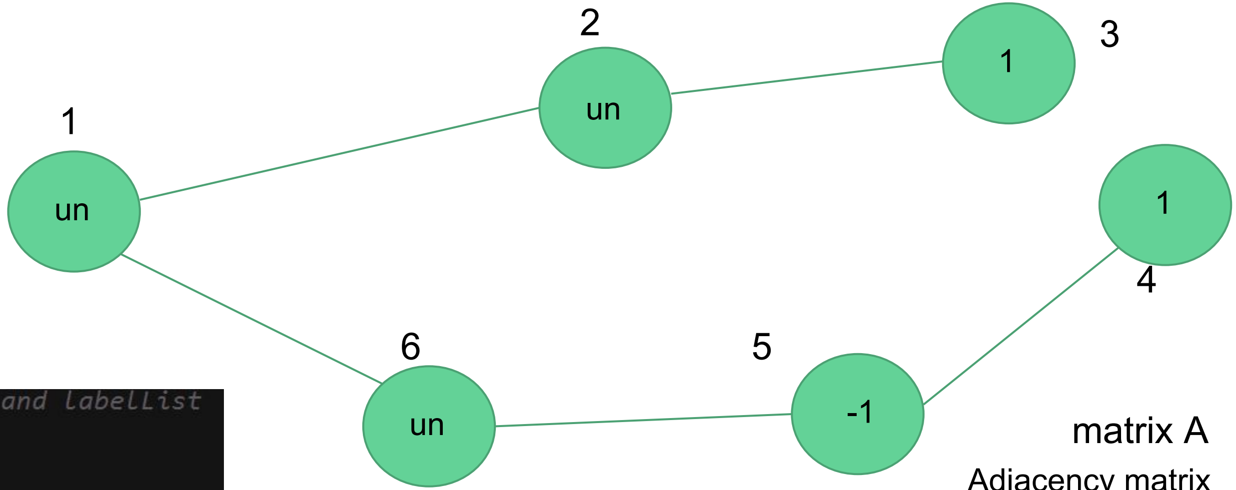

Question 2
random walk




```
load simpleGraph.mat % Loads adjacency A and labelList

n = length(A);
p = zeros(n,2);
r = 100;


for i = 2:n
    for j = 1:r
        % Run random walk
        yhat = runRandomWalk(A,labelList,i);
        if yhat == 1
            p(i,1) = p(i,1) + 1;
        elseif yhat == -1
            p(i,2) = p(i,2) + 1;
        end
    end
end

% Output final probabilities
probabilities = p/r
```

matrix labelList

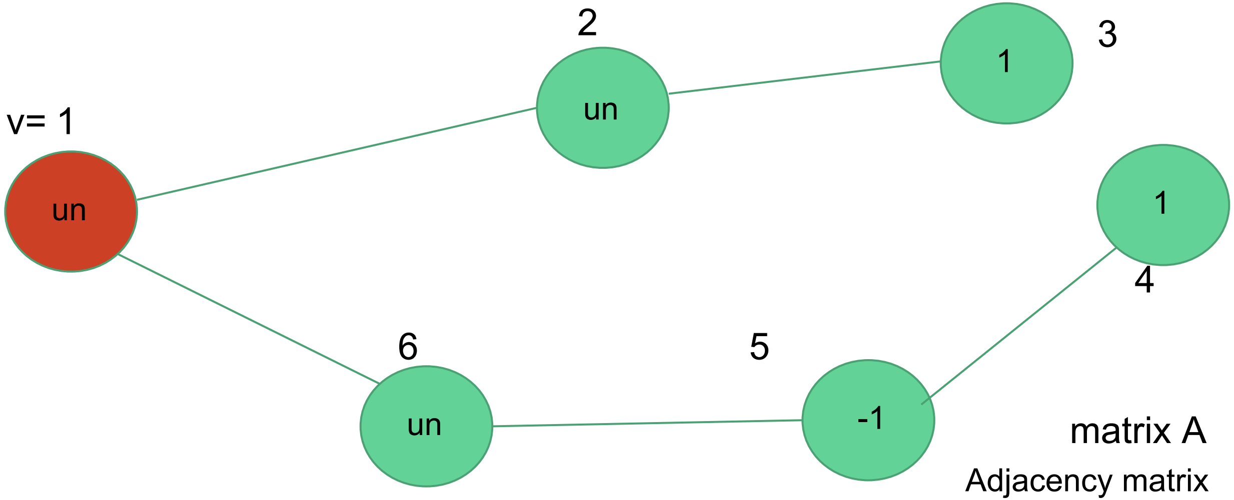
3	1
4	1
5	-1

matrix A

Adjacency matrix

	1	2	3	4	5	6
1	0	1	0	0	0	1
2	1	0	1	0	0	0
3	0	1	0	0	0	0
4	0	0	0	0	1	0
5	0	0	0	1	0	1
6	1	0	0	0	1	0

Question 2
random walk



matrix A

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matrix labelList

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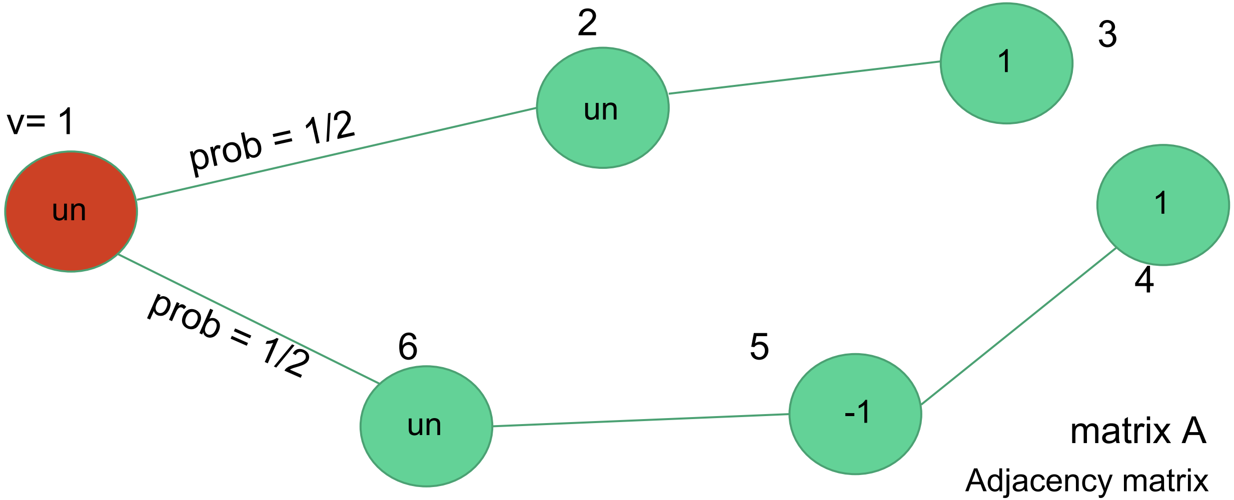

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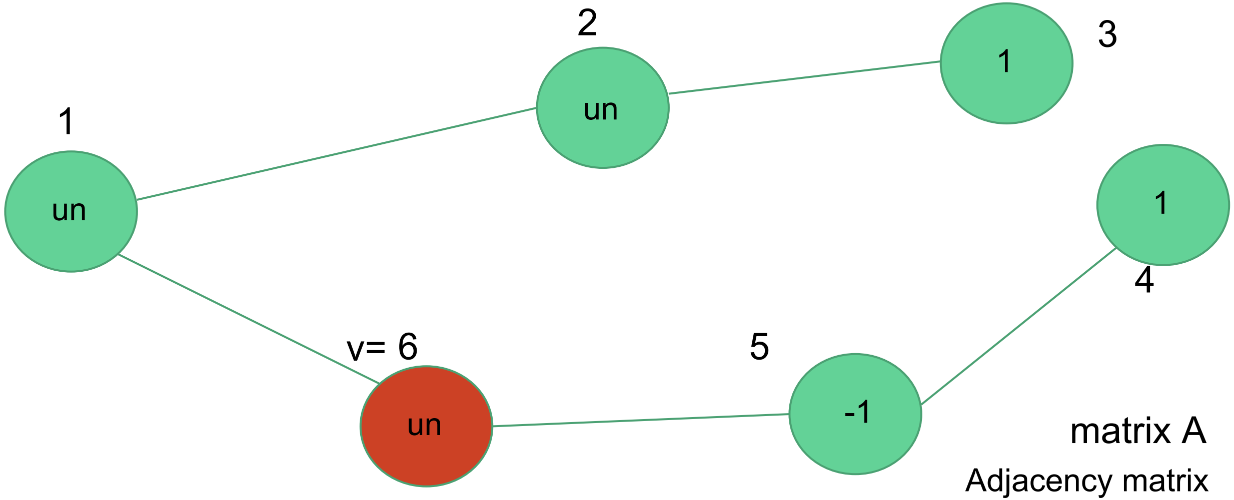
random walk



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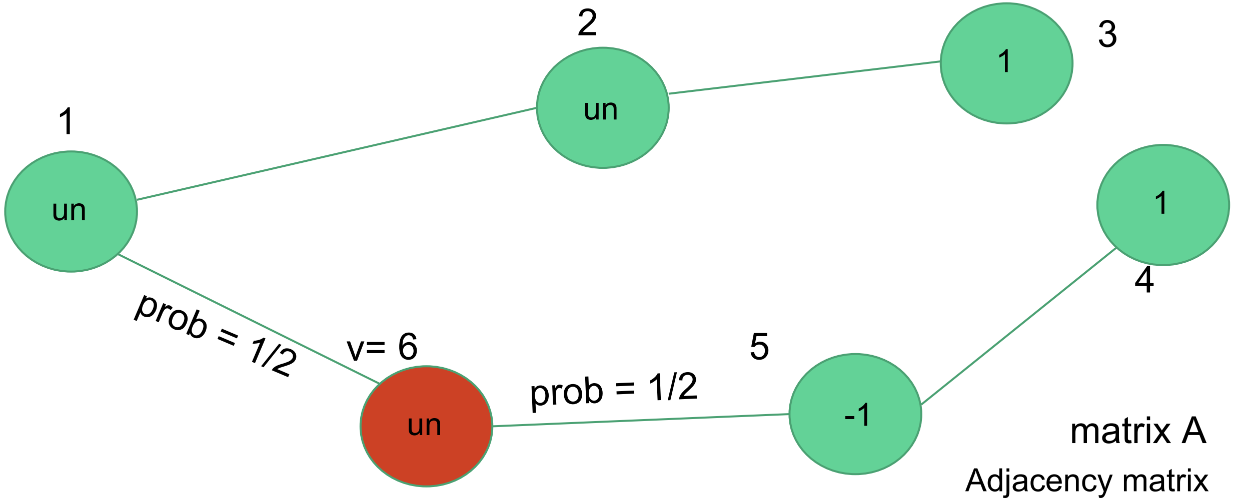

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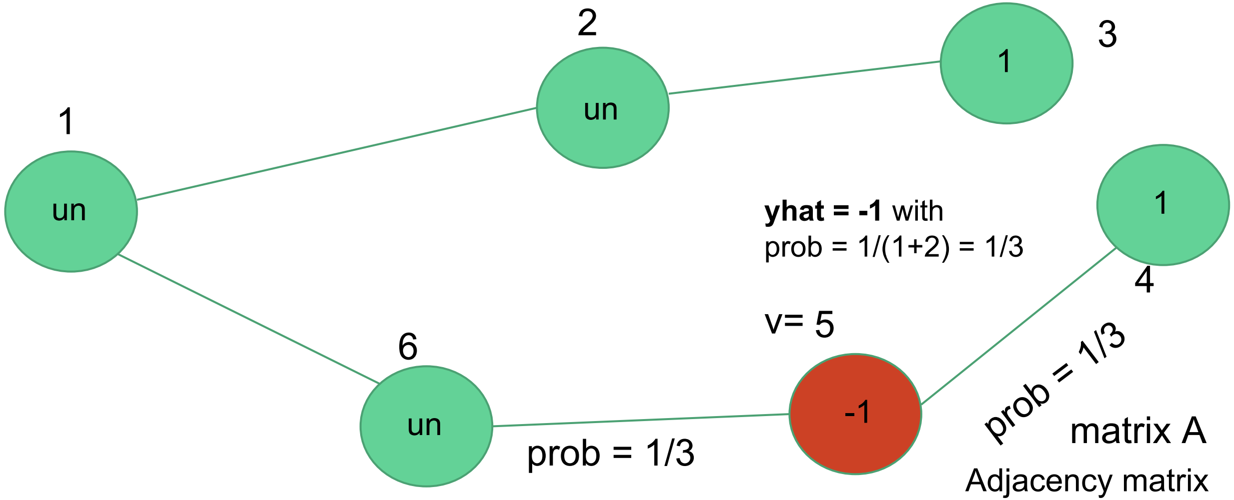
random walk



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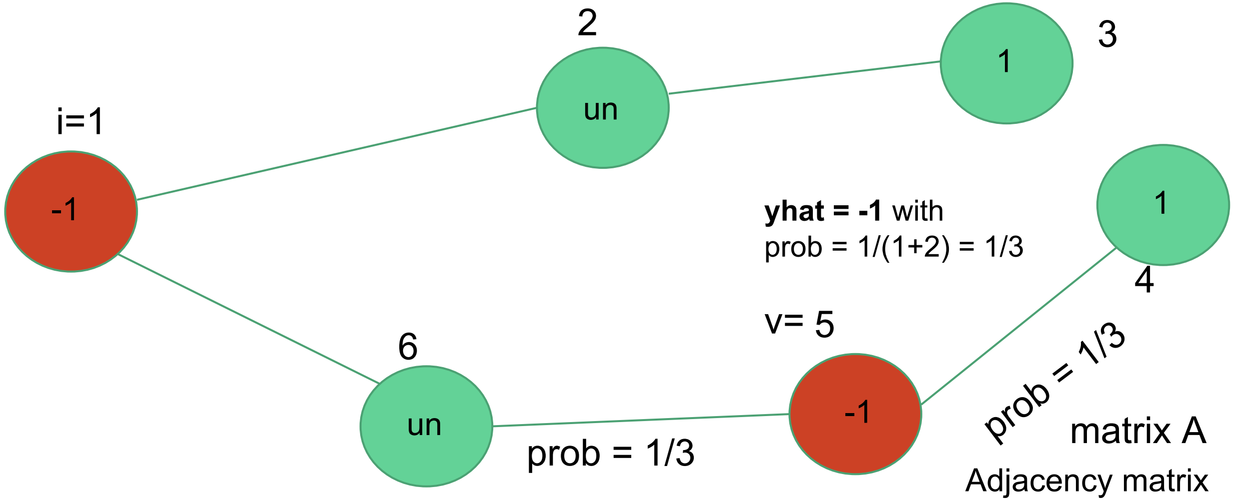
random walk



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