Proof of a Forward-Reasoning Principle for B

Wednesday 20th March, 2024

Proposition 1. $\forall r_1, r_2, r_3 \in \text{TREE}[\text{ATOM}].\, r_1, r_2, r_3 \in \text{TERM} \Rightarrow \textit{if } r_1 \textit{ then } r_2 \textit{ else } r_3 \in \text{TERM}$.

PROOF:

1. SUFFICES ASSUME:
   1. r_1 set,
   2. r_2 set,
   3. r_3 set,
   4. $r_1 \in \text{TREE}[\text{ATOM}]$,
   5. $r_2 \in \text{TREE}[\text{ATOM}]$,
   6. $r_3 \in \text{TREE}[\text{ATOM}]$,
   7. $r_1 \in \text{TERM}$,
   8. $r_2 \in \text{TERM}$,
   9. $r_3 \in \text{TERM}$.

   PROVE: if r_1 then r_2 else $r_3 \in \text{TERM}$.

   by $\Rightarrow$I and $\forall$I, repeatedly.
2. PICK
   1. $\mathcal{D}_1$ set,
   2. $\mathcal{D}_1 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}]$,
   3. $\mathcal{D}_1 :: r_1$.

   2.1. $\forall Q.\, Q \in \text{TERM} \Leftrightarrow Q \in \text{TREE}[\text{ATOM}] \wedge \exists \mathcal{D}_1 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}].\, \mathcal{D}_1 :: Q$ by def'n TERM

   2.2. $r_1 \in \text{TERM} \Leftrightarrow r_1 \in \text{TREE}[\text{ATOM}] \wedge \exists \mathcal{D}_1 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}].\, \mathcal{D}_1 :: r_1$ by $\forall$E with 2.1 and assumption 1:1.

   2.3. $r_1 \in \text{TERM} \Rightarrow r_1 \in \text{TREE}[\text{ATOM}] \wedge \exists \mathcal{D}_1 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}].\, \mathcal{D}_1 :: r_1$ by $\wedge$E1 with 2.2.

   2.4. $r_1 \in \text{TREE}[\text{ATOM}] \wedge \exists \mathcal{D}_1 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}].\, \mathcal{D}_1 :: r_1$ by $\Rightarrow$E with 2.3 and assumption 1:7

   2.5. $\exists \mathcal{D}_1 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}].\, \mathcal{D}_1 :: r_1$ by $\wedge$E2 with 2.4

   2.6. Q.E.D. by $\exists$E with 2.5
3. PICK
   1. $\mathcal{D}_2$ set,
   2. $\mathcal{D}_2 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}]$,
   3. $\mathcal{D}_2 :: r_2$.

   Analogous to the proof of 2
4. PICK
   1. $\mathcal{D}_3$ set,
   2. $\mathcal{D}_3 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}]$,
   3. $\mathcal{D}_3 :: r_3$.

   Analogous to the proof of 2
5. if r_1 then r_2 else $r_3 \in \text{TREE}[\text{ATOM}]$ by def'n of TREE[ATOM]

LET: $$\mathcal{D}_4 := \frac{\overset{\mathcal{D}_1}{r_1} \quad \overset{\mathcal{D}_2}{r_2} \quad \overset{\mathcal{D}_3}{r_3}}{\text{if } r_1 \text{ then } r_2 \text{ else } r_3}$$

6. 1. $\mathcal{D}_4$ set
   2. $\mathcal{D}_4 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}]$

   by def'n of $\text{DERIV}[\mathcal{R}_{\text{TERM}}]$ with assumptions 2:2,3:2,4:2
7. $\mathcal{D}_4 ::$ if r_1 then r_2 else r_3 by def'n of $\mathcal{D}_4$

8. $\exists \mathcal{D}_4 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}].\, \mathcal{D}_4 :: \textsf{if}\ r_1\ \textsf{then}\ r_2\ \textsf{else}\ r_3$ by $\exists$I with 6,7
9. $\textsf{if}\ r_1\ \textsf{then}\ r_2\ \textsf{else}\ r_3 \in \text{TERM}$
   9.1. $\forall Q.\, Q \in \text{TERM} \Leftrightarrow Q \in \text{TREE}[\text{ATOM}] \wedge \exists \mathcal{D} \in \text{DERIV}[\mathcal{R}_{\text{TERM}}].\, \mathcal{D} :: Q$ by def'n TERM
   LET: $r_4 := \textsf{if}\ r_1\ \textsf{then}\ r_2\ \textsf{else}\ r_3$
   9.2. $r_4 \in \text{TERM} \Leftrightarrow r_4 \in \text{TREE}[\text{ATOM}] \wedge \exists \mathcal{D} \in \text{DERIV}[\mathcal{R}_{\text{TERM}}].\, \mathcal{D} :: r_4$ by $\forall$E with 9.1 and assumption 1:1.
   9.3. $r_4 \in \text{TREE}[\text{ATOM}] \wedge \exists \mathcal{D} \in \text{DERIV}[\mathcal{R}_{\text{TERM}}].\, \mathcal{D} :: r_4 \Rightarrow r_4 \in \text{TERM}$ by $\wedge$E2 with 9.2.
   9.4. $\mathcal{D}_4 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}] \wedge \mathcal{D}_4 :: r_4$
   9.5. $\exists \mathcal{D}_4 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}].\, \mathcal{D}_4 :: r_4$ by $\exists$I with 6,9.4
   9.6. $r_4 \in \text{TREE}[\text{ATOM}] \wedge \exists \mathcal{D}_4 \in \text{DERIV}[\mathcal{R}_{\text{TERM}}].\, \mathcal{D}_4 :: r_4$ by $\wedge$I with 5,9.5
   9.7. Q.E.D. by $\Rightarrow$E with 9.3, 9.6