B: Language of Boolean Expressions
Abstract Syntax, Abstracted
CPSC 509: Programming Language Principles

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Syntax

Presented as BNF

$$t \in \textsc{Term}, \quad v \in \textsc{Value},$$
$$t \quad ::= \quad \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}\ t\ \mathsf{then}\ t\ \mathsf{else}\ t$$

Equivalent Presentation as Inductive Definitions

$$a \in \textsc{Atom},$$
$$\mathsf{true}, \mathsf{false}, \mathsf{if} \in \textsc{Atom}, \quad \mathsf{true} \neq \mathsf{false}, \quad \mathsf{true} \neq \mathsf{if\text{-}then\text{-}else}, \quad \mathsf{false} \neq \mathsf{if\text{-}then\text{-}else}$$
$$r \in \textsc{Tree}[\textsc{Atom}]$$

$\boxed{\textsc{Term} \subseteq \textsc{Tree}[\textsc{Atom}]}$

$$\frac{}{\mathsf{true} \in \textsc{Term}}\ (\text{rtrue}) \qquad \frac{}{\mathsf{false} \in \textsc{Term}}\ (\text{rfalse}) \qquad \frac{r_1 \in \textsc{Term} \quad r_2 \in \textsc{Term} \quad r_3 \in \textsc{Term}}{\mathsf{if}\ r_1\ \mathsf{then}\ r_2\ \mathsf{else}\ r_3 \in \textsc{Term}}\ (\text{rif})$$

Same Syntax, but Factored

$$\begin{aligned} \textit{ttrue} &: \{\,()\,\} \to \textsc{Tree}[\textsc{Atom}], \\ \textit{tfalse} &: \{\,()\,\} \to \textsc{Tree}[\textsc{Atom}], \\ \textit{tif} &: \textsc{Tree}[\textsc{Atom}] \times \textsc{Tree}[\textsc{Atom}] \times \textsc{Tree}[\textsc{Atom}] \to \textsc{Tree}[\textsc{Atom}] \end{aligned}$$

$$\begin{aligned} \textit{ttrue}() &= \textsf{true} \\ \textit{tfalse}() &= \textsf{false} \\ \textit{tif}(r_1, r_2, r_3) &= \textsf{if } r_1 \textsf{ then } r_2 \textsf{ else } r_3 \end{aligned}$$

Some Properties

Proposition 1 (Tree Functions Serve As Constructors)**.**

1. $\forall r_1, r_2, r_3 \in \textsc{Tree}[\textsc{Atom}].\ \textit{ttrue}() \neq \textit{tfalse}() \wedge \textit{ttrue}() \neq \textit{tif}(r_1, r_2, r_3) \wedge \textit{tfalse}() \neq \textit{tif}(r_1, r_2, r_3)$;
2. $\forall r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{23}.\ \textit{tif}(r_{11}, r_{12}, r_{13}) = \textit{tif}(r_{21}, r_{22}, r_{23}) \Rightarrow r_{11} = r_{21} \wedge r_{12} = r_{22} \wedge r_{13} = r_{23}$.

$\boxed{\textsc{TTerm} \subseteq \textsc{Tree}[\textsc{Atom}]}$

$$\frac{}{\textit{ttrue}() \in \textsc{TTerm}}\ (\text{ttrue}) \qquad \frac{}{\textit{tfalse}() \in \textsc{TTerm}}\ (\text{tfalse}) \qquad \frac{r_1 \in \textsc{TTerm} \quad r_2 \in \textsc{TTerm} \quad r_3 \in \textsc{TTerm}}{\textit{tif}(r_1, r_2, r_3) \in \textsc{TTerm}}\ (\text{tif})$$

Different Syntax, but Who's Counting?

$$\begin{aligned} \textit{ntrue} &: \{\,()\,\} \to \mathbb{N}, \\ \textit{nfalse} &: \{\,()\,\} \to \mathbb{N}, \\ \textit{nif} &: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N} \end{aligned}$$

$$\begin{aligned} \textit{ntrue}() &= 2^0 = 1 \\ \textit{nfalse}() &= 2^1 = 2 \\ \textit{nif}(n_1, n_2, n_3) &= 2^2 * 3^{n_1} * 5^{n_2} * 7^{n_3} \end{aligned}$$

Some Properties

Proposition 2 (Number Functions Serve As Constructors)**.**

1. $\forall n_1, n_2, n_3 \in \mathbb{N}.\ \textit{ntrue}() \neq \textit{nfalse}() \wedge \textit{ntrue}() \neq \textit{nif}(n_1, n_2, n_3) \wedge \textit{nfalse}() \neq \textit{nif}(n_1, n_2, n_3)$;
2. $\forall n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}.\ \textit{nif}(n_{11}, n_{12}, n_{13}) = \textit{nif}(n_{21}, n_{22}, n_{23}) \Rightarrow n_{11} = n_{21} \wedge n_{12} = n_{22} \wedge n_{13} = n_{23}$.

$\boxed{\textsc{NTerm} \subseteq \mathbb{N}}$

$$\frac{}{\textit{ntrue}() \in \textsc{NTerm}}\ (\text{ntrue}) \qquad \frac{}{\textit{nfalse}() \in \textsc{NTerm}}\ (\text{nfalse})$$

$$\frac{n_1 \in \textsc{NTerm} \quad n_2 \in \textsc{NTerm} \quad n_3 \in \textsc{NTerm}}{\textit{nif}(n_1, n_2, n_3) \in \textsc{NTerm}}\ (\text{nif})$$

Term and NTerm are Equivalent in All Ways That Matter

$$toNTerm : \text{TTERM} \to \text{NTERM}$$
$$\begin{aligned} toNTerm(ttrue()) &= ntrue() \\ toNTerm(tfalse()) &= nfalse() \\ toNTerm(tif(r_1, r_2, r_3)) &= nif(toNTerm(r_1), toNTerm(r_2), toNTerm(r_3)) \end{aligned}$$

$$toTTerm : \text{TTERM} \to \text{NTERM}$$
$$\begin{aligned} toTTerm(ttrue()) &= ntrue() \\ toTTerm(tfalse()) &= nfalse() \\ toTTerm(tif(n_1, n_2, n_3)) &= nif(toTTerm(n_1), toTTerm(n_2), toTTerm(n_3)) \end{aligned}$$

Proposition 3 (TTERM and NTERM are Isomorphic)**.**

1. $\forall n \in \text{NTERM}.\ toNTerm(toTTerm(n)) = n$;
2. $\forall r \in \text{TTERM}.\ toTTerm(toNTerm(r)) = r$.

1 *Initial Algebra* Interpretation of BNF

Presented as BNF

$$t \in \text{TERM}, \quad v \in \text{VALUE},$$
$$t \quad ::= \quad \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}\ t\ \mathsf{then}\ t\ \mathsf{else}\ t$$

Interpretation Let A be a set

$$\begin{aligned} \mathsf{true} &: \{\,()\,\} \to A, \\ \mathsf{false} &: \{\,()\,\} \to A, \\ \mathsf{if} &: A \times A \times A \to A \end{aligned}$$

1. $\forall a_1, a_2, a_3 \in A.\ \mathsf{true}() \neq \mathsf{false}() \wedge \mathsf{true}() \neq \mathsf{if}\,(a_1, a_2, a_3) \wedge \mathsf{false}() \neq \mathsf{if}\,(a_1, a_2, a_3)$;
2. $\forall a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}.\ \mathsf{if}\,(a_{11}, a_{12}, a_{13}) = \mathsf{if}\,(a_{21}, a_{22}, a_{23}) \Rightarrow a_{11} = a_{21} \wedge a_{12} = a_{22} \wedge a_{13} = a_{23}$.

$\boxed{\text{TERM} \subseteq A}$

$$\frac{}{\mathsf{true}() \in \text{TERM}}\ \text{(atrue)} \qquad \frac{}{\mathsf{false}() \in \text{TERM}}\ \text{(afalse)} \qquad \frac{a_1 \in \text{TERM} \quad a_2 \in \text{TERM} \quad a_3 \in \text{TERM}}{\mathsf{if}\,(a_1, a_2, a_3) \in \text{TERM}}\ \text{(aif)}$$

Proposition 4 (Principle of Size Induction for $t \in \text{TERM}$)**.**
Let Φ be a predicate on TERMs t. Then the following holds:

$$\Big(\forall t_1 \in \text{TERM}.\ \Big(\forall t_2 \in \text{TERM}.\ size(t_2) < size(t_1) \Rightarrow \Phi(t_2)\Big) \Rightarrow \Phi(t_1)\Big) \Rightarrow \forall t \in \text{TERM}.\ \Phi(t)$$