B: Language of Boolean Expressions
Big-step Semantics
CPSC 509: Programming Language Principles

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21 September 2015
(**Time Stamp**: 14:39, Tuesday 12th March, 2024)

Syntax

$$t \in \textsc{Term}, \quad v \in \textsc{Value},$$
$$t ::= \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}\ t\ \mathsf{then}\ t\ \mathsf{else}\ t$$
$$v ::= \mathsf{true} \mid \mathsf{false}$$

Big-step Semantics

$\boxed{\Downarrow \subseteq \textsc{Term} \times \textsc{Value}}$

$$\frac{}{\mathsf{true} \Downarrow \mathsf{true}}\ \text{(etrue)} \qquad \frac{}{\mathsf{false} \Downarrow \mathsf{false}}\ \text{(efalse)} \qquad \frac{t_1 \Downarrow \mathsf{true} \quad t_2 \Downarrow v_2}{\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Downarrow v_2}\ \text{(eif-t)}$$

$$\frac{t_1 \Downarrow \mathsf{false} \quad t_3 \Downarrow v_3}{\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Downarrow v_3}\ \text{(eif-f)}$$

$$\textsc{Pgm} = \textsc{Term}, \quad \textsc{Obs} = \textsc{Value}$$

$$eval_B : \textsc{Pgm} \to \textsc{Obs}$$
$$eval_B(t) = \mathsf{true} \text{ if } t \Downarrow \mathsf{true}$$
$$eval_B(t) = \mathsf{false} \text{ if } t \Downarrow \mathsf{false}$$

Reasoning Principles

Proposition 1 (Forward-Reasoning for $\textsc{Term} \subseteq \textsc{Tree}[\textsc{Atom}]$).

(rtrue) $\textsf{true} \in \textsc{Term}$;

(rfalse) $\textsf{false} \in \textsc{Term}$;

(rif) $\forall r_1, r_2, r_3 \in \textsc{Tree}[\textsc{Atom}].\, r_1, r_2, r_3 \in \textsc{Term} \Rightarrow \textsf{if}\, r_1\, \textsf{then}\, r_2\, \textsf{else}\, r_3\, \textsf{in}\, \textsc{Term}$.

Proposition 2 (Backward-Reasoning for $\textsc{Term} \subseteq \textsc{Tree}[\textsc{Atom}]$).
$\forall r \in \textsc{Tree}[\textsc{Atom}].\, r \in \textsc{Term} \Rightarrow$
$r = \textsf{true} \lor r = \textsf{false} \lor (\exists r_1, r_2, r_3 \in \textsc{Tree}[\textsc{Atom}].\, r_1, r_2, r_3 \in \textsc{Term} \land r = \textsf{if}\, r_1\, \textsf{then}\, r_2\, \textsf{else}\, r_3\, \textsf{in}\, \textsc{Term})$.

Proposition 3 (Backward-Reasoning for $\textsc{Term} \subseteq \textsc{Tree}[\textsc{Atom}]$, Distinguishing $\textsc{Term}$).

(inv-t-true) $\textsf{true} \in \textsc{Term} \Rightarrow \top$;

(inv-t-false) $\textsf{false} \in \textsc{Term} \Rightarrow \top$;

(inv-t-if) $\forall r_1, r_2, r_3 \in \textsc{Tree}[\textsc{Atom}].\, \textsf{if}\, r_1\, \textsf{then}\, r_2\, \textsf{else}\, r_3\, \textsf{in}\, \textsc{Term} \Rightarrow r_1, r_2, r_3 \in \textsc{Term}$.

Proposition 4 (Principle of Derivation Induction for $\textsc{Term} \subseteq \textsc{Tree}[\textsc{Atom}]$).
Let Φ be a predicate on derivations $\mathcal{D} \in \textsc{Deriv}[\mathcal{R}_{\textsc{Term}}]$. Then $\Phi(\mathcal{D})$ holds for all derivations $\mathcal{D}$ if:

1. $\Phi\left(\dfrac{}{\textsf{true} \in \textsc{Term}}\ \textit{(rtrue)}\right)$ *holds;*
2. $\Phi\left(\dfrac{}{\textsf{false} \in \textsc{Term}}\ \textit{(rfalse)}\right)$ *holds;*
3. $\forall \mathcal{D}_1, D_2, D_3 \in \textsc{Deriv}, r_1, r_2, r_3 \in \textsc{Tree}[\textsc{Atom}].$
$\mathcal{D}_1 :: r_1 \land \mathcal{D}_2 :: r_2 \land \mathcal{D}_3 :: r_3 \land \Phi(\mathcal{D}_1) \land \Phi(\mathcal{D}_2) \land \Phi(\mathcal{D}_3) \Rightarrow$
$$\Phi\left(\dfrac{\begin{matrix}\mathcal{D}_1 \\ r_1 \in \textsc{Term}\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ r_2 \in \textsc{Term}\end{matrix} \quad \begin{matrix}\mathcal{D}_3 \\ r_3 \in \textsc{Term}\end{matrix}}{\textsf{if}\, r_1\, \textsf{then}\, r_2\, \textsf{else}\, r_3 \in \textsc{Term}}\ \textit{(rif)}\right).$$

Proposition 5 (Principle of Rule Induction for $\textsc{Term} \subseteq \textsc{Tree}[\textsc{Atom}]$).
Let Φ be a predicate on $\textsc{Tree}[\textsc{Atom}]$. Then $\Phi(r)$ holds for all $t \in \textsc{Term}$:

1. $\Phi(\textsf{true})$ *holds;*
2. $\Phi(\textsf{false})$ *holds;*
3. *For all $r_1, r_2, r_3 \in \textsc{Tree}[\textsc{Atom}]$, if $\Phi(r_1)$, $\Phi(r_2)$, and $\Phi(r_3)$ hold then $\Phi(\textsf{if}\, r_1\, \textsf{then}\, r_2\, \textsf{else}\, r_3)$ holds.*

Proposition 6 (Forward-Reasoning for $\textsc{Value} \subseteq \textsc{Tree}[\textsc{Atom}]$).

(vtrue) $\textsf{true} \in \textsc{Value}$;

(vfalse) $\textsf{false} \in \textsc{Value}$.

Proposition 7 (Backward-Reasoning for $\textsc{Value} \subseteq \textsc{Tree}[\textsc{Atom}]$). $\forall r \in \textsc{Tree}[\textsc{Atom}].\, r \in \textsc{Value} \Rightarrow r = \textsf{true} \lor r = \textsf{false}$.

Proposition 8 (Principle of Derivation Induction for $\textsc{Value} \subseteq \textsc{Tree}[\textsc{Atom}]$).
Let Φ be a predicate on derivations $\mathcal{D} \in \textsc{Deriv}[\mathcal{R}_{\textsc{Value}}]$. Then $\Phi(\mathcal{D})$ holds for all derivations $\mathcal{D}$ if:

1. $\Phi\left(\dfrac{}{\textsf{true} \in \textsc{Value}}\ \textit{(vtrue)}\right)$ *holds;*
2. $\Phi\left(\dfrac{}{\textsf{false} \in \textsc{Value}}\ \textit{(vfalse)}\right)$ *holds.*

Proposition 9 (Principle of Rule Induction for $\textsc{Value} \subseteq \textsc{Tree}[\textsc{Atom}]$).
Let Φ be a predicate on $\textsc{Tree}[\textsc{Atom}]$. Then $\Phi(r)$ holds for all $v \in \textsc{Value}$ if:

1. $\Phi(\textsf{true})$ *holds;*
2. $\Phi(\textsf{false})$ *holds.*

Proposition 10 (Forward Reasoning for $(\cdot \Downarrow \cdot) \subseteq \textsc{Term} \times \textsc{Value}$)**.**

(etrue) $\textit{true} \Downarrow \textit{true}$;

(efalse) $\textit{false} \Downarrow \textit{false}$;

(eif-t) $\forall t_1, t_2, t_3 \in \textsc{Term}, v_2 \in \textsc{Value}.\, t_1 \Downarrow \textit{true} \wedge t_2 \Downarrow v_2 \Rightarrow \textit{if}\ t_1\ \textit{then}\ t_2\ \textit{else}\ t_3 \Downarrow v_2.$

(eif-f) $\forall t_1, t_2, t_3 \in \textsc{Term}, v_e \in \textsc{Value}.\, t_1 \Downarrow \textit{false} \wedge t_3 \Downarrow v_3 \Rightarrow \textit{if}\ t_1\ \textit{then}\ t_2\ \textit{else}\ t_3 \Downarrow v_3.$

Proposition 11 (Backward Reasoning for $(\cdot \Downarrow \cdot) \subseteq \textsc{Term} \times \textsc{Value}$)**.**

$$\begin{aligned}\forall t \in \textsc{Term}, v \in \textsc{Value}.\, t \Downarrow v \Rightarrow\ & \\ & (t = \textit{true} \wedge v = \textit{true}) \vee \\ & (t = \textit{false} \wedge v = \textit{false}) \vee \\ & (\exists t_1, t_2, t_3 \in \textsc{Term}.\, t = \textit{if}\ t_1\ \textit{then}\ t_2\ \textit{else}\ t_3 \wedge t_1 \Downarrow \textit{true} \wedge t_2 \Downarrow v) \vee \\ & (\exists t_1, t_2, t_3 \in \textsc{Term}.\, t = \textit{if}\ t_1\ \textit{then}\ t_2\ \textit{else}\ t_3 \wedge t_1 \Downarrow \textit{false} \wedge t_3 \Downarrow v).\end{aligned}$$

Proposition 12 (Backward Reasoning for $(\cdot \Downarrow \cdot) \subseteq \textsc{Term} \times \textsc{Value}$, Distinguishing $\textsc{Term}$)**.**

1. $\forall v \in \textsc{Value}.\, \textit{true} \Downarrow v \Rightarrow v = \textit{true}$;
2. $\forall v \in \textsc{Value}.\, \textit{false} \Downarrow v \Rightarrow v = \textit{false}$;
3. $\forall t_1, t_2, t_3 \in \textsc{Term}, v \in \textsc{Value}.\, \textit{if}\ t_1\ \textit{then}\ t_2\ \textit{else}\ t_3 \Downarrow v \Rightarrow (t_1 \Downarrow \textit{true} \wedge t_2 \Downarrow v) \vee (t_1 \Downarrow \textit{false} \wedge t_3 \Downarrow v).$

Proposition 13 (Backward Reasoning for $(\cdot \Downarrow \cdot) \subseteq \textsc{Term} \times \textsc{Value}$, Distinguishing $\textsc{Value}$)**.**[1]

1. $\forall t \in \textsc{Term}.\, t \Downarrow \textit{true} \Rightarrow (t = \textit{true}) \vee$
   $(\exists t_1, t_2, t_3 \in \textsc{Term}.\, t = \textit{if}\; t_1 \;\textit{then}\; t_2 \;\textit{else}\; t_3 \wedge (t_1 \Downarrow \textit{true} \wedge t_2 \Downarrow \textit{true}) \vee (t_1 \Downarrow \textit{false} \wedge t_3 \Downarrow \textit{true})$
2. $\forall t \in \textsc{Term}.\, t \Downarrow \textit{false} \Rightarrow (t = \textit{false}) \vee$
   $(\exists t_1, t_2, t_3 \in \textsc{Term}.\, t = \textit{if}\; t_1 \;\textit{then}\; t_2 \;\textit{else}\; t_3 \wedge (t_1 \Downarrow \textit{true} \wedge t_2 \Downarrow \textit{false}) \vee (t_1 \Downarrow \textit{false} \wedge t_3 \Downarrow \textit{false})$

Proposition 14 (Principle of Derivation Induction for $(\cdot \Downarrow \cdot) \subseteq \textsc{Term} \times \textsc{Value}$)**.** *Let Φ be a predicate on derivations $\mathcal{D} \in \textsc{Deriv}[\mathcal{R}_{\Downarrow}]$. Then $\Phi(\mathcal{D})$ holds for all $\mathcal{D} \in \textsc{Deriv}$ if:*

1. $\Phi\left(\overline{\textit{true} \Downarrow \textit{true}}\right)$;
2. $\Phi\left(\overline{\textit{false} \Downarrow \textit{false}}\right)$;
3. $\forall \mathcal{D}_1, D_2 \in \textsc{Deriv}, t_1, t_2, t_3 \in \textsc{Term}, v_2 \in \textsc{Value}.$
   $$(\mathcal{D}_1 :: t_1 \Downarrow \textit{true}) \wedge (\mathcal{D}_2 :: t_2 \Downarrow v_2) \wedge \Phi(\mathcal{D}_1) \wedge \Phi(\mathcal{D}_2) \Rightarrow \Phi\left(\frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ t_1 \Downarrow \textit{true} & t_2 \Downarrow v_2\end{array}}{\textit{if}\; t_1 \;\textit{then}\; t_2 \;\textit{else}\; t_3 \Downarrow v_2}\right).$$
4. $\forall \mathcal{D}_1, D_2 \in \textsc{Deriv}, t_1, t_2, t_3 \in \textsc{Term}, v_2 \in \textsc{Value}.$
   $$(\mathcal{D}_1 :: t_1 \Downarrow \textit{false}) \wedge (\mathcal{D}_2 :: t_3 \Downarrow v_3) \wedge \Phi(\mathcal{D}_1) \wedge \Phi(\mathcal{D}_2) \Rightarrow \Phi\left(\frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ t_1 \Downarrow \textit{false} & t_3 \Downarrow v_3\end{array}}{\textit{if}\; t_1 \;\textit{then}\; t_2 \;\textit{else}\; t_3 \Downarrow v_3}\right).$$

Proposition 15 (Principle of Rule Induction for $(\cdot \Downarrow \cdot) \subseteq \textsc{Term} \times \textsc{Value}$)**.**
Let Φ be a predicate on $\textsc{Term} \times \textsc{Value}$. Then $\Phi(t, v)$ holds for all $\langle t, v \rangle \in (\cdot \Downarrow \cdot)$ if:

1. $\Phi(\textit{true}, \textit{true})$;
2. $\Phi(\textit{false}, \textit{false})$;
3. $\forall t_1, t_2, t_3 \in \textsc{Term}, v_2 \in \textsc{Value}.$
   $t_1 \Downarrow \textit{true} \wedge t_2 \Downarrow v_2 \wedge \Phi(t_1, \textit{true}) \wedge \Phi(t_2, v_2) \Rightarrow \Phi(\textit{if}\; t_1 \;\textit{then}\; t_2 \;\textit{else}\; t_3, v_2)$.
4. $\forall t_1, t_2, t_3 \in \textsc{Term}, v_3 \in \textsc{Value}.$
   $t_1 \Downarrow \textit{false} \wedge t_3 \Downarrow v_3 \wedge \Phi(t_1, \textit{false}) \wedge \Phi(t_3, v_3) \Rightarrow \Phi(\textit{if}\; t_1 \;\textit{then}\; t_2 \;\textit{else}\; t_3, v_3)$.

[1] This principle is weak, but it is meaningful, and demonstrates that *some* useful reasoning can come of knowing v.