So far in class we have mostly been proving that something is true, for example that “There is a program in Vapid1 with undefined result.”

Sometimes, we want to prove that something is not true though, for example, “There is no Vapid 0 program with undefined result.” Proving something of the form “not P” is common, so we should make sure we understand how to do that.

Suppose I have some proposition $P$. I may want to prove that “$P$ is false” or “not $P$.” In symbolic notation, this is written $\neg P$.

To prove something of this form, the standard practice is to prove that “If $P$ is true then absurdity follows.” In logic, we represent absurdity with the symbol $\bot$, which is typically given the name “bottom.” So for our purposes, $\neg P$ is just an abbreviation for $P \Rightarrow \bot$. The intuition is that if $P$ is true then something is really broken in the world.

Though we haven’t explicitly stated it before, there are a lot of things that we already know are not true, meaning that they imply $\bot$. For instance, we know that the atom $true$ is not the same as the atom $false$. In our typical mathematical notation we write this as

$$true \neq false$$

But this is shorthand for

$$\neg (true = false) \quad (i.e., \text{“it is not the case that } true = false\text{.”})$$

and that is shorthand for

$$(true = false) \Rightarrow \bot \quad (i.e., \text{“if } true = false \text{ then the world is broken.”})$$

We can use knowledge of this proposition to prove that something is false about our language of Boolean Expressions:

**Proposition 1.** $true \downarrow false$.

Rewriting this symbolically, we are proving that $\neg (true \downarrow false)$, i.e., that $(true \downarrow false) \Rightarrow \bot$. We are proving an implication and we already know how to do that: assume the premise and use that to prove the conclusion.

**Proof.** Suppose that $true \downarrow false$. By inversion on $t \downarrow v$, we that for all values $v$, if $true \downarrow v$ then $v = true$. Specializing this for our assumption, it follows that $false = true$. But that’s absurd (i.e., we apply $(true = false) \Rightarrow \bot$ to deduce absurdity $\bot$).

Thus we’ve proven that it’s absurd that $true \downarrow false$ or rather

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1You may have heard the word “contradiction” as a synonym for absurdity. For technical reasons I’m avoiding that word, and I also want to assure you that what I am about to demonstrate is not “proof by contradiction.”