Boolean Expressions
Structural Operational Semantics
CPSC 509: Programming Language Principles

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Syntax

$$t \in \textsc{Term}, \quad v \in \textsc{Value}$$
$$t ::= \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}\ t\ \mathsf{then}\ t\ \mathsf{else}\ t$$
$$v ::= \mathsf{true} \mid \mathsf{false}$$

Small-step Semantics

$\boxed{\longrightarrow\ \subseteq \textsc{Term} \times \textsc{Term}}$

$$\frac{t_1 \longrightarrow t_1'}{\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \longrightarrow \mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3}\ \text{(sif)}$$

$$\frac{}{\mathsf{if}\ \mathsf{true}\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \longrightarrow t_2}\ \text{(sif-t)} \qquad \frac{}{\mathsf{if}\ \mathsf{false}\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \longrightarrow t_3}\ \text{(sif-f)}$$

$\boxed{\longrightarrow^*\ \subseteq \textsc{Term} \times \textsc{Term}}$

$$\frac{}{t_1 \longrightarrow^* t_2}\ \text{(incl)}\ \ t_1 \longrightarrow t_2 \qquad \frac{}{t \longrightarrow^* t}\ \text{(refl)} \qquad \frac{t_1 \longrightarrow^* t_2 \quad t_2 \longrightarrow^* t_3}{t_1 \longrightarrow^* t_3}\ \text{(trans)}$$

$$\textsc{Pgm} = \textsc{Term}, \quad \textsc{Obs} = \textsc{Value}$$

$$eval_{ss} : \textsc{Pgm} \to \textsc{Obs}$$
$$eval_{ss}(t) = \mathsf{true} \text{ if } t \longrightarrow^* \mathsf{true}$$
$$eval_{ss}(t) = \mathsf{false} \text{ if } t \longrightarrow^* \mathsf{false}$$

Reasoning Principles

Proposition 1 (Forward Reasoning for $t \longrightarrow t'$).

(sif) $\forall t_{11}, t_{12}, t_2, t_3 \in \textsc{Term}.\, t_{11} \longrightarrow t_{12} \Rightarrow \textit{if}\, t_{11}\, \textit{then}\, t_2\, \textit{else}\, t_3 \longrightarrow \textit{if}\, t_{12}\, \textit{then}\, t_2\, \textit{else}\, t_3;$

(sif-t) $\forall t_2, t_3 \in \textsc{Term}.\, \textit{if true then}\, t_2\, \textit{else}\, t_3 \longrightarrow t_2;$

(sif-f) $\forall t_2, t_3 \in \textsc{Term}.\, \textit{if false then}\, t_2\, \textit{else}\, t_3 \longrightarrow t_3;$

Proposition 2 (Backward Reasoning for $t \longrightarrow t'$, Distinguishing t).

(inv-ss-true) $\forall t' \in \textsc{Term}.\, \textit{true} \not\longrightarrow t';$

(inv-ss-false) $\forall t' \in \textsc{Term}.\, \textit{false} \not\longrightarrow t';$

(inv-ss-if) $\forall t_1, t_2, t_3, t_4 \in \textsc{Term}.\, \textit{if}\, t_1\, \textit{then}\, t_2\, \textit{else}\, t_3 \longrightarrow t_4 \Rightarrow$
$(t_1 = \textit{true} \wedge t_4 = t_2) \vee (t_1 = \textit{false} \wedge t_4 = t_3) \vee (\exists t_5 \in \textsc{Term}.\, t_1 \longrightarrow t_5 \wedge t_4 = \textit{if}\, t_5\, \textit{then}\, t_2\, \textit{else}\, t_3).$

Proposition 3 (Forward Reasoning for $t \longrightarrow^* t'$).

(incl) $\forall t_1, t_2 \in \textsc{Term}.\, t_1 \longrightarrow t_2 \Rightarrow t_1 \longrightarrow^* t_2;$

(refl) $\forall t \in \textsc{Term}.\, t \longrightarrow^* t;$

(trans) $\forall t_1, t_2, t_3 \in \textsc{Term}.\, t_1 \longrightarrow^* t_2 \wedge t_2 \longrightarrow^* t_3 \Rightarrow t_1 \longrightarrow^* t_3;$

Proposition 4 (Backward Reasoning for $t \longrightarrow^* t'$.).
$\forall t, t' \in \textsc{Term}.\, t \longrightarrow^* t' \Rightarrow t = t' \vee t \longrightarrow t' \vee \exists t_0 \in \textsc{Term}.\, t \longrightarrow^* t_0 \wedge t_0 \longrightarrow^* t'.$

Proposition 5 (Backward Reasoning for $t \longrightarrow^* t'$, Distinguishing t.). [1]

(inv-ms-true) $\forall t' \in \textsc{Term}.\, \textit{true} \longrightarrow^* t' \Rightarrow \textit{true} = t' \vee \textit{true} \longrightarrow t' \vee \exists t_0 \in \textsc{Term}.\, \textit{true} \longrightarrow^* t_0 \wedge t_0 \longrightarrow^* t'.$

(inv-ms-false) $\forall t' \in \textsc{Term}.\, \textit{false} \longrightarrow^* t' \Rightarrow \textit{false} = t' \vee \textit{false} \longrightarrow t' \vee \exists t_0 \in \textsc{Term}.\, \textit{false} \longrightarrow^* t_0 \wedge t_0 \longrightarrow^* t'.$

(inv-ms-if) $\forall t_1, t_2, t_3, t' \in \textsc{Term}.\, \textit{if}\, t_1\, \textit{then}\, t_2\, \textit{else}\, t_3 \longrightarrow^* t' \Rightarrow$
$\textit{if}\, t_1\, \textit{then}\, t_2\, \textit{else}\, t_3 = t' \vee \textit{if}\, t_1\, \textit{then}\, t_2\, \textit{else}\, t_3 \longrightarrow t' \vee$
$\exists t_0 \in \textsc{Term}.\, \textit{if}\, t_1\, \textit{then}\, t_2\, \textit{else}\, t_3 \longrightarrow^* t_0 \wedge t_0 \longrightarrow^* t'.$

Proposition 6 (Principle of Induction on Derivations $\mathcal{D} \in \textsc{Deriv}[\mathcal{R}_{\longrightarrow^*}]$).
Let Φ be a property of derivations $\mathcal{D} \in \textsc{Deriv}[\mathcal{R}_{\longrightarrow^*}]$. Then

$$\left(\forall t_1, t_2 \in \textsc{Term}.\, t_1 \longrightarrow t_2 \Rightarrow \Phi\left(\frac{t_1 \longrightarrow t_2}{t_1 \longrightarrow^* t_2}\ \textit{(incl)}\right)\right) \wedge$$
$$\left(\forall t \in \textsc{Term}.\, \Phi\left(\frac{}{t \longrightarrow^* t}\ \textit{(refl)}\right)\right) \wedge$$
$$\Bigg(\forall t_1, t_2, t_3 \in \textsc{Term}.\, \forall \mathcal{D}_1, \mathcal{D}_2 \in \textsc{Deriv}[\mathcal{R}_{\longrightarrow^*}].\, (\mathcal{D}_1 :: t_1 \longrightarrow^* t_2) \wedge (\mathcal{D}_2 :: t_2 \longrightarrow^* t_3) \wedge$$
$$\Phi(\mathcal{D}_1) \wedge \Phi(\mathcal{D}_2) \Rightarrow \Phi\left(\frac{\overset{\mathcal{D}_1}{t_1 \longrightarrow^* t_2} \quad \overset{\mathcal{D}_2}{t_2 \longrightarrow^* t_3}}{t_1 \longrightarrow^* t_3}\ \textit{(trans)}\right)\Bigg)$$
$$\Rightarrow \forall \mathcal{D} \in \textsc{Deriv}[\mathcal{R}_{\longrightarrow^*}].\Phi(\mathcal{D}).$$

Proposition 7 (Principle of Rule Induction on $\longrightarrow^*$).
Let Φ be a property of $\textsc{Term} \times \textsc{Term}$. *Then*

$$(\forall t_1, t_2 \in \textsc{Term}.\, t_1 \longrightarrow t_2 \Rightarrow \Phi(t_1, t_2)) \wedge$$
$$(\forall t \in \textsc{Term}.\, \Phi\,(t, t)) \wedge$$
$$(\forall t_1, t_2, t_3 \in \textsc{Term}.\, \Phi(t_1, t_2) \wedge \Phi(t_2, t_3) \Rightarrow \Phi\,(t_1, t_3))$$
$$\Rightarrow \forall t_1, t_2 \in \textsc{Term}. t_1 \longrightarrow^* t_2 \Rightarrow \Phi(t_1, t_2).$$

[1] Notice how essentially *nothing* was gained by distinguishing t. Deeper reasoning will get us better principles.