CPSC 340: Machine Learning and Data Mining

Conditional Probability and Generative Models

September 18, 2015

Admin

- Assignment 1 was due at 3pm.
  - 1 late day if you hand it in before Monday at 3pm.
  - 2 late days if you hand it in before Wednesday at 3pm.
  - You've used all your late days if you hand it in before next Friday at 3pm.
  - Mark of 0 after that.
- No tutorials Monday, there will be office hours Tuesday.
- Assignment 2 out by Monday, due in 2 weeks.
  - Start early!
- Registration in tutorials:
  - You need to be registered in a tutorial section to stay enrolled.
- Auditors:
  - Will not be able to register while students are on the waiting list (currently: 6).

Should you trust them?

- Scenario 1:
  - "I built a model based on the data you gave me."
  - "It classified your data with 98% accuracy."
  - "It should get 98% accuracy on the rest of your data."

- Probably not:
  - They are reporting training error.
  - This might have nothing to do with test error.
  - E.g., they could have fit a very deep decision tree.
- Why 'probably'?
  - If they only tried a few very simple models, the 98% might be reliable.
  - E.g., they only considered decision stumps with simple 1-variable rules.

Should you trust them?

- Scenario 2:
  - "I built a model based on half of the data you gave me."
  - "It classified the other half of the data with 98% accuracy."
  - "It should get 98% accuracy on the rest of your data."

- Probably:
  - They computed the validation error once.
  - This is an unbiased approximation of the test error.
  - Trust them if you believe they didn't violate the golden rule.

Should you trust them?

- Scenario 3:
  - "I built 10 models based on half of the data you gave me."
  - "One of them classified the other half of the data with 98% accuracy."
  - "It should get 98% accuracy on the rest of your data."

- Probably:
  - They computed the validation error a small number of times.
  - Maximizing over these errors is a biased approximation of test error.
  - But they only maximized it over 10 models, so bias is probably small.
  - They probably know about the golden rule.

Should you trust them?

- Scenario 4:
  - "I built 1 billion models based on half of the data you gave me."
  - "One of them classified the other half of the data with 98% accuracy."
  - "It should get 98% accuracy on the rest of your data."

- Probably not:
  - They computed the validation error a huge number of times.
  - Maximizing over these errors is a biased approximation of test error.
  - They tried so many models, one of them is likely to work by chance.
- Why 'probably'?
  - If the 1 billion models were all extremely-simple, 98% might be reliable.

Should you trust them?

- Scenario 5:
  - "I built 1 billion models based on the first third of the data you gave me."
  - "One of them classified the second third of the data with 98% accuracy."
  - "It also classified the last third of the data with 98% accuracy."
  - "It should get 98% accuracy on the rest of your data."

- Probably:
  - They computed the first validation error a huge number of times.
  - But they had a second validation set that they only looked at once.
  - The second validation set gives unbiased test error approximation.
  - This is ideal, as long as they didn't violate golden rule on second set.
  - And assuming you are using IID data in the first place.

The ‘Best’ Machine Learning Model

- Decision trees are not always most accurate.
- What is the ‘best’ machine learning model?
- No free lunch theorem:
  - There is no ‘best’ model that achieves the best test error for every problem.
  - If model A works better than model B on one dataset, there is another dataset where model B works better.
- This question is kind of like asking which is ‘best’ among “rock”, “paper”, and “scissors”.

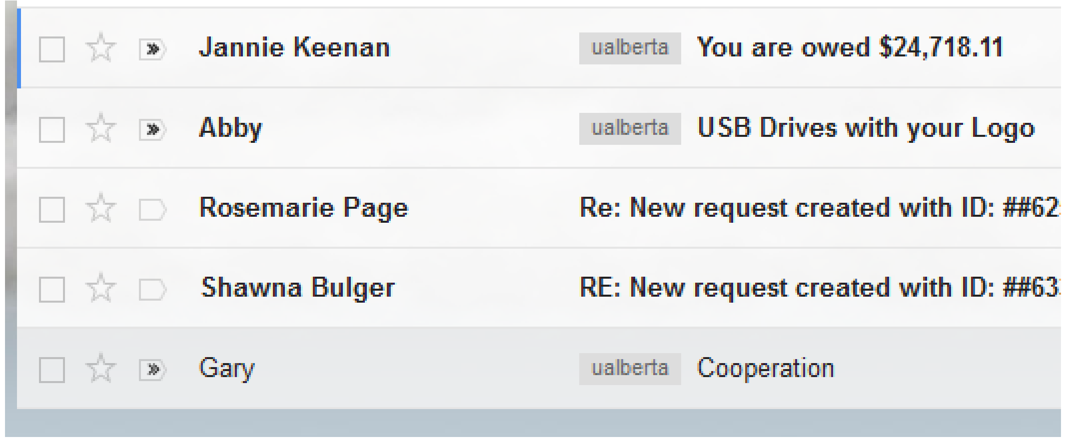
The ‘Best’ Machine Learning Model

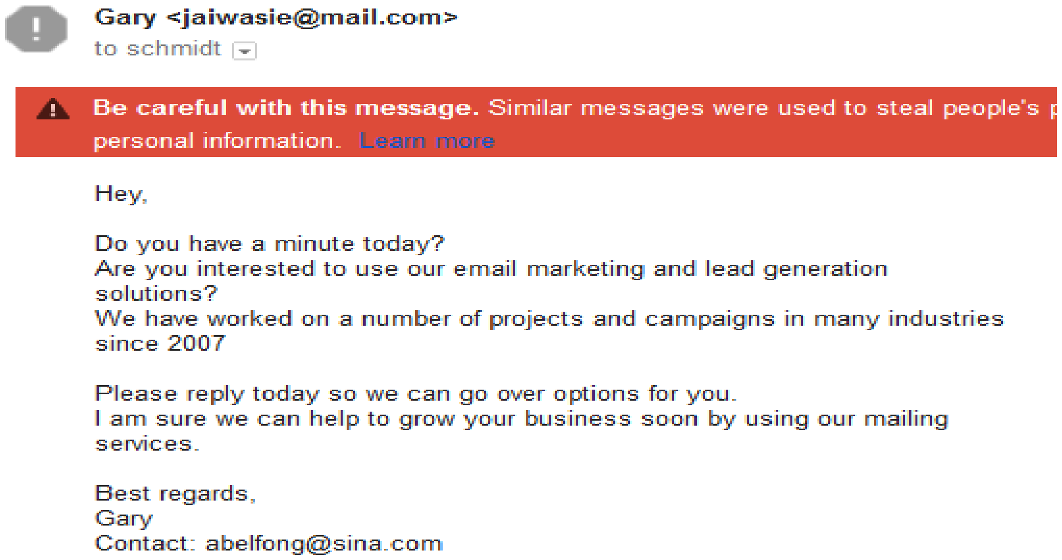
- Implications of the lack of a ‘best’ model:
  - We need to learn about and try out multiple models.
- So which ones to study in CPSC 340?
  - We’ll usually motivate a method by a specific application.
  - But we’ll focus on models that are effective in many applications.

- Caveat of no free lunch (NFL) theorem:
  - The world is very structured.
  - Some datasets are more likely than others.
  - Model A really could be better than model B on every real dataset in practice.
- Machine learning research:
  - Large focus on models that are useful across many applications.

Application: E-mail Spam Filtering

- Want a build a system that filters spam e-mails.

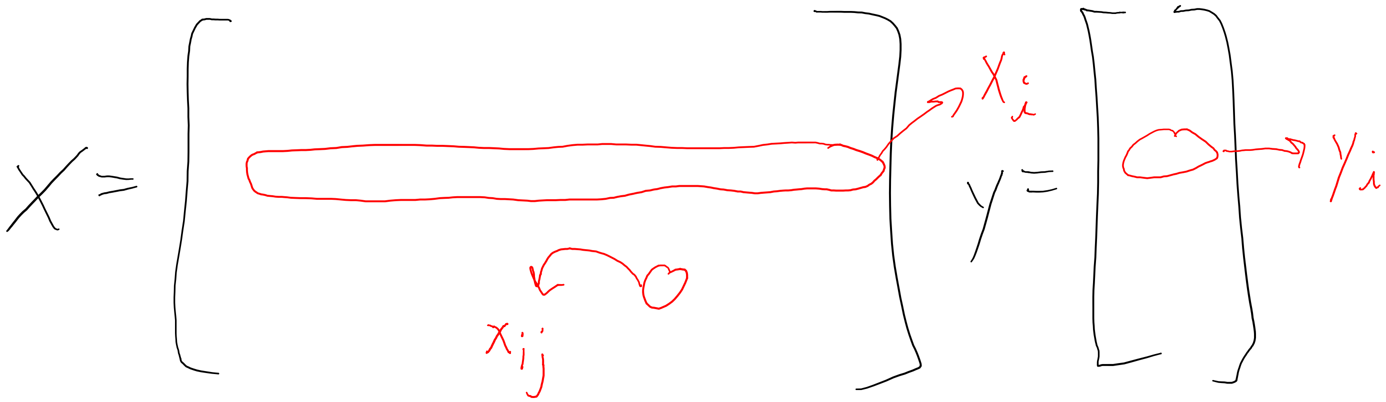




- We have a big collection of e-mails, labeled by users.
- Can we formulate as supervised learning?

First a bit more supervised learning notation

- We have been using the notation ‘X’ and ‘y’ for supervised learning:



- X is matrix of all features, y is vector of all labels.
- Need a way to refer to the features and label of specific object ‘i’.
  - We use y_i for the label of object ‘i’ (element ‘i’ of ‘y’).
  - We use x_i for the features object ‘i’ (row ‘i’ of ‘X’).
  - We use x_{ij} for feature ‘j’ of object ‘i‘.

Feature Representation for Spam

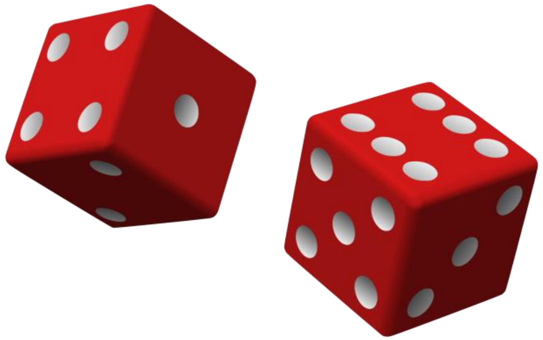
- How do we make label 'y_i' of an individual e-mail?
  - (y_i = 1) means 'spam', (y_i = 0) means 'not spam'.
- How do we construct features 'x_i' for an e-mail?
  - Use bag of words:
    - "hello", "vicodin", "$".
    - "vicodin" feature is 1 if "vicodin" is in the message, and 0 otherwise.
  - Could add phrases:
    - "be your own boss", "you're a winner", "CPSC 340".
  - Could add regular expressions:
    - <recipient>, <sender domain == "mail.com">

Probabilistic Classifiers

- For years, best spam filtering methods used naïve Bayes.
  - Naïve Bayes is probabilistic classifier based on Bayes rule.
  - It's 'naïve' because it makes a strong independence assumption.
  - But it tends to work well with bag of words.

- Probabilistic classifiers model a conditional probability, $p(y_i \mid x_i)$.
  - "If a message has words x_i, what is probability that message is spam?"
- If $p(y_i = \text{'spam'} \mid x_i) > p(y_i = \text{'not spam'} \mid x_i)$, classify as spam.

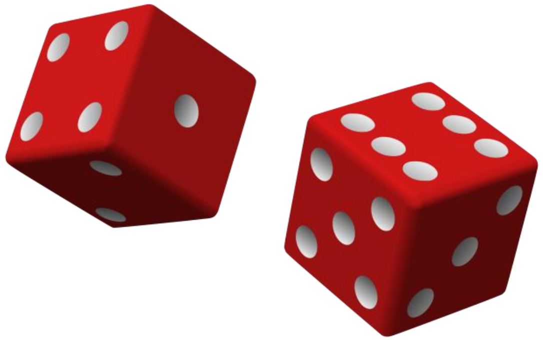
Digression to Review Probabilities…



https://en.wikipedia.org/wiki/Dice_throw_%28review%29

Digression to Review Probabilities…

- Dungeons & Dragons scenario:
  - You roll dice 1:
    - Roll 5 or 6 you sneak past monster.
    - Otherwise, you are eaten.
  - If you survive, you roll dice 2:
    - Roll 4-6, find pizza.
    - Otherwise, you find nothing.



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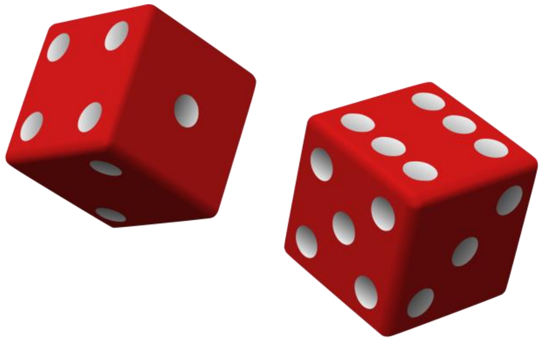

- Probabilities defined on ‘event space’:

D1\D2	1	2	3	4	5	6
1						
2						
3		$D_1=3,D_2=2$				
4						
5						
6						

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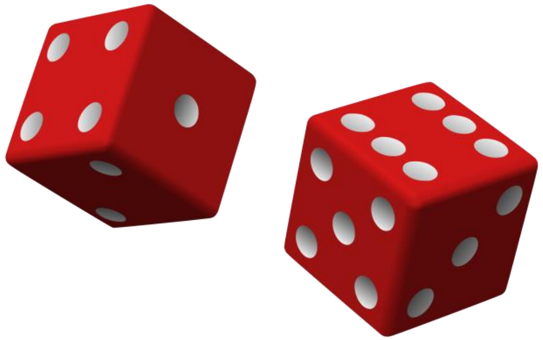

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D1\D2	1	2	3	4	5	6
1						
2						
3			¬Survive			
4						
5						
6			Survive		Pizza	

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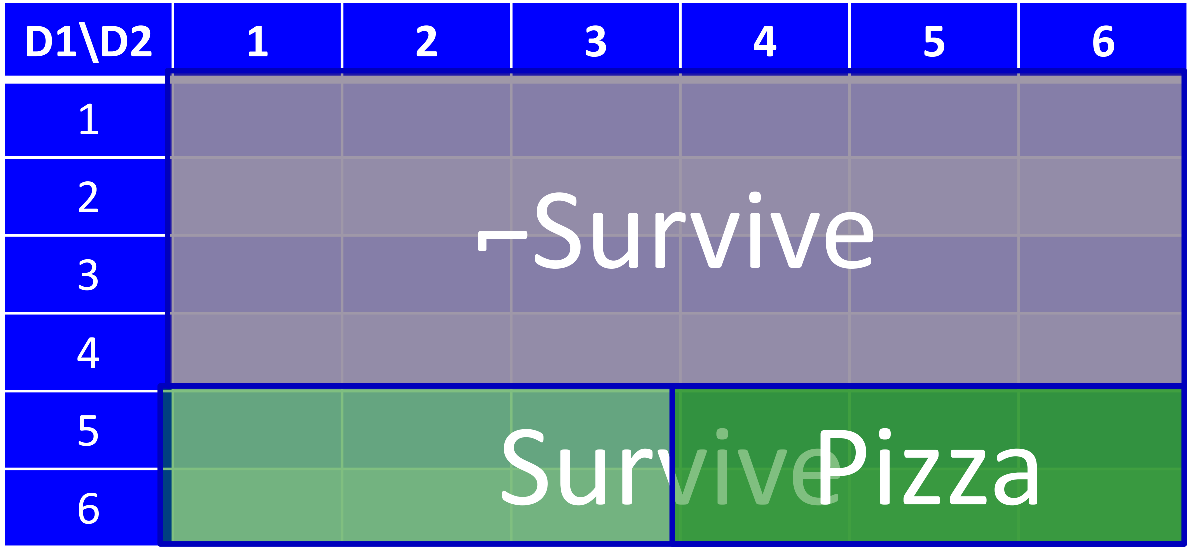
Calculating Basic Probabilities

- Probability of event ‘A’ is ratio:
  - p(A) = Area(A)/TotalArea.
  - ‘Likelihood’ that ‘A’ happens.
- Examples:
  - p(Survive) = 12/36 = 1/3.
  - p(Pizza) = 6/36 = 1/6.
  - p(¬Survive) = 1 – p(Survive) = 2/3.



Calculating Basic Probabilities

- Probability of event 'A' is ratio:
  - $p(A) = \text{Area}(A)/\text{TotalArea}$.
  - 'Likelihood' that 'A' happens.
- Examples:
  - $p(\text{Survive}) = 12/36 = 1/3$.
  - $p(\text{Pizza}) = 6/36 = 1/6$.
  - $p(\neg\text{Survive}) = 1 - p(\text{Survive}) = 2/3$.
  - $p(D_1 \text{ is even}) = 18/36 = ½$.

D1\D2	1	2	3	4	5	6
1						
2	D_1 is even					
3						
4	D_1 is even					
5						
6	D_1 is even					

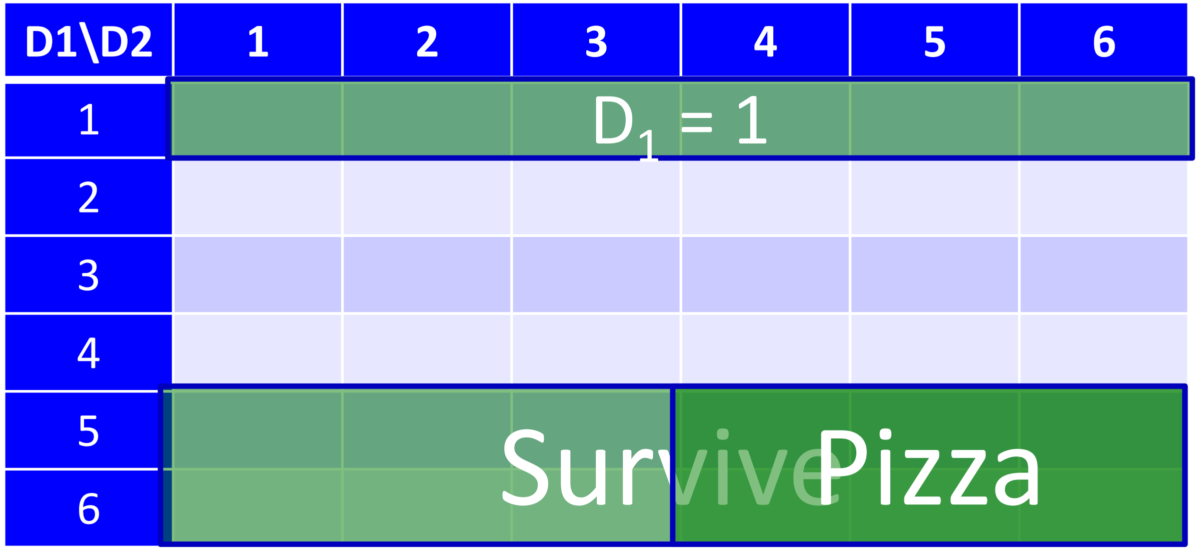
Random Variables and ‘Sum to 1’ Property

- Random variable: variable whose value depends on probability.
- Example: event (D_1 = x) depends on random variable D_1.
- Convention:
  - Often use p(x) to mean p(X = x), when random variable X is obvious.
- Sum of probabilities of random variable over entire domain is 1:
  - $\sum_x p(x) = 1$.
  - E.g, $\sum_i p(D_1 = i)$ = 1/6+1/6 + …
    = 1.

D1\D2	1	2	3	4	5	6
1	D_1 =1					
2	D_1 =2					
3	D_1 = 3					
4	D_1 = 4					
5	D_1 = 5					
6	D_1 = 6					

Joint Probability

- Joint probability: probability that A and B happen, written ‘p(A,B)’.
  - Intersection of Area(A) and Area(B).
- Examples:
  - $p(D_1 = 1, \text{Survive}) = 0$.
  - $p(\text{Survive}, \text{Pizza}) = 6/36 = 1/6$.



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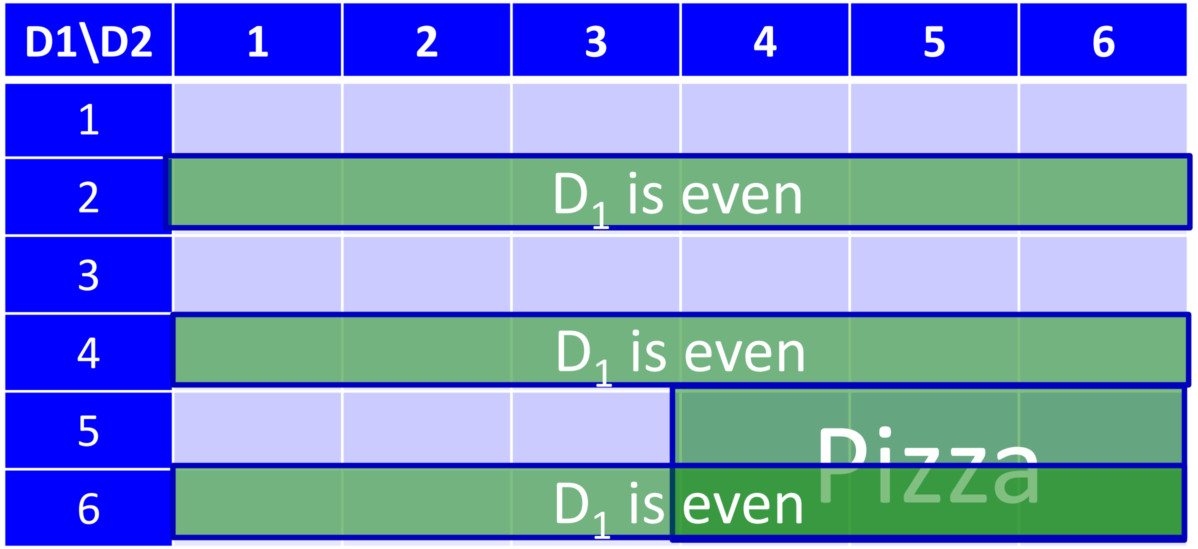
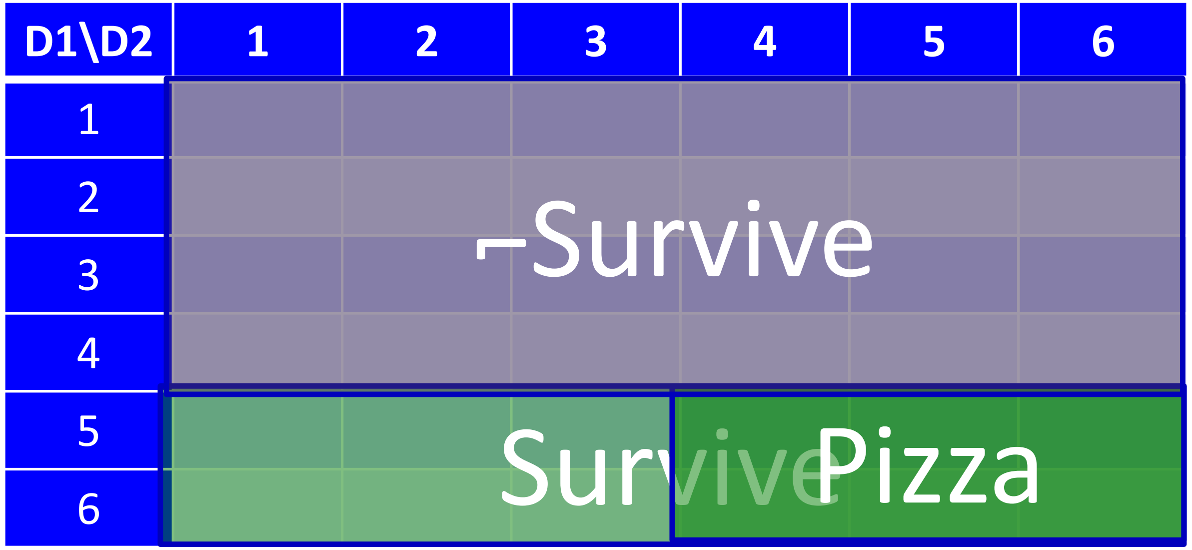
- Examples:
  - $p(D_1 = 1, \text{Survive}) = 0$.
  - $p(\text{Survive}, \text{Pizza}) = 6/36 = 1/6$.
  - $p(D_1 \text{ even}, \text{Pizza}) = 3/36 = 1/12$.



- Note: order of A and B does not matter

Conditional Probability

- Conditional probability: probability of A, if know B happened.
  - probability that A will happen *if we know* that B happens.
  - “probability of A *restricted* to scenarios where B happens”.
  - Written p(A|B), said “probability of A given B”.
- Calculation:
  - Within area of B:
    - Compute Area(A)/TotalArea.
  - p(Pizza | Survive) =



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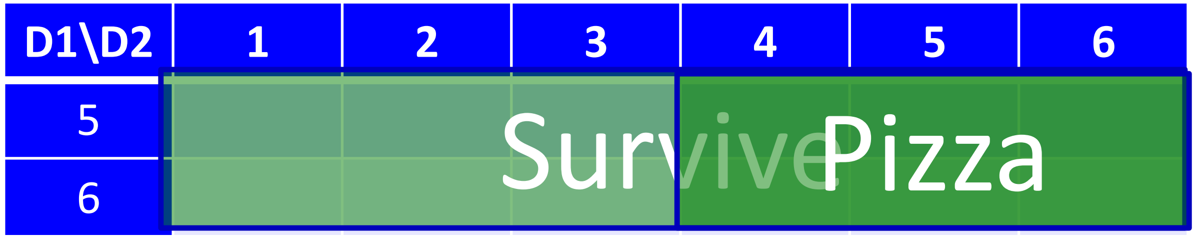
- Calculation:
  - Within area of B:
    - Compute Area(A)/TotalArea.
  - p(Pizza | Survive) =

    p(Pizza, Survive)/p(Survive) = 6/12 = ½.
  - Higher than p(Pizza, Survive) = 6/36 = 1/6.
  - More generally, p(A | B) = p(A,B)/p(B).

Geometrically: compute area of A on new space where B happened.



'Sum to 1' Properties and Bayes Rule.

- Conditional probability P(A | B) sums to one over all A:
  - $\sum_x P(x \mid B) = 1.$
  - P(Pizza | Survive) + P(¬ Pizza | Survive) = 1.
  - P(Pizza | Survive) + P(Pizza | ¬Survive) ≠ 1.
- Bayes Rule:

$$p(A \mid B) = \frac{p(B \mid A)\, p(A)}{p(B)}$$

  - P(Pizza | Survive) = P(Survive | Pizza)P(Pizza)/P(Survive)
    = (1)(1/6)/(1/3) = ½.

Back to E-mail Spam Filtering…

- Recall our spam filtering setup:
  - y_i: whether or not the e-mail was spam.
  - x_i: words/phrases/expressions in the e-mail.
- To model conditional probability, naïve Bayes uses Bayes rule:

$$p(y_i = \text{'spam'} \mid x_i) = \frac{p(x_i \mid y_i = \text{'spam'})\, p(y_i = \text{'spam'})}{p(x_i)}$$

- Easy part: $p(x_i)$ does not depend on y_i, we can ignore it.
- Easy part: $p(y_i$ = 'spam') is the probability that an e-mail is spam.
  - Count of number of times (y_i = 'spam') divided by number of objects 'n'.
  - For (complicated) proof of this (simple) fact, see:
    - http://www.cs.ubc.ca/~schmidtm/Courses/540-F14/naiveBayes.pdf

Generative Classifiers

- Hard part: $p(x_i \mid y_i = \text{'spam'})$ is the probability of seeing the words/expressions x_i if the e-mail is spam.
- This is called a generative classifier:
  - It needs to know the probability of the features, given the class.
  - You need one model that knows what spam messages look like.
  - You need a second model that knows what non-spam messages look like.
- Generative classifiers tend to work well when:
  - We have a huge number of features compared to number of objects.
- But does it need to know language to model $p(x_i \mid y_i)$???

Generative Classifiers

- To fit generative models, usually make BIG assumptions:
  - Gaussian discriminant analysis (GDA):
    - Assume that $p(x_i \mid y_i)$ follows a multivariate normal distribution.
  - Naïve Bayes (NB):
    - Assume that variables in x_i are independent of each other given y_i.
- Events A and B are independent if $p(A,B) = p(A)p(B)$.
  - Equivalently: $p(A|B) = p(A)$.
  - "Knowing B happened tells you nothing about A".
  - We use the notation:

$$A \perp B$$

Independence of Random Variables

- Random variables are independent if $p(x,y) = p(x)p(y)$ for all x and y.
  - Flipping two coins:

$$p(C_1 = \text{'heads'}, C_2 = \text{'heads'}) = p(C_1 = \text{'heads'})p(C_2 = \text{'heads'}).$$
$$p(C_1 = \text{'tails'}, C_2 = \text{'heads'}) = p(C_1 = \text{'tails'})p(C_2 = \text{'heads'}).$$
$$p(C_1 = \text{'heads'}, C_2 = \text{'tails'}) = p(C_1 = \text{'heads'})p(C_2 = \text{'tails'}).$$
$$p(C_1 = \text{'tails'}, C_2 = \text{'tails'}) = p(C_1 = \text{'tails'})p(C_2 = \text{'tails'}).$$

$$C_1 \perp C_2$$

Summary

1. Reviewed scenarios where you should trust test error estimates.
2. No free lunch theorem: there is no ‘best’ ML model.
3. Joint probability: probability of A and B happening.
4. Conditional probability: probability of A if we know B happened.
5. Generative classifiers: build a probability of seeing the features.
6. Independent variables: variables do not affect each other.

- Monday:
  - Conditional independence and naïve Bayes assumption.
  - Models that whose complexity grows with the data.