TFL: A Tiny Functional Language
Structural Operational Semantics
CPSC 509: Programming Language Principles

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November 5, 2019

Syntax

$x \in \textsc{Var}, \quad n \in \mathbb{Z}, \quad t \in \textsc{Term}, \quad v \in \textsc{Value} \subseteq \textsc{Pgm}, \quad F \in \textsc{Frame}, \quad r \in \textsc{Redex} \subseteq \textsc{Pgm}$
$f \in \textsc{Faulty} \subseteq \textsc{Redex}, \quad err \in \textsc{Error}, \quad c \in \textsc{Config}, \quad w \in \textsc{Canon}$
$p \in \textsc{Pgm} = \{\, t \in \textsc{Term} \mid FV(t) = \emptyset \,\}, \quad \textsc{Obs} = \{\, \mathsf{procedure}, \infty \,\} \cup \mathbb{Z} \cup \textsc{Error}$

$$
\begin{array}{rcll}
t & ::= & x \mid n \mid t\ t \mid \lambda x.t \mid \mathsf{let}\ x = t\ \mathsf{in}\ t \mid t + t \\
v & ::= & n \mid \lambda x.t \\
F & ::= & \square\ p \mid v\ \square \mid \mathsf{let}\ x = \square\ \mathsf{in}\ t \mid \square + p \mid v + \square \\
r & ::= & v\ v \mid \mathsf{let}\ x = v\ \mathsf{in}\ t \mid v + v \\
f & ::= & v_1\ v_2 & v_1 \notin \{\, \lambda x.t \in \textsc{Term} \,\} \\
 & \mid & v_1 + v_2 & \{\, v_1, v_2 \,\} \not\subseteq \mathbb{Z} \\
err & ::= & \mathsf{mismatch} \\
c & ::= & p \mid err \\
w & ::= & v \mid err
\end{array}
$$

Note: Terms are identified up to α-equivalence

$\boxed{[t/x] : \textsc{Term} \to \textsc{Term}}$ **Substitution (w/ Barendregt's Variable Convention)**

$$
\begin{aligned}
[t/x]x &= t \\
[t/x]x_0 &= x_0 \quad \text{if } x_0 \neq x \\
[t/x]n &= n \\
[t/x]t_1\ t_2 &= ([t/x]t_1)\ ([t/x]t_2) \\
[t/x]t_1 + t_2 &= ([t/x]t_1) + ([t/x]t_2) \\
[t/x]\mathsf{let}\ x_0 = t_1\ \mathsf{in}\ t_2 &= \mathsf{let}\ x_0 = ([t/x]t_1)\ \mathsf{in}\ ([t/x]t_2) \text{ if } x \neq x_0 \\
[t/x]\lambda x_0.t_0 &= \lambda x_0.[t/x]t_0 \text{ if } x \neq x_0
\end{aligned}
$$

$\boxed{(\bullet \longrightarrow \bullet) \subseteq \textsc{Pgm} \times \textsc{Config}}$ **Single-step Reduction**

$$
\frac{}{(\lambda x.t)\ v \longrightarrow [v/x]t}\ (\text{sapp}) \qquad
\frac{}{\mathsf{let}\ x = v\ \mathsf{in}\ t \longrightarrow [v/x]t}\ (\text{slet}) \qquad
\frac{n_3 = n_1 + n_2}{n_1 + n_2 \longrightarrow n_3}\ (\text{splus})
$$

$$
\frac{}{f \longrightarrow \mathsf{mismatch}}\ (\text{serr}) \qquad
\frac{p_1 \longrightarrow p_2}{F[p_1] \longrightarrow F[p_2]}\ (\text{Fstep}) \qquad
\frac{p_1 \longrightarrow err}{F[p_1] \longrightarrow err}\ (\text{Ferr})
$$

$\boxed{(\bullet \xrightarrow{\infty}\ ) \subseteq \textsc{Pgm}}$ **Divergence**

$$\frac{p_1 \longrightarrow p_2 \quad p_2 \xrightarrow{\infty}}{p_1 \xrightarrow{\infty}}$$

$$eval : \textsc{Pgm} \stackrel{\text{dens}}{\rightarrow} \textsc{Obs}$$

$$\begin{aligned} eval(p) &= n && \text{if } p \longrightarrow^* n \\ eval(p) &= \textsf{procedure} && \text{if } p \longrightarrow^* \lambda x.t \\ eval(p) &= err && \text{if } p \longrightarrow^* err \\ eval(p) &= \infty && \text{if } p \xrightarrow{\infty} \end{aligned}$$

Proposition 1 (Principle of Rule Coinduction for $(\bullet \xrightarrow{\infty}\ )$)**.**
Let $S \subseteq \textsc{Pgm}$*. Then* $\forall p \in S. p \xrightarrow{\infty}$ *if* $\forall p_1 \in S. \exists p_2 \in \textsc{Pgm}.\, p \longrightarrow p_2 \wedge p_2 \in S$.

$\boxed{\textsc{Safe} \subseteq \textsc{Config}}$ **Safety**

$$\frac{}{v \in \textsc{Safe}} \qquad \frac{}{err \in \textsc{Safe}} \qquad \frac{p_1 \longrightarrow c_2 \quad c_2 \in \textsc{Safe}}{p_1 \in \textsc{Safe}}$$

Proposition 2 (Principle of Rule Coinduction for $\textsc{Safe}$)**.** *Let* $S \subseteq \textsc{Config}$*. Then* $S \subseteq \textsc{Safe}$ *if for every* $c \in S$ *one of the following holds:*

1. $c \in \textsc{Error}$;
2. $c \in \textsc{Value}$;
3. $\exists p_1 \in \textsc{Pgm}, c_2 \in \textsc{Config}.\, c = p_1 \wedge p_1 \to c_2 \wedge c_2 \in S$.

Proposition 3 (Safety)**.** $\textsc{Pgm} \subseteq \textsc{Safe}$.

$\boxed{(\bullet \xrightarrow{\star} \bullet) \subseteq \textsc{Config} \times \textsc{Canon}}$ **Convergence**

$$\frac{}{v \xrightarrow{\star} v} \qquad \frac{}{err \xrightarrow{\star} err} \qquad \frac{p_1 \longrightarrow c_2 \quad c_2 \xrightarrow{\star} w_3}{p_1 \xrightarrow{\star} w_3}$$

Proposition 4. $\forall c_1 \in \textsc{Config}, w_2 \in \textsc{Canon}.\, c_1 \longrightarrow^* w_2 \Leftrightarrow c_1 \xrightarrow{\star} w_2$.