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The Logic of Description and Existence

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CONTENTS

1. Introduction.

1.1. There is one sense in which the problems of the descriptive phrase "the so-and-so" are not so important from a logical point of view. Recent developments in mathematical logic indicate that descriptions are dispensable at least as far as one is just interested in expressing truths. Descriptions which do refer to an object seem always to be eliminable from sentences containing them. Empty descriptions do not seem to be needed either as a means for expressing truths.

A rigorous proof that descriptions are dispensable within some framework must, however, be based on some general theory of descriptions. And if one regards logic as concerned not only with the analysis of truth-conditions of sentences but also with analysis of informal reasoning and informal proof the situation is different. It is a fact that the descriptive phrase occurs frequently in informal reasoning. So there is still the problem of interpreting sentences containing descriptions. In particular sentences containing empty descriptions which we recognize as meaningful and sometimes even as true such as the classical example "the present King of France does not exist".

Under the name of "free logic" there has been a renewed interest in the problem of descriptions in recent years.* Several systems of free logic have been constructed in which non-referential singular terms in general and empty descriptions in particular are treated as genuine singular terms. The effort to construct logical systems which admit non-referential singular terms has also been combined with the effort to eliminate "existential

*See e.g. Grandy 1972 and Lambert 1972 and the references therein.

presuppositions" implicit in the classical notion of validity in all non-empty domains. The motivation for this kind of work is often said to be a desire to get logical systems more adequate to capture portions of ordinary language and to get insight into the philosophically important notion of existence. At the same time a free description theory is proposed as a logical analysis of the phrase "the object x such that ..x.." in the same sense as classical quantification theory provides a logical analysis of the phrase "there is an object x such that ..x..".

It seems to me that many of the systems of free logic that have been proposed are unsatisfactory to the same extent as it has not been realized that these two aims may be incompatible. To reflect better what is said in ordinary discourse is one thing and to provide a logical analysis is another, and the former may not always be helpful in carrying out the latter. Ordinary usage seems to give little guidance on questions of the truth or falsity of many sentences containing empty descriptions. The large number of (more or less artificial) conventions on this point, seems to me to be an indication that there is something in the role of the descriptive phrase in informal reasoning which is not understood. According to one line of development our use of non-referential singular terms imposes strong ontological assumptions. In the systems of Cocchiarella 1966 and Scott 1970 the bound variables are intended to range over two different kinds of objects; "actual objects" and "possible objects". I do not think that our informal use of non-referential singular terms has such ontological implications.

1.2. The purpose of this paper is to present a logical or, rather, a proof-theoretical analysis of the descriptive phrase "the (one) object x such that ..x.." within the framework of classical first order predicate logic. The theory of descriptions to be presented below is based not so much on an analysis of truth-conditions of sentences containing descriptions (as most theories of descriptions are), but rather on an analysis of the role of descriptions in informal reasoning.

As the starting point for explaining the ideas behind the theory I shall consider the classical theories of descriptions of Frege , Russell and Hilbert and Bernays 1968. These theories all agree when confined to proper descriptions, i.e. descriptions

$$\iota x A(x)$$

whose existence condition

$$\exists x A(x)$$

and uniqueness condition

$$\forall x \forall y (A(x) \& A(y) \rightarrow x=y),$$

which together can be stated

$$\exists y \forall x (A(x) \leftrightarrow y=x),$$

are valid. They agree in the sense that $\iota x A(x)$ denotes the unique object x auch that A(x) and the biconditional

$$B(\iota x A(x)) \leftrightarrow \exists y \forall x (A(x) \leftrightarrow y=x \& B(y))$$

is valid. Thus the truth expressed by a sentence containing a proper description can be expressed by means of a sentence not containing that description. In Russell's theory of descriptions this is an immediate consequence of the doctrine of contextual definition of descriptions and in the theories of Frege and Hilbert and Bernays it has to be verified. This seems to me to be the only reasonable interpretation of the descriptive phrase when the existence and uniqueness conditions are satisfied.

The classical theories differ, however, in their treatment of empty descriptions. Both Frege and Russell realized that a sentence containing an empty description can be significant or meaningful. But, according to Frege 1892 each name-like expression in a precise formal language ought to have a reference. He therefore adopted the convention of assigning a more or less arbitrary chosen object as the reference of empty descriptions. This device is perhaps convenient from a strictly technical point of view, but it does not explain how sentences containing empty descriptions can be significant.

For Russell definitions are interpreted merely as typographical abbreviations and not as introducing names (in the object language) of the objects described by the definitions. A descriptive phrase denotes nothing at all but has a meaning only in a context; a meaning which is explained by eliminating the description. And, as is well-known, Russell defined the meaning of a sentence of the form

$$P \iota x A(x)$$

to be the conjunction of the sentences

$$\exists y \forall x (A(x) \rightarrow x = y)$$

and

$$\forall x (A(x) \rightarrow Px).$$

Russell was forced to this conclusion by his realist view that sentences (or propositions as he would have said) are meaningful if true or false and true if they express facts and false when they express nothing; together with his view that descriptive phrases have no meaning in isolation.

I think, however, that there is somethimg with the classical examples like "the present King of France is bald" which is not correctly explained by Russell's account. I think it is reasonable to say that the descriptive phrase

(1) "the present King of France"

is quite as meaningful as the sentence

(2) "the present King of France is bald".

(1) and (2) are certainly both meaningful in the sense that they are understood (by anyone who knows a certain portion of English). Russell would perhaps not have denied this either, but I think that this fact should be taken seriously. To talk about the truth and falsity of the sentence (2) seems to me to presuppose that the phrase (1) is meaningful in isolation. So I prefer to follow Frege in treating descriptive phrases as true singular terms on a pair with proper names.

Another well-known disadvantage of Russell's theory of descriptions is that it becomes necessary to distinguish between different scopes of a description. The formula

$$\neg P \,\iota x A(x)$$

can be translated either as

$$\exists y \forall x(A(x) \rightarrow x = y \;\&\; \neg Py)$$

or as

$$\neg \exists y \forall x(A(x) \rightarrow x = y \;\&\; Py).$$

The formal complications which arise from this circumstance seems to me to indicate that there is something wrong with the idea.

1.3. I shall give another account for what makes sentences containing empty descriptions sometimes meaningful. I think that Geach 1950 has already indicated the way to do it.

According to Russell the sentence (2) is defined to mean that

There is exactly one present King of France

and

Whoever is a present King of France is bald.

I shall instead interpret the sentence (2) conditionally:

Assume that the sentence

(3) "there is exactly one individual which is a present King of France"

is true, then the sentence (2) is true if the individual of (3) is bald and (2) is false if the individual of (3) is not bald.

Now since (3) is as a matter of fact false, the question of the truth and falsity of (2) (in "the actual world") does not arise, i.e. (2) has no truth-value.

More generally, if P is a (primitive) predicate then the question of the truth and the falsity of the sentence

(4) $P\, \imath x A(x)$

arises only under the presupposed truth of the sentence

(5) $\exists y\, \forall x (A(x) \leftrightarrow y = x)$.

Thus, on this interpretation the truth of the sentence (4) presupposes the truth of (5), while according to Russell the truth of (4) implies the truth of (5).

Now suppose that (5) happen to be false so (4) has no truth-value. How do we then explain the fact that the sentence (4) can still be significant and meaningfully used? The answer is simply that we are free to adopt any assumption we want. If we introduce (5) as an assumption we are free to treat $\imath x A(x)$ as an ordinary singular term with reference, but everything which we are then able to say about our assumed reference of $\imath x A(x)$ does of course depend on our assumption and is not true or false in some absolute sense. This is how (4) can be meaningful even though $\imath x A(x)$ may be an empty description.

So the idea is that empty descriptions are to be treated precisely like proper descriptions under the assumption of their existence and uniqueness conditions or under assumptions which imply these conditions. This is to understood as a kind of operational interpretation of descriptions rather than as an interpretation in terms of truth-conditions of sentences containing them. In the final section I shall, however, state what consequences this interpretation has for truth-conditions by developing a model theory for the formal system.

Ideas closely related to the one just described have appeared elsewhere but, as far as I know, noone has made the idea precise which I shall attempt to do below.

1.4. Let us finally consider the theory of descriptions of Hilbert and Bernays to which my theory is most closely related. In their theory empty descriptions are considered meaningless: They are dismissed altogether from the language. An expression $\iota xA(x)$ is a term only if the existence and uniqueness condition $\exists y \forall x(A(x) \rightarrow y = x)$ has already been proved, and $P\iota xA(x)$ is a formula only if $\iota xA(x)$ is a term. So what expressions are terms and sentences depend on what formulas are provable.

There are two main arguments against the Hilbert and Bernays theory of descriptions (cf. Carnap 1947 and Scott 1967). Since the formation rules depend upon the rules of proof, the class of terms and formulas will hardly ever be recursive. What expressions are well-formed or meaningful will in general depend on extralinguistic facts.

Now it is quite obvious that the meaningfulness of a term in the sense of Hilbert and Bernays must be understood as its having a reference. From this point of view it is obvious that the class of terms must be non-recursive. This is even true on the informal level. Take an arithmetical sentence $A(x)$ such that noone has yet been able to answer the question wether $\exists xA(x)$ is true or false. We have no recursive method to answer the question wether the phrase "the number x such that $A(x)$" has a reference or not. Interpreted this way as dealing only with referential descriptions, the Hilbert and Bernays theory is quite natural.

According to the other objection of Scott 1967 the use of descriptions in mathematics (which is Hibert and Bernays´ concern) is quite common even <u>before</u> the existence and uniqueness conditions have been proved. For example in set theory one often defines a function by transfinite recursion and only afterwards proves the existence and uniqueness conditions by transfinite induction. This objection is perhaps not so serious regarded as a criticism of the adequacy of Hilbert and Bernays' formal system as a codification of arithmetic. Because even if it is common to use descriptions in mathematics before the existence and uniqueness conditions have been proved, it does not follow that it is essential (for expressing mathematical truths). The objection is, however, justified in so far as Hilbert and Bernays' system does not adequately codify informal mathematical reasoning. In particular it provides no interpretation of sentences containing descriptions whose existence and uniqueness conditions are not known to be true and it offers no explanation of the role of such descriptions in informal reasoning.

The purpose of this paper is to make Hilbert and Bernays' theory adequate also in these respects, and it will be verified in section 5 that the use of descriptions is not essential as far as one is just interested in expressing truths since descriptions turn out to be eliminable.

1.5. The theory of the present paper can be regarded as an extension of the theory of Hilbert and Bernays in the following respect. What expressions are formulas and terms will depend on what formulas are provable but our class of closed formulas will be wider than theirs. Also as in their theory a term has always a reference and a formula is either true or false.

To be able to deal with descriptions whose existence and uniqueness conditions have not been proved (and perhaps never can be proved) I shall distinguish between terms and <u>term-expressions</u> and formulas and <u>formula-expressions</u>. The class of term-expressions and formula-expressions are defined by ordinary recursive formation rules. So wether an expression is a term-expression or a formula-expression or not is a purely linguistic matter that does not depend on the rules of proof. Each term is a term-expression and each formula is a formula-expression but the converse is not true. The intended interpretation is that a term-expression is a term if it has a reference and a formula-expression is a formula if it is either true or false. All term-expressions and formula-expressions are however meaningful in the sense that they can be understood by anyone who knows the meaning of the primitive symbols of the language. Using Frege's distinction between Sinn and Bedeutung we can say that all term-expressions and formula-expressions have a Sinn but only terms and formulas have a Bedeutung.

It follows that not all formula-expressions have a truth-value and there is in general no mechanical method to find out wether a formula-expression has a truth-value or not. Is this reasonable? I think it is. Let A(n) be an (informal) arithmetical sentence for which it is not known wether A(n) is true for exactly one natural number n. If a mathematician, confronted with the question wether the sentence

"the natural number n such that A(n) is a prime number"

is true or false, finds out that there is no n such that A(n), it seems likely that his answer will not be "it is true" or "it is false" but rather that the problem was not well-defined. And we have no mechanical method to decide which questions of this form formulates well-defined problems even though the questions are perfectly "well-defined" from a grammatical point of view.

The formulas are not simply (as in the theory of Hilbert and Bernays) those formula-expressions whose descriptions (if any) have provable existence and uniqueness conditions. There are certainly formula-expressions containing descriptions which we recognize as true only by virtue of their form. A simple example is

$$\exists y \forall x(A(x) \leftrightarrow y = x) \rightarrow A(\imath x A(x))$$

which will be among our theorems for any formula $A(x)$. And it is a formula since, as will be verified, only formulas are theorems.

As already pointed out the present theory of descriptions is developed within classical first order logic. From an intuitionistic point of view descriptions seem to be less problematic. The reason is that the notion of proof rather than the notion of truth is basic intuitionistically. Proofs are treated as objects. The descriptive phrase receives a stronger interpretation intuitionistically. For example, the number n such that $A(n)$ is given not only as a number but as a number n together with a proof of the proposition $\forall x(A(x) \rightarrow x = n)$. (Cf. the codification of intuitionistic abstractions of Martin-Löf 1972). On this interpretation the problems of the descriptive phrase disappears.

The deductive system that will be presented in section 3 is such that only formulas valid in all domains including the empty domain are derivable as theorems (if not axioms to the contrary are introd-used of course). This is achieved not simply by the introduction of some (suitable) formal restrictions that give us this class of theorems. It results rather as a biproduct from a more general and natural formulation of the inference rules for quantification. I shall give a device for introducing and cancelling assumptions of the form "let x be an individual ...". In section 4 I shall try to explain what this means for the notion of existence and how it makes clear the logical role of variables and names.

2. Language.

We consider a language of first order predicate logic based on the following primitive logical symbols:

$\rightarrow$	(material implication)
&	(conjunction)
$\perp$	(absurdity)
$\forall$	(universal quantification)
ι	(description operator)
=	(identity).

There may be a list of n-place function symbols, $n \geq 0$. If $n = 0$, the function symbols are called individual constants or names. There is a list of n-place predicate symbols and an infinite list of individual variables.

We shall often assume that the language contains only one one-place function symbol f and only one one-place predicate symbol P (other than identity). When this is done there will be no loss of generality. As syntactical notations for individual variables we shall use $x, y, z, u, v, w, \ldots$.

2.1. The term-expressions and the formula-expressions are defined by a simultaneous induction:

2.1.1. Each individual variable is a term-expression.

2.1.2. If $t_1, \ldots, t_n$ are term-expressions and if f is an n-place function symbol, then $ft_1 \ldots t_n$ is a term-expression.

2.1.3. If $t_1, \ldots, t_n$ are term-expressions and P is an n-place predicate symbol, then $Pt_1 \ldots t_n$ is a formula-expression. In particular $t = s$ is a formula-expression if t and s are term-expressions.

2.1.4. $\perp$ is a formula-expression and if A and B are formula-expressions, then so are $(A \& B)$ and $(A \rightarrow B)$.

2.1.5. If A is a formula-expression and x an individual variable, then $\forall xA$ is a formula-expression and ιxA is a term-expression.

2.2. We shall use A, B, C, ... as syntactical notations for formula-expressions and t, s, u, ... as syntactical notations for term-expressions. The formula-expression $\perp$ and those of 2.1.3 are the atomic formula-expressions, the others are composite.

Free and bound variables are defined as usual and we shall adopt the convention of not distinguishing between formula-expressions and and term-expressions that differ only in the naming of their bound variables. We shall write

$$A(x_1, \ldots, x_n)$$

to indicate that some (possibly none) of the free variables in the formula-expression A are among the variables in the list $x_1, \ldots, x_n$. If $t_1, \ldots, t_n$ are term-expressions, then $A(t_1, \ldots, t_n)$ denotes the result of simultaneously substituting the term-expressions $t_1, \ldots, t_n$ for all free occurences of the variables $x_1, \ldots, x_n$, respectively, in A. The described substitutions are defined provided that no free variable of $t_1, \ldots, t_n$ becomes bound in $A(t_1, \ldots, t_n)$. This can always be accomplished by renaming the bound variables. We shall always assume that such renamings have been made when we indicate a substitution.

2.3. We shall also adopt the following abbreviations:

$\neg A$	for	$A \to \bot$
$A \vee B$	for	$\neg(\neg A \mathbin{\&} \neg B)$
$A \leftrightarrow B$	for	$(A \to B) \mathbin{\&} (B \to A)$
$\exists x A$	for	$\neg \forall x \neg A$
$\exists_1 x A$	for	$\exists x A \mathbin{\&} \forall x \forall y (A(x) \mathbin{\&} A(y) \to x = y)$.

We shall also adopt the usual conventions for omitting parenthesis. In particular

$$A_1 \to A_2 \to \dots \to A_n$$

is to be understood as

$$(A_1 \to (A_2 \to \dots \to (A_{n-1} \to A_n) \dots)).$$

3. Rules of inference.

3.1. I shall now present a formal deductive system which will codify the ideas described informally in the introduction. In order to do that I introduce the following (metalogical) symbols

$$I, \quad F, \quad \in.$$

The derivations of the system are certain tree arrangements of expressions of the form

$$t \in I, \quad A \in F, \quad A,$$

where t is a term-expression and A is a formula-expression. The rules of inference allow us to infer expressions of each of these forms. The informal meaning of $t \in I$ is that t denotes an individual or t has a reference. $A \in F$ means informally that A is either true or false and when we have a derivation ending with A we interpret this as meaning that the proposition (expressed by the formula) A has been proved. A derivation will be said to codify a piece of reasoning, an informal proof or an argument.

The system of inference rules is an extension and a modification of the system of natural deduction described in Prawitz 1965, to which the reader is referred for a more complete and rigorous presentation. As in systems of natural deduction a derivation is started from an axiom or by making an assumption and proceeds downwards by means of the rules of inference. Some of the rules of inference are such that when one passes from the premisses to the conclusion, certain assumptions are discharged or cancelled. I shall indicate this by enclosing the assumptions in square brackets. In order to make more explicit at what stage in a derivation a certain assumption is discharged, I shall sometimes use the device of indexing the assumptions as in Prawitz 1965. The assumptions are of two forms. Either it is of the form

$$x \in I$$

where x is an individual variable or else, if we have a derivation (possibly from other assumptions) of $A \in F$, then we may introduce

$$A$$

as an assumption. A derivation is closed if all of its assumptions have been discharged. Otherwise it is open.

If there is a closed derivation of

$$t \in I$$

we say that the term-expression t is a term. If there is a closed derivation of

$$A \in F$$

we say that A is a formula and, finally, if there is a closed derivation with the conclusion

$$A$$

we say that A is a (formal) theorem.

3.2. I shall divide the rules of inference into three groups: the

I-rules the F-rules and the remaining rules which include the introduction and the elimination rules for the logical symbols. It should be noted, however, that rules within different groups can be linked with each other in a derivation.

3.2.1. The I-rules are the following:

(f-rule)

$$\frac{t_1 \in I, \ldots, t_n \in I}{ft_1 \ldots t_n \in I}$$

(ιI-rule)

$$\frac{\exists_1 x A}{\iota x A \in I}$$

It is to be understood that when f is a 0-place function symbol, i.e. an individual constant then the f-rule is the axiom

$$f \in I.$$

3.2.2. The F-rules are the following:

($\bot$)

$$\bot \in F$$

(P-rules)

$$\frac{t_1 \in I, \ldots, t_n \in I}{Pt_1 \ldots t_n \in F} \qquad \frac{t \in I, s \in I}{t = s \in F}$$

(&-rule)

$$\frac{A \in F, \quad B \in F}{A \& B \in F}$$

$$(\to\text{-rule})\qquad \frac{A \in F \qquad \begin{matrix}[A]\\ \vdots \\ B \in F\end{matrix}}{A \to B \in F}$$

$$(\forall\text{-rule})\qquad \frac{\begin{matrix}[x \in I]\\ \vdots \\ A(x) \in F\end{matrix}}{\forall x\, A(x) \in F}$$

The $\forall$-rule is subject to the restriction that x must not occur free in any assumption in the derivation of $A(x) \in F$ other than $x \in I$.

3.2.4. Among the remaining rules we have first the following rule for descriptions:

$$(\iota\text{-rule})\qquad \frac{\exists_1 x\, A(x)}{A(\iota x\, A(x))}$$

and the rule of indirect proof:

$$\frac{\neg A \in F \qquad \begin{matrix}[\neg A]\\ \vdots \\ \bot\end{matrix}}{A}$$

The remaining rules are the introduction and elimination rules for the logical symbols (including identity):

$$\frac{A \qquad B}{A \mathbin{\&} B} \qquad\qquad \frac{A \mathbin{\&} B}{A} \qquad \frac{A \mathbin{\&} B}{B}$$

$$\frac{A \in F \qquad \begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \rightarrow B} \qquad\qquad \frac{A \rightarrow B \qquad A}{B}$$

$$\frac{\begin{matrix}[x \in I] \\ \vdots \\ A(x)\end{matrix}}{\forall x\, A(x)} \qquad\qquad \frac{\forall x\, A(x) \qquad t \in I}{A(t)}$$

$$\frac{t \in I}{t = t} \qquad\qquad \frac{t = s \qquad A(t)}{A(s)}$$

The introduction rule for $\forall$ is subject to the restriction that x must not occur free in any assumption in the derivation of $A(x)$ other than $x \in I$. Note that in the $\rightarrow$-rule and the introduction rule for $\rightarrow$, when passing to the conclusion $A \rightarrow B \in F$ and $A \rightarrow B$, respectively, all occurences of the assumption A become discharged and the conclusion becomes dependent on the assumptions in the derivation of $A \in F$ (if any). The same thing is true in the case of the assumption $\neg A$ of the rule of indirect proof.

3.2.5. When disjunction and existence are defined as in section 2.3, the introduction and elimination rules for disjunction and existence hold as derived rules in the following forms:

$$\frac{A \quad B \in F}{A \vee B} \qquad \frac{A \in F \quad B}{A \vee B} \qquad \frac{A \vee B \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}$$

$$\frac{t \in I \quad A(t)}{\exists x A(x)} \qquad \frac{\exists x A(x) \quad \begin{matrix}[x \in I],[A(x)] \\ \vdots \\ C\end{matrix}}{C}$$

It is easy to see that terms and formulas as we have defined them are always closed. As already pointed out when we have a derivation of $A \in F$ we interpret this as meaning that A is true or false and when we have a derivation ending with A, it means informally that we have a proof of A and hence A is true. If A is true, then A is true or false, so we must require that if we have a derivation of A we must also have a derivation of $A \in F$. In other words only formulas are theorems. That our deductive system is adequate in this respect follows from the following stronger result.

3.2.6. THEOREM. *If we have a derivation of A, then we can find a derivation of $A \in F$.*

Proof. The proof is by induction on the length of the derivation of A.

Case 1. A is an assumption. Then the result is immediate since we cannot introduce A as an assumption unless we have a derivation of $A \in F$.

Case 2.

$$\frac{\begin{matrix}\vdots\\ \exists_1 x A(x)\end{matrix}}{A(\iota x A(x))}$$

By the induction hypothesis we have a derivation of $\exists_1 x A(x)$. This implies in particular that

$$\exists x A(x) \in F$$

i.e.

$$\neg \forall x \neg A(x) \in F.$$

It is easy to see that $A \in F$ iff $\neg A \in F$. Hence $\forall x \neg A(x) \in F$, so we have derivations

$$\begin{matrix} x \in I \\ \vdots \\ \neg A(x) \in F \end{matrix}$$

and

$$\begin{matrix} x \in I \\ \vdots \\ A(x) \in F . \end{matrix}$$

Substituting $\iota x A(x)$ for x everywhere in this derivation we obtain

$$\begin{matrix} \vdots \\ \exists_1 x A(x) \\ \hline \iota x A(x) \in I \\ \vdots \\ A(\iota x A(x)) \in F \end{matrix}$$

and the result follows.

Case 3.

$$\dfrac{\begin{matrix} \vdots \\ \neg A \in F \end{matrix} \qquad \begin{matrix} [\neg A] \\ \vdots \\ \bot \end{matrix}}{A}$$

Immediate, since $A \in F$ iff $\neg A \in F$.

Case 4.

$$\dfrac{\begin{matrix} \vdots \\ A \end{matrix} \qquad \begin{matrix} \vdots \\ B \end{matrix}}{A \,\&\, B}$$

By the induction hypothesis we have derivations

$$\begin{matrix} \vdots \\ A \in F \end{matrix} \quad \text{and} \quad \begin{matrix} \vdots \\ B \in F \end{matrix}$$

Hence, $A \,\&\, B \in F$ follows by the &-rule.

Case 5.

$$\dfrac{\begin{matrix} \vdots \\ A \,\&\, B \end{matrix}}{A}$$

By the induction hypothesis we have a derivation of $A \,\&\, B \in F$, which must look like this

$$\dfrac{\begin{matrix} \vdots \\ A \in F \end{matrix} \qquad \begin{matrix} \vdots \\ B \in F \end{matrix}}{A \,\&\, B \in F}$$

and the result follows.

Case 6.

$$\dfrac{\begin{array}{c}\vdots\\ A \in F\end{array} \qquad\qquad \begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \to B}$$

By the induction hypcthesis we have a derivation

$$\begin{array}{c}A\\ \vdots\\ B \in F\end{array}$$

which together with the given derivation

$$\begin{array}{c}\vdots\\ A \in F\end{array}$$

gives us $A \to B \in F$ by the $\to$-rule.

Case 7.

$$\dfrac{\begin{array}{c}\vdots\\ A \to B\end{array} \qquad \begin{array}{c}\vdots\\ A\end{array}}{B}$$

By the induction hypothesis we have a derivation of $A \to B \in F$, which must look like this

$$\dfrac{\begin{array}{c}\vdots\\ A \in F\end{array} \qquad\qquad \begin{array}{c}[A]\\ \vdots\\ B \in F\end{array}}{A \to B \in F}$$

The derivation

$$\begin{array}{c}\vdots\\ B \in F\end{array}$$

taken together with the given derivation

$$\begin{array}{c}\vdots\\ A\end{array}$$

gives us

$$\begin{array}{c}\vdots\\ A\\ \vdots\\ B \in F\end{array}$$

and the result follows.

Case 8.

$$\frac{\begin{array}{c}[x \in I]\\ \vdots\\ A(x)\end{array}}{\forall x A(x)}$$

By the induction hypothesis we have a derivation

$$\begin{array}{c}x \in I\\ \vdots\\ A(x) \in F\end{array}$$

Which by the $\forall$-rule gives the desired result.

Case 9.

$$\frac{\begin{matrix}\vdots & & \vdots \\ \forall x A(x) & & t \in I\end{matrix}}{A(t)}$$

By the induction hypothesis we have a derivation of $\forall x A(x) \in F$, which must have the form

$$\frac{\begin{matrix}[x \in I] \\ \vdots \\ A(x) \in F\end{matrix}}{\forall x A(x) \in F}$$

Substituting t for x in this derivation and using the given derivation

$$\begin{matrix}\vdots \\ t \in I\end{matrix}$$

we have

$$\begin{matrix}\vdots \\ t \in I \\ \vdots \\ A(t) \in F.\end{matrix}$$

Case 10.

$$\frac{t \in I}{t = t}$$

Since we have $t \in I$, the result follows immediately by the P-rule.

Case 11. It remains only to consider a derivation which ends like this

$$\frac{\begin{array}{c}\vdots \\ t = s\end{array} \qquad \begin{array}{c}\vdots \\ A(t)\end{array}}{A(s)}$$

By the induction hypothesis we have derivations

$$\begin{array}{c}\vdots \\ t = s \in F\end{array} \quad \text{and} \quad \begin{array}{c}\vdots \\ A(t) \in F .\end{array}$$

The former derivation must have the form

$$\frac{\begin{array}{c}\vdots \\ t \in I\end{array} \qquad \begin{array}{c}\vdots \\ s \in I\end{array}}{t = s \in F}$$

so we have derivations

$$\begin{array}{c}\vdots \\ t \in I\end{array} \quad \text{and} \quad \begin{array}{c}\vdots \\ s \in I.\end{array}$$

We shall prove that $A(s) \in F$ by induction over $A(x)$, using the following result.

LEMMA. If we have derivations of $t = s$, $t \in I$, $s \in I$ and $u(t) \in I$ from certain assumptions, then we can find a derivation of $u(s) \in I$ from the same assumptions.

We first prove this lemma by induction over $u(x)$.

(i) $u(x)$ is an individual constant. Then $u(t) = u(s)$ and there is nothing to prove.

(ii) $u(x)$ is a variable. The result is immediate.

(iii) $u(x)$ is $fu'(x)$. Now, $fu'(t) \in I$ means that $u'(t) \in I$. By the induction hypothesis $u'(s) \in I$, so $u(s) = fu'(s) \in I$.

(iv) $u(x)$ is $\iota y\, B(y,x)$. Then $u(t) \in I$ means that we have a derivation

$$\begin{array}{c} \vdots \\ \exists_1 y\, B(y,t) \end{array}$$

hence

$$\dfrac{\dfrac{t = s \qquad \begin{array}{c} \vdots \\ \exists_1 y\, B(y,t) \end{array}}{\exists_1 y\, B(y,s)}}{u(s) \in I}$$

and the lemma follows. We now continue case 11.

<u>Subcase 11.1.</u> $A(x)$ is $Pu(x)$. $Pu(t) \in F$ means that $u(t) \in I$ and by the lemma $u(s) \in I$, so $Pu(s) \in F$ follows by the P-rule.

<u>Subcase 11.2.</u> $A(x)$ is $\bot$. Nothing to prove.

<u>Subcase 11.3.</u> $A(x)$ is $B(x)\ \&\ C(x)$. $A(t) \in F$ means that we have derivations

$$\begin{array}{ccc} \vdots & & \vdots \\ B(t) \in F & \text{and} & C(t) \in F. \end{array}$$

By the induction hypothesis we can find derivations

$$\begin{array}{ccc} \vdots & & \vdots \\ B(s) \in F & \text{and} & C(s) \in F \end{array}$$

and the result follows by the &-rule.

<u>Subcase 11.4.</u> A(x) is B(x) → C(x). Since A(t) ∈ F, we have derivations

$$\begin{array}{ccc} & & B(t) \\ \vdots & & \vdots \\ B(t) \in F & \text{and} & C(t) \in F . \end{array}$$

By the induction hypothesis we have

$$\begin{array}{ccc} & & B(t) \\ \vdots & & \vdots \\ B(s) \in F & \text{and} & C(s) \in F \end{array}$$

and, hence

$$\begin{array}{c} \dfrac{t = s \qquad B(s)}{B(t)} \\ \vdots \\ C(s) \in F \end{array}$$

and the result follows by the →-rule.

<u>Subcase 11.5.</u> A(x) is ∀y B(y,x). Since A(t) ∈ F, we have a derivation

$$\begin{array}{c} y \in I \\ \vdots \\ B(y,t) \in F \end{array}$$

By the induction hypothesis we have

$$\begin{array}{c} y \in I \\ \vdots \\ B(y,s) \in F \end{array}$$

and the result follows by the $\forall$-rule.

This completes the proof of theorem 3.2.6.

4. Examples and discussion.

4.1. The purpose of this section is to explain the meaning and the significance of the deductive system of section 3 by means of some examples.

4.1.1. The following is a derivation of the formula $\forall x(Px \to Px)$.

$$\dfrac{\dfrac{\dfrac{\overset{(1)}{[x \in I]}}{Px \in F} \qquad \overset{(2)}{[Px]}}{Px \to Px}\,(2)}{\forall x(Px \to Px)}\,(1)$$

Here we have assigned a number to the two assumptions in the order in which they are introduced and we have indicated at what step in the derivation they are discharged. Both assumptions in this derivation are discharged so we have a closed derivation of the formula $\forall x(Px \rightarrow Px)$. The truth of the formula does not depend on the truth of some other formulas. Its truth does neither rest on any existential assumptions (that there is at least one individual) nor on any assumption about what is the case (that certain individuals, if they exist, have a certain property). The formula is true in all domains including the empty one. In this respect there is nothing special about the formula $\forall x(Px \rightarrow Px)$. It will be proved in section 6 that there is a closed derivation of a formula A if and only if it is valid in all domains including the empty domain. Since we also permit terms with no reference to occur both in our language and in derivations, the deductive system of section 3 can be said to be a "quantification theory without existential presuppositions" (Cf. Hintikka 1959), or, rather, a deductive system where all (existential) assumptions have to be made explicit.

4.1.2. To take another example. Let Px mean that x is a present King of France and let Qx mean that x is bald. Since P is a primitive predicate we can easily find a closed derivation

$$\begin{array}{c}[x \in I] \\ \vdots \\ \exists_1 x\, Px \in F\end{array}$$

We can therefore introduce $\exists_1 x\, Px$ as an assumption and obtain, for example, the following derivation

$$(1) \qquad \cfrac{\cfrac{\exists_1 x Px}{\iota x Px \in I}}{Q \iota x Px \in F}$$

which means that under the assumption that there is exactly one present King of France, the sentence "the present King of France is bald" is either true or false. We can continue the reasoning codified by the derivation (1) as follows

$$(2) \qquad \cfrac{\begin{matrix} \overset{(1)}{[x \in I]} \\ \vdots\,(1) \\ \exists_1 x Px \in F \end{matrix} \qquad \cfrac{\cfrac{\overset{(2)}{\exists_1 x Px}}{\iota x Px \in I}}{Q \iota x Px \in F} \qquad \overset{(3)}{[Q \iota x Px]}}{Q \iota x Px \rightarrow Q \iota x Px}\ (3)$$

which means that under the assumption that there is exactly one present King of France, the present King of France is bald only if the present King of France is bald. It is important to note that the derivation (2) in not closed because the assumption (2) is not discharged. The closed formula-expression

$$(3) \qquad Q \iota x Px \rightarrow Q \iota x Px$$

is not derivable from no assumptions. On the informal level this means that the formula is true in those "worlds" where the formula $\exists_1 x Px$ is true. In other "possible worlds" it has no truth-value.

As already pointed out in the introduction, the meaningfulness of empty descriptions or descriptions whose existence and uniqueness conditions are not known to be true, I take to be their role in arguments like (1) and (2). On Russell´s interpretation a descriptive phrase has a meaning only in the context of a (true or false) sentence while on this interpretation its meaningfulness comes from its role in arguments where the corresponding existence and uniqueness condition are assumed to be true.

An argument involving a description which as a matter of fact has no reference, such as the argument codified in the derivation (2), is of course vacuous and useless. It can never be used to infer truths about reality. But nevertheless it is valid as an argument and within this argument the description $\imath x Px$ behaves like an ordinary proper description.

4.1.3. Inspection shows that the I-rules and the F-rules almost coincide with the formation rules of section 2.1 except for the rules that involve the description operator. It follows that all closed formula-expressions that do not contain terms of the form $\imath x A(x)$ are formulas. But as pointed out in the introduction there are also formulas and even theorems containing descriptions whose existence and uniqueness conditions are not derivable. An example is the formula

(4) $$\exists_1 x Px \rightarrow P\imath x Px .$$

A derivation to the effect that it is a formula goes as follows

$$\dfrac{\dfrac{\overset{(1)}{[x \in I]}}{\exists_1 x Px \in F}\,\vdots(1) \qquad \dfrac{\dfrac{\overset{(2)}{[\exists_1 x Px]}}{\imath x Px \in I}}{P\imath x Px \in F}}{(\exists_1 x Px \rightarrow P\imath x Px) \in F}\,(2)$$

A derivation of the formula (4) as a theorem is as follows

$$\dfrac{\dfrac{\overset{(1)}{[x \in I]}\quad\vdots(1)}{\exists_1 x Px} \qquad \dfrac{\overset{(2)}{[\exists_1 x Px]}}{P\imath x Px}}{\exists_1 x Px \rightarrow P\imath x Px}\ (2)$$

The existence and uniqueness conditions $\exists_1 x Px$ of the description $\imath x Px$ is certainly not derivable. It will be true on some interpretations and false on others.

4.2. Despite the inference rules which govern the description operator, there is another feature of our formulation of the inference rules of section 3 which is more satisfactory from a theoretical point of view than usual formulations. Namely, that $\forall$-introduction is formulated as follows

$$\dfrac{\begin{matrix}[x \in I]\\ \vdots \\ A(x)\end{matrix}}{\forall x A(x)}$$

with $x \in I$ as an assumption which is discharged by this inference and $\forall$-elimination has $t \in I$ as an additional premiss.

4.2.1. First of all this corresponds more closely to the meaning meaning of the quantifiers in informal reasoning. For example, suppose that our individuals are natural numbers and that we want to prove a universal arithmetical statement

$$\forall x A(x).$$

Such an argument might begin like this

(5) Suppose that x is an arbitrary natural number ...
..., then ... hence $A(x)$

This is what we symbolize by writing

$$\begin{array}{c} x \in I \\ \vdots \\ A(x) . \end{array}$$

When the argument ends with the conclusion

(6) $A(x)$ for all natural numbers x

it does no longer depend on the assumption made in (5). The conclusion (6) means that whenever we encounter a natural number x we can carry out the argument in (5). It does not mean that if there exists a natural number x, then, whenever we encounter a natural number x we can etc.

Conversely, when we pass from $\forall x A(x)$ to $A(t)$ by $\forall$-elimination this inference is incompletely specified without $t \in I$ as an additional premiss.

Corresponding remarks apply to the derived forms of $\exists$-introduction and $\exists$-elimination in section 3.2.5.

4.2.2. Another theoretical advantage of our formulation of the inference rules for quantification is that we obtain full generality: The formulas of which there is a closed derivation are precisely those which are valid in all domains including the empty domain.*

*The first system of natural deduction with this property was constructed by Jáskowski (cf. Prawitz 1965, p 99).

To see how our deductive system works in this respect consider the formula

(7) $$\forall x\, Px \rightarrow \exists x\, Px$$

which by the usual formulations of Gentzen´s rules of natural deduction can be derived as follows

(8) $$\dfrac{\dfrac{\dfrac{\overset{(1)}{[\forall x\, Px]}}{Px}}{\exists x\, Px}}{\forall x\, Px \rightarrow \exists x\, Px}\,(1)$$

Using our rules of inference we have

$$\dfrac{\dfrac{\dfrac{\overset{(1)}{[x \in I]}}{Px \in F}}{\forall x\, Px \in F}\,(1) \qquad \dfrac{\dfrac{\overset{(2)}{[\forall x\, Px]} \quad \overset{(3)}{x \in I}}{Px} \quad \overset{(3)}{x \in I}}{\exists x\, Px}}{\forall x\, Px \rightarrow \exists x\, Px}\,(2)$$

Since there is an undischarged assumption, namely

$$x \in I$$

we have not a closed derivation of the formula (7). The assumption $x \in I$ is precisely the (existential) assumption that there is at least one individual.

We can of course get the same class of theorems as in usual formulations of the (classical) predicate calculus. We have only to introduce an individual constant e and

$e \in I$

as an axiom. Putting $e \in I$ for the assumption $x \in I$ in the derivation (8) we have a closed derivation of the formula (7). The resulting class of formulas for which there is a closed derivation are precisely those which are valid in all non-empty domains. So to introduce an individual constant with $e \in I$ as an axiom is to restrict one's attention to non-empty domains.

It is sometimes maintained that the effort to remove existential assumptions implicit in the classical notion of validity in non-empty domains has philosophical relevance for the notion of existence. I do not want to make such a claim. Validity in the empty domain is trivial and rather uninteresting and to include it in one's notion of validity involves certain technical complications. So one have had good reason not to include it in classical formulations of quantification theory.

With the formulation of the rules of quantification of section 3 I only hope to have made more clear how the assumption of non-emptyness is introduced in the deductive system. It shows that it is a matter of convention and convenience rather than based on some philosophical doctrine of necessary existence.

4.2.3. Still another theoretical advantage of our formulation of quantification is that it reflects very clearly the difference between the role of variables and individual constants or names. Variables are used to make assumptions within an argument and these assumptions are discharged when the variables are bound by a quantifier. Individual constants or names on the other hand are used to refer to individuals. Thus e $\in$ I is an axiom when e is a name. This role of names in logic was pointed long ago by Russell 1919, p 179. The introduction of names presupposes the corresponding existence and uniqueness conditions.

The difference between the existence assumptions in the case of names and those we make by means of variables is the following: In the case of names the assumptions are made on the meta-level. They are presupposed by the theory we have decided to consider and are implicit in the intended interpretation of the language. They impose a restriction on the individual domains which are possible models. The existential assumptions implicit in the introduction of names are thus not discharged but stay as long as we do not change the intended interpretation.

The assumptions made by means of variables on the other hand are made within an already presupposed universe of discourse and for the sake of an argument. Variables do therefore not denote or refer to individuals, they are only a tool for making assumptions in arguments and, when they are bound, for expressing propositions.

In standard formulations of the predicate calculus this difference between the role of variables and names is often confused. Variables play here a double role. They are treated as variables when they are bound, but as free variables they figure also as names. The reason for this is of course that when a variable is used as a "dummy symbol" in a derivation it is actually used as an individual constant because when the variable is bound one has no device for discharging the (tacit) assumption x $\in$ I. This assumption then stays as an axiom which means that the (free) variable is treated as an individual constant and one restricts the possible models to those with a non-empty domain.

Some logicians also allow for empty names in their systems i.e. names without reference. I think that this is a mistake. From a logical point of view there are no empty names. The mistake results from a confusion of namehood in the grammatical sense and namehood in the referential sense i.e. in the sense of "that which names". It is only names in the latter referential sense that we should construe as names in a logically well-written language. What looks like a name from a grammatical point of view can not always be construed as a name in the logical sense. As is well-known grammatical form does not always coincide with logical form.

We could of course allow for empty names in our language. This would amount to introducing individual constants e, ... without introducing axioms $e \in I$, ... But these names would be superfluous in the theory since they could never appear in true or alse statements and in arguments. If we allowed them to appear in arguments in the form of assumptions, they would still be superfluous since the variables could be used for the same purposes.

What then about the classical example "Pegasus exists". Are we to treat "Pegasus" as a name or not? The answer depends on the intended interpretation. If we intended to make a logical analysis of Greek mythology, then certainly "Pegasus" is to be treated as a name and "Pegasus exists" is an axiom. Precisely as we introduce numerals as names of numbers in a logical analysis of arithmetic. The more likely situation is perhaps that we intend our universe of discourse to comprise only beings of flesh and blood that have actually lived at some time and then the situation is different. Quine 1960 has already shown how to paraphrase the sentence

"Pegasus exists" in this case. "Pegasus" is to be construed as a general term.

4.3. As already pointed out the problems of the empty domain and empty singular terms have often been related to the notion of existence. Is (singular) existence as in "Pegasus exists" a predicate? If so, what kind of predicate? Some authors have introduced existence as a predicate and treated it as a predicate of the same sort as other primitive predicates. In Schock 1968 this is done explicitly and in Scott 1970 it is implicit in the distinction between actual and possible objects. This seems to me also to be wrong. The mistake comes from a confusion of use and mention.

What corresponds to the existence predicate in our deductive system of section 3 is the predicate I, because $t \in I$ means that t denotes an individual i.e. there is such a thing as t, and when $t \in I$ is false there is no such thing as t. Now, couldn't we treat I as a primitive predicate in the object language i.e. as an existence predicate, writing

$$It$$

instead of $t \in I$ and define It to be a formula-expression whenever t is a term-expression. This would be pointless if we did not allow It to be a formula (i.e. something true or false) even when t is a term-expression which is not a term. Because otherwise It would be true for any t and the predicate I would be superfluous in the theory. So suppose then that we defined It to be a formula whenever t is a term-expression, then obviously the formula

$$\neg I \iota x(x \neq x)$$

would be true because there is no object which is not identical with itself. Now, if I is to be treated as an ordinary primitive predicate we are justified in asking: What is the object which does not have the property I of existing? We would be forced to admit non-existing objects such as the object which is not identical with itself. This is clearly nonsense.

Existence can be thought of as a significant predicate. But it is not a predicate of the same type as the primitive predicates P, Q, ... of the object language. If it is to be significantly asserted or denied of something it must be interpreted as asserting reference of term-expressions. This is precisely what our predicate I of section 3 does. A statement $t \in I$ is not a statement of the object language. It is a metalogical statement about the term-expression t and not about the object which t may denote. Otherwise expressed, the term-expression t is mentioned but not used in the statement $t \in I$.

When we reason in a specific (formal) language, we are doing this relative to an intended universe of discourse. As a matter of fact, the choice of the universe proceeds the specification of the language. But it makes no sense to talk about an objects being of not being in this universe within the object language. All objects which can be "talked about" within the given language are presupposed to belong to that universe. Existence, treated as a significant predicate, must therefore be thought of as a predicate of singular terms and not of the objects which these singular terms may denote. In this respect the existence predicate is a predicate of the same kind as the truth-predicate and the derivability predicate with the only difference that they apply to sentences and not to singular terms.

The predicate I is, however, dispensable as a means for expressing truths. As will be proved in the next section we can find for each term-expression t a formula I(t) such that

$$t \in I \quad \text{iff} \quad I(t) \text{ is derivable}$$

This does not mean that the "existence predicate" I is definable in the object language. It means only that each sentence $t \in I$ can be translated into an equivalent sentence in the object language.
To take an example: Let Px mean that x is a present King of France. Then the sentence "the present King of France does not exist" which I interpret as "$\iota x Px$ has no reference" i.e. $\iota x Px \notin I$, can be translated as

$$\neg \exists_1 x Px .$$

Hintikka 1969 p. 34 has proposed the formula

$$\exists y(x = y)$$

as a definition of the existence predicate "x exists". In the system of section 3 it can be verified that for each term-expression t, $t \in I$ holds if and only if $\exists y(t = y)$ is derivable. Should we then say that the formula $\exists y(t = x)$ defines the metalogical existence predicate I in the object language? No, because what we have is only a characterization of the metalogical predicate $t \in I$ in terms of the metalogical derivability predicate via the formula $\exists y(t = y)$. If we read $t \in I$ as "t exists" we have informally

(9) t exists iff "$\exists y(t = y)$" is true

but we do <u>not</u> have

(10) $$t \text{ exists iff } \exists y(t = y)$$

which we would require of a definition of the existence predicate. But (10) does not even make sense because on the left hand side t is mentioned and on right hand side t is used. If "t exists" would be false i.e. if there were no such thing as t in the universe of discourse the formula $\exists y(t = y)$ would be neither true nor false since a formula in the object language can only "talk about" objects in that universe. On our interpretation, the truth or falsity of $\exists y(t = y)$ presupposes the truth of "t exists", i.e. of $t \in I$. The statement $\exists y(t = y)$ does not ascribe the existence property to some object. It says of the already presupposed existing objects in the universe of discourse (if any), that at least one of them is identical with the object t.

Hence, to find out what objects are presupposed as existing when one is given a language with an intended interpretation is to find out what objects the bound variables are intended to range over. But it makes no sense to talk about the existence or non-existence of these objects in the same language. If by a theory we mean a language together with an intended interpretation we can say that the ontology of a given theory does not belong to that theory but to its meta-theory. What can be talked about in that meta-theory is what term-expressions of the language refer to objects in the universe. This is what we have done by means of the I-rules of section 3.

5. Eliminability.

5.1. By the description-free system we shall understand the system obtained by omitting the clause "$\iota x A(x)$ is a term-expression if $A(x)$ is a formula-expression" from the formation rules and by omitting the inference rules for the description operator. The description-free system is of course a subsystem of the original system. When we say that a formula is derivable we are in general referring to derivability in the original system unless the contrary is not explicitly stated.

As promised in the introduction we shall prove that descriptions are eliminable by showing how to associate with each formula A a description-free formula A^o such that A is derivable if and only if A^o is derivable in the description-free system.

5.1.1. Let A be a formula-expression. We define a formula-expression A^o by induction on the construction of A as follows:

(i) $(\forall x A(x))^o = \forall x A^o(x)$

(ii) $(A \rightarrow B)^o = A^o \rightarrow B^o$

(iii) $(A \& B)^o = A^o \& B^o$

(iv) $\bot^o = \bot$

(v) $(Pt_1 \ldots t_n)^o = Pt_1 \ldots t_n$, if none of $t_1, \ldots t_n$ contains a term-expression of the form $\iota x C$.

(vi) If A is $Pt_1 \ldots t_n$, let $P(\iota x A_1(x), \ldots, \iota y A_n(y))$ stand for $Pt_1 \ldots t_n$, where $\iota x A_1(x), \ldots, \iota y A_n(y)$ are all outermost term-expressions of the form

$\imath x\, C$ occuring in $t_1, \ldots, t_n$ from left to right.
i.e. at least one occurence of each of $\imath x A_1(x), \ldots, \imath y A_n(y)$ in $Pt_1 \ldots t_n$ is not within a term of the form $\imath x C$, and it is these occurences which are indicated with $x, \ldots, y$ in $P(x, \ldots, y)$. Then we put

$$(Pt_1 \ldots t_n)^\circ = \forall x \ldots \forall y (A_1^\circ(x) \,\&\, \ldots \,\&\, A_n^\circ(y) \rightarrow P(x, \ldots, y))$$

We first establish the following result.

5.1.2. THEOREM. Given a derivation of A in the original system, we can find a derivation of A° in the description-free system.

Proof. The proof is by induction on the length of the proof of A. Since the $^\circ$-transformation is compatible with the logical constants $\bot$, $\&$, $\rightarrow$, $\forall$, the result follows immediately by the induction hypothesis for all inference rules except the following ones:

(a) $$\frac{\exists_1 x A(x)}{A(\imath x A(x))}$$

(b) $$\frac{\forall x A(x) \qquad t \in I}{A(t)}$$

(c) $$\frac{t = s \qquad A(t)}{A(s)}$$

where t and s contain at least one occurence of a term-expression of the form $\imath x C$ (otherwise the result follows immediately in the case of (b) and (c), too).

Instead of dealing with (c) we shall replace it by each instance of the following formula-expression

$$(c') \qquad \forall z_1 \dots z_n \forall x \forall y (x = y \to A(x) \to A(y))$$

where $z_1, \dots, z_n$ are all free variables in $A(x)$ distinct from x and y. Having (c′) the rule (c) can now be derived using only $\forall$-eliminations and $\to$-eliminations. It is also easy to see that each instance of (c′) in the description-free system is derivable in that system. Since the $^{\circ}$-transform of the formula-expression (c′) has the same form, i.e. is an instance of (c′), it remains for us only to consider the rules (a) and (b) in the case where t of (b) contains at least one term-expression of the form $\iota x B$. We may in fact assume that t always has this form. Suppose for example that t has the form $f \iota x B(x)$, then an application of

$$\dfrac{\forall x A(x) \qquad \dfrac{\begin{matrix}\vdots \\ \iota x B(x) \in I\end{matrix}}{f \iota x B(x) \in I}}{A(f \iota x B(x))}$$

could be replaced by

$$\dfrac{\dfrac{\dfrac{\forall x A(x) \qquad \dfrac{[z \in I]}{fz \in I}}{A(fz)}}{\forall z A(fz)} \qquad \begin{matrix}\vdots \\ \iota x B(x) \in I\end{matrix}}{A(f \iota x B(x))}$$

This example can clearly be generalized. It remains for us to consider the rule (a) and the rule

$$(b') \qquad \frac{\forall x\, A(x) \qquad \iota x\, B(x) \in I}{A(\iota x\, B(x))}$$

Instead of treating the rules (a) and (b′) it is convenient to treat the following rule

$$(d) \qquad \frac{\exists_1 x\, A(x)}{\forall z_1 \dots \forall z_n (\forall x (A(x) \rightarrow B(x)) \leftrightarrow B(\iota x\, A(x)))}$$

where $z_1, \dots, z_n$ are all free variables in $A(x)$ and $B(x)$ other than x. (For simplicity we shall in general ignore the variables $z_1, \dots, z_n$ below). From the rule (d), the rule (a) follows immediately by putting $A(x)$ for $B(x)$ and noting that $\forall x(A(x) \rightarrow A(x))$ is provable from no assumptions other than those on which $\exists_1 x\, A(x)$ depends. The rule (b′) can be derived as follows: Assume that we have derivations

$$\begin{array}{ccc} \vdots & & \vdots \\ \forall x\, A(x) & \text{and} & \iota x\, B(x) \in I \end{array}$$

then we must have derivations

$$\begin{array}{ccc} x \in I & & \\ \vdots & & \vdots \\ B(x) \in F & \text{and} & \exists_1 x\, B(x) . \end{array}$$

Then we proceed as follows

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[x\in I]\\ \vdots\\ B(x)\in F\end{matrix}\qquad [B(x)]\qquad \dfrac{\begin{matrix}\vdots\\ \forall x A(x)\end{matrix}\quad [x\in I]}{A(x)}}{B(x)\to A(x)}}{\forall x(B(x)\to A(x))}\qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \exists_1 x B(x)\end{matrix}}{\forall x(B(x)\to A(x))\leftrightarrow A(\iota x B(x))}}{\forall x(B(x)\to A(x))\to A(\iota x B(x))}}{A(\iota x B(x))}$$

So it remains for us to prove that if we have a derivation in the description-free system of

$$\exists_1 x A^{\circ}(x)$$

then there is a description-free derivation of

$$\forall x(A^{\circ}(x) \to B^{\circ}(x)) \leftrightarrow B(\iota x A(x))^{\circ} .$$

The proof is by induction on the number of logical symbols in $B(x)$, counting the description operator as a logical symbol.

Case 1. B is an atomic formula-expression.

Case 1.1. $B(x)$ contains no logical symbols. Then

$$B(\iota x A(x))^{\circ} = \forall x(A^{\circ}(x) \to B(x))$$

which is

$$\forall x(A^{\circ}(x) \to B^{\circ}(x))$$

and the result follows.

Caso 1.2. $B(x)$ contains at least one description none of which is $\iota x A(x)$ and none of which contains the variable x free.

Then $\iota x A(x)$ is an outermost description in $B(\iota x A(x))$ which can be written

$$B'(\iota x A(x), \iota y C(y))$$

Where $B'(x,y)$ is a description-free atomic formula (assuming for simplicity that $B(x)$ contains only one description). Then we have

$$B'(\iota x A(x), \iota y C(y))^{o} = \forall x \forall y (A^{o}(x) \& C^{o}(y) \rightarrow B'(x,y))$$

In the description-free system this is clearly interdeducible with

$$\forall x (A^{o}(x) \rightarrow \forall y (C^{o}(y) \rightarrow B'(x,y))$$

which is

$$\forall x (A^{o}(x) \rightarrow B^{o}(x)) .$$

Case 1.3. $B(x)$ contains at least one description one of which coincides with $\iota x A(x)$ but none of which contains x free.

Let

$$B(\iota x A(x)) = B'(\iota x A(x), \iota y C(y), \iota z D(z))$$

where $\iota x A(x)$ and $\iota y C(y)$ coinside. Then $\forall x (A^{o}(x) \rightarrow B^{o}(y))$ is

$$\forall x (A^{o}(x) \rightarrow \forall y \forall z (C^{o}(y) \& D^{o}(z) \rightarrow B'(x,y,z,)))$$

which is interdeducible with

(1) $$\forall x \forall y \forall z (A^{o}(x) \& A^{o}(y) \& D^{o}(z) \rightarrow B'(x,y,z))$$

The formula-expression $B(\iota x A(x))^{o}$ is

(2) $$\forall x \forall z (A^{o}(x) \& D^{o}(z) \rightarrow B'(x,x,z)).$$

By the induction hypothesis we have

$$\exists_1 x A^o(x)$$

from which we can derive $A^o(x) \,\&\, A^o(y) \rightarrow x = y$, and since

$$x = y \rightarrow (B'(x,x,z) \leftrightarrow B'(x,y,z))$$

we have

$$A^o(x) \,\&\, A^o(y) \rightarrow (B'(x,x,z) \leftrightarrow B'(x,y,z))$$

by means of which the interderivability of (1) and (2) is easy.

Case 1.4. $B(x)$ contains at least one description and x is free only in at least one of these descriptions none of which coinsides with $\imath x A(x)$.

Then we can write

$$B(x) = B'(\imath y\, C(y,x),\ \imath z D(z)).$$

Then $\forall x(A^o(x) \rightarrow B^o(x))$ is

$$\forall x(A^o(x) \rightarrow \forall y \forall z(C^o(y,x) \,\&\, D^o(z) \rightarrow B'(y,z)))$$

or equivalently

(3) $$\forall x \forall y \forall z(A^o(x) \,\&\, C^o(y,x) \,\&\, D^o(z) \rightarrow B'(y,z))$$

and $B(\imath x A(x))^o$ is

(4) $$\forall y \forall z(C(y, \imath x A(x))^o \,\&\, D^o(z) \rightarrow B'(y,z)).$$

By the induction hypothesis applied to $C(y,x)$ we have

$$\forall x(A^o(x) \rightarrow C^o(y,x)) \leftrightarrow C^o(y, \imath x A(x))$$

The formula-expression (3) is interderivable with

$$\forall y \forall z(\exists x(A^{o}(x) \mathbin{\&} C^{o}(y,x)) \mathbin{\&} D^{o}(z) \rightarrow B'(y,z))$$

so to establish the interderivability of (3) and (4) it is sufficient to have

$$\exists x(A^{o}(x) \mathbin{\&} C^{o}(y,x)) \leftrightarrow \forall x(A^{o}(x) \rightarrow C^{o}(y,x))$$

which follows from $\exists_{1} x A^{o}(x)$.

<u>Case 1.5.</u> $B(x)$ contains at least one description and x is free in at least one of these and in at least one occurence not in one of these descriptions (none of which coinsides with $\iota x A(x)$).

Then we can write

$$B(x) = B'(x, \iota y C(y,x), \iota z D(z))$$

and $\forall x(A^{o}(x) \rightarrow B^{o}(x))$ is

$$\forall x(A^{o}(x) \rightarrow \forall y \forall z(C^{o}(y,x) \mathbin{\&} D^{o}(z) \rightarrow B'(x,y,z)))$$

or equivalently

$$\forall x \forall y \forall z(A^{o}(x) \mathbin{\&} C^{o}(y,x) \mathbin{\&} D^{o}(z) \rightarrow B'(x,y,z)),$$

and $B(\iota x A(x))^{o}$ is

$$\forall x \forall y \forall z(A^{o}(x) \mathbin{\&} C(y, \iota x A(x))^{o} \mathbin{\&} D^{o}(z) \rightarrow B'(x,y,z))$$

By the induction hypothesis we have

$$\forall x(A^{o}(x) \rightarrow C^{o}(y,x)) \leftrightarrow C(y, \iota x A(x))^{o}$$

so it is sufficient to prove that

$$A^{o}(x) \ \& \ C^{o}(y,x) \leftrightarrow A^{o}(x) \ \& \ \forall x(A^{o}(x) \rightarrow C^{o}(y,x))$$

which follows easily using $\exists_1 x A(x)$ and the elimination rule for identity.

Case 1.6. $B(x)$ contains at least one description and x is free in at least one of these and one of the descriptions in $B(x)$ coincides with $\iota x A(x)$. If we put

$$B(x) = B'(\iota y C(y,x), \iota z D(z))$$

it is clear that the description $\iota y C(y,x)$ can not coincide with $\iota x A(x)$, so it must be $\iota z D(z)$. The result follows then by a combination of case 1.3 and 1.5.

Case 2. $B(x)$ is composite.

Case 2.1. $B(x)$ is $B_1(x) \ \& \ B_2(x)$. We have a description-free derivation of $\exists_1 x A^{o}(x)$ and we want to derive

$$\forall x(A^{o}(x) \rightarrow B_1^{o}(x) \ \& \ B_2^{o}(x)) \leftrightarrow B_1(\iota x A(x))^{o} \ \& \ B_2(\iota x A(x))^{o}$$

By the induction hypothesis we have

$$\forall x(A^{o}(x) \rightarrow B_1^{o}(x)) \leftrightarrow B_1(\iota x A(x))^{o}$$

and

$$\forall x(A^{o}(x) \rightarrow B_2^{o}(x)) \leftrightarrow B_2(\iota x A(x))^{o}$$

and the result follows then easily.

Case 2.2. $B(x)$ is $B_1(x) \rightarrow B_2(x)$. We have a description-free derivation of $\exists_1 x A^o(x)$ and we want to give a description-free derivation of

$$\forall x(A^o(x) \rightarrow (B_1^o(x) \rightarrow B_2^o(x)) \leftrightarrow (B_1(\imath x A(x))^o \rightarrow B_2(\imath x A(x))^o)$$

By the induction hypothesis we have

$$\forall x(A^o(x) \rightarrow B_1^o(x)) \leftrightarrow B_1(\imath x A(x))^o$$

and

$$\forall x(A^o(x) \rightarrow B_2^o(x)) \leftrightarrow B_2(\imath x A(x))^o.$$

The formula-expression $\forall x(A^o(x) \rightarrow B_1^o(x) \rightarrow B_2^o(x))$ is interderivable with

$$\exists x(A^o(x) \rightarrow B_1^o(x)) \rightarrow \forall x(A^o(x) \rightarrow B_2^o(x)).$$

Using $\exists_1 x A^o(x)$ we can derive

$$\forall x(A^o(x) \rightarrow B_1^o(x)) \leftrightarrow \exists x(A^o(x) \rightarrow B_1^o(x))$$

and the result follows.

Case 2.3. $B(x)$ is $\forall y B_1(y,x)$. We want to prove that

$$\forall x(A^o(x) \rightarrow \forall y B_1^o(y,x)) \leftrightarrow \forall y B_1(y, \imath x A(x))^o$$

this follows easily by the induction hypothesis:

$$\forall x(A^o(x) \rightarrow B_1^o(y,x)) \leftrightarrow B_1(y, \imath x A(x))^o.$$

This completes the proof of theorem 5.1.2.

5.1.3. <u>THEOREM. For each formula-expression A, if we have a derivation of $A \in F$, then we can find a derivation of $A \leftrightarrow A^{o}$.</u>

Proof. The proof is by induction on the number of logical symbols in A.

Case 1. A is atomic. If A does not contain descriptions the result is immediate. Let

$$A = A'(\iota x B(x))$$

where $\iota x B(x)$ is the outermost description in A (assuming for simplicity that A contains only one such description, which is no restriction). Then

$$A^{o} = \forall x (B^{o}(x) \rightarrow A'(x)).$$

By the induction hypothesis there is a derivation of

(5) $$B(x) \leftrightarrow B^{o}(x).$$

We have a derivation of $A'(\iota x B(x)) \in F$. This derivation must contain a derivation of $\exists_1 x B(x)$. We can then derive

$$B(x) \leftrightarrow x = \iota x B(x)$$

as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists_1 x B(x)}{\iota x B(x) \in I} \quad x \in I}{x = \iota x B(x) \in F} \quad [x = \iota x B(x)] \quad \dfrac{\exists_1 x B(x)}{B(\iota x B(x))}}{B(x)}}{x = \iota x B(x) \rightarrow B(x)}$$

and in the other direction we have

$$\dfrac{\dfrac{\exists_1 x\, B(x)}{\forall x \forall y (B(x) \,\&\, B(y) \to x = y)} \quad x \in I}{\forall y (B(x) \,\&\, B(y) \to x = y)} \qquad \dfrac{\exists_1 x\, B(x)}{\iota x\, B(x) \in I}$$

$$B(x) \,\&\, B(\iota x\, B(x)) \to x = \iota x\, B(x)$$

$$\vdots$$

$$\dfrac{\dfrac{\exists_1 x\, B(x)}{B(\iota x\, B(x))} \qquad B(\iota x\, B(x)) \to (B(x) \to x = \iota x\, B(x))}{B(x) \to x = \iota x\, B(x)}$$

By (5) we have

$$B^{\circ}(x) \leftrightarrow x = \iota x\, B(x)$$

By this and the elimination rule for indentity we have

$$B^{\circ}(x) \to (A'(\iota x\, B(x)) \to A'(x))$$

or

$$A'(\iota x\, B(x)) \to (B^{\circ}(x) \to A'(x))$$

which is

$$A \to A^{\circ}.$$

Conversely assume A°, i.e.

$$\forall x (B(x) \to A'(x))$$

By the induction hypothesis (5) we can find a derivation of $\exists_1 x\, B^{\circ}(x)$. Hence we can derive $\iota x\, B^{\circ}(x) \in I$ and then, using $\forall$-elimination, we have

$$B^{\circ}(\iota x\, B^{\circ}(x)) \to A$$

By the $\supset$-rule, modus ponens A follows, hence

$$A^{o} \rightarrow A.$$

Case 2. A is composite. The result follows in these cases immediately by the induction hypothesis and the proof is complete.

Using theorem 5.1.2 and 5.1.3 we now have the promised result. From theorem 5.1.3. it follows that A is derivable in the original system iff A^{o} is so derivable and from theorem 5.1.2 it follows that A^{o} is derivable in the original system iff A^{o} is derivable in the description-free system. Hence, we have:

5.1.4. THEOREM (Eliminability). If there is a derivation of $A \in F$, then A is derivable in the original system if and only if A^{o} is derivable in the description-free system.

5.2. As mentioned in section 4, it is possible to give a characterization of the meta-logical predicate I in terms of the derivability predicate. In this section we shall prove more than that. We shall associate with each term-expression t a description-free formula-expression I(t) and with each formula-expression A a description-free formula-expression F(A) such that

$$t \in I \quad \text{iff} \quad I(t) \text{ is derivable}$$

and

$$A \in I \quad \text{iff} \quad F(A) \text{ is derivable.}$$

By the result on eliminability it follows that derivability can here be understood to refer to derivability in the description-free system.

We define I(t) and F(A) by induction on the construction of t and A as follows:

5.2.1. $I(t) = (t = t)$, if t is a variable or an individual constant

5.2.2. $I(ft_1 \ldots t_n) = I(t_1) \mathbin{\&} \ldots \mathbin{\&} I(t_n)$

5.2.3. $I(\imath x A(x)) = \forall x F(A(x)) \mathbin{\&} \exists_1 x A^o(x)$

5.2.4. $F(\bot) = (\bot \rightarrow \bot)$

5.2.5. $F(Pt_1 \ldots t_n) = I(t_1) \mathbin{\&} \ldots \mathbin{\&} I(t_n)$

5.2.6. $F(A \mathbin{\&} B) = F(A) \mathbin{\&} F(B)$

5.2.7. $F(A \rightarrow B) = F(A) \mathbin{\&} (A^o \rightarrow F(B))$

5.2.8. $F(\forall x A(x)) = \forall x F(A(x))$.

It is easy to see that I(t) and F(A) are always description-free.

5.2.9. THEOREM. *For any term-expression t and any formula-expression A we have*

$$t \in I \quad \text{iff} \quad I(t) \text{ is derivable}$$

and

$$A \in F \quad \text{iff} \quad F(A) \text{ is derivable.}$$

Proof. The proof is by induction over t and A.

Case 1. t is a variable or an individual constant. If we have a derivation

$$\begin{array}{c} \vdots \\ t \in I \end{array},$$

the following is a derivation of I(t)

$$\dfrac{\begin{array}{c}\vdots\\ t \in I\end{array}}{t = t}\ .$$

Conversely, assume that we have a derivation of $I(t) = (t = t)$. By theorem 3.2.6 we can find a derivation of $t = t \in F$. This derivation must end like this

$$\dfrac{\begin{array}{c}\vdots\\ t \in I\end{array} \qquad \begin{array}{c}\vdots\\ t \in I\end{array}}{t = t \in F}$$

and the result follows.

<u>Case 2.</u> Suppose that we have a derivation of $ft_1 \ldots t_n \in I$. This derivation must look like this

$$\dfrac{\begin{array}{c}\vdots\\ t_1 \in I\end{array} \quad \ldots \quad \begin{array}{c}\vdots\\ t_n \in I\end{array}}{ft_1 \ldots t_n \in I}$$

so we have $t_1 \in I, \ldots, t_n \in I$. By the induction hypothesis we have derivations

$$\begin{array}{c}\vdots\\ I(t_1)\end{array}, \ldots , \begin{array}{c}\vdots\\ I(t_n)\end{array}$$

and $I(ft_1 \ldots t_n)$ is derivable by &-introductions.

Conversely, if we have a derivation of $I(ft_1 \ldots t_n)$, we can derive

$I(t_1), \ldots, I(t_n)$ by &-eliminations. By the induction hypothesis we have $t_1 \in I, \ldots, t_n \in I$ and $ft_1 \ldots t_n \in I$ follows by the f-rule.

Case 3. If we have a derivation of $\iota x A(x) \in I$, it must end like this

$$\frac{\begin{array}{c}\vdots \\ \exists_1 x A(x)\end{array}}{\iota x A(x) \in I}$$

so we have a derivation of $\exists_1 x A(x)$. By theorem 5.1.4, we have then a derivation

$$\begin{array}{c}\vdots \\ \exists_1 x A^{\circ}(x) .\end{array}$$

By theorem 3.2.6, we have also a derivation

$$\begin{array}{c}\vdots \\ \exists_1 x A(x) \in F\end{array}$$

which must contain a derivation

$$\begin{array}{c} x \in I \\ \vdots \\ A(x) \in F .\end{array}$$

By the induction hypothesis we have a derivation

$$\begin{array}{c} x \in I \\ \vdots \\ F(A(x)).\end{array}$$

$I(\iota x A(x))$ is now derived as follows

$$\dfrac{\begin{array}{c}\vdots \\ \exists_1 x A^{o}(x)\end{array} \qquad \dfrac{\begin{array}{c}[x \in I] \\ \vdots \\ F(A(x))\end{array}}{\forall x F(A(x))}}{\exists_1 x A^{o}(x) \,\&\, \forall x F(A(x))} \;.$$

Conversely, assume that we have a derivation of $I(\iota x A(x))$. By &-elimination we have derivations

$$\begin{array}{c}\vdots \\ \exists_1 x A^{o}(x)\end{array} \quad \text{and} \quad \begin{array}{c}\vdots \\ \forall x F(A(x))\end{array}$$

We want to use theorem 5.1.4 to find a derivation of $\exists_1 x A(x)$ in order to conclude that $\iota x A(x) \in I$. But to be able to use theorem 5.1.4, we must first have a derivation of $\exists_1 x A(x) \in F$, or, what amounts to the same thing, a derivation of $\forall x A(x) \in F$ from no new assumptions. We can find such a derivation as follows.

$$\dfrac{\begin{array}{c}\vdots \\ \forall x F(A(x))\end{array} \qquad x \in I}{F(A(x))}$$

By the induction hypothesis we have then a derivation

$$\begin{array}{c} x \in I \\ \vdots \\ A(x) \in F \end{array}$$

and $\forall x A(x) \in F$ follows by the $\forall$-rule.

Case 4. A is $\bot$. Immediate.

Case 5. A is $Pt_1 \ldots t_n$. Like case 2.

Case 6. A is $B \& C$. The result follows as in case 2 easily by the induction hypothesis.

Case 7. Suppose we have a derivation of $A \to B \in F$. This derivation must end like this

$$\frac{\begin{array}{c} \\ \vdots \\ A \in F \end{array} \qquad \begin{array}{c} A \\ \vdots \\ B \in F \end{array}}{A \to B \in F}$$

so we have derivations

$$\begin{array}{c} \\ \vdots \\ A \in F \end{array} \quad \text{and} \quad \begin{array}{c} A \\ \vdots \\ B \in F \end{array}$$

By the induction hypothesis we have then derivations

$$\begin{array}{c} \\ \vdots \\ F(A) \end{array} \quad \text{and} \quad \begin{array}{c} A \\ \vdots \\ F(B) \end{array}$$

Since $A^o \in F$, we can use theorem 5.1.3 to derive $F(B)$ from the assumption of A^o like this

$$\dfrac{A^{o} \rightarrow A \quad A^{o}}{A} \\ \vdots \\ F(B)$$

and $A^{o} \rightarrow F(B)$ follows by $\rightarrow$-introduction. A &-introduction then gives us $F(A \rightarrow B)$.

Conversely, assume that we have a derivation of

$$F(A) \mathbin{\&} (A^{o} \rightarrow F(B)) .$$

By &-elimination we have derivations

$$\vdots \qquad\qquad \vdots \\ F(A) \quad \text{and} \quad A^{o} \rightarrow F(B)$$

By the induction hypothesis we have $A \in F$, so we can use A as an assumption to derive F(B) as follows

$$\dfrac{A^{o} \rightarrow F(B) \quad \dfrac{A \rightarrow A^{o} \quad A}{A^{o}}}{F(B)}$$

By the induction hypothesis we have a derivation

$$A \\ \vdots \\ B \in F$$

and the result follows by the $\rightarrow$-rule.

Case 8. If we have a derivation of $\forall x A(x) \in F$, we must have a derivation

$$\begin{array}{c} x \in I \\ \vdots \\ A(x) \in F \end{array}$$

and by the induction hypothesis we have

$$\begin{array}{c} x \in I \\ \vdots \\ F(A(x)) \end{array}$$

and $\forall x F(A(x))$ follows by $\forall$-introduction.

Conversely, if we have a derivation of $\forall x F(A(x))$, we can derive $F(A(x))$ from the assumption of $x \in I$ as follows

$$\frac{\forall x F(A(x)) \qquad x \in I}{F(A(x))}$$

By the induction hypothesis we have a derivation

$$\begin{array}{c} x \in I \\ \vdots \\ A(x) \in F \end{array}$$

and $\forall x A(x) \in F$ follows by the $\forall$-rule.

This completes the proof.

6. Model theory

6.1. The purpose of this final section is to develop a model theory for the formal system of section 2 and 3. We shall introduce the notion of a structure and define what it means for a closed term-expression to have a reference in a structure and what it means for a closed formula-expression to have a truth-value and to be valid in a structure.

For simplicity we shall assume that the language has only one one-place function symbol f and only one one-place predicate symbol P. The development can be generalized in an obvious way so this is no loss of generality.

With respect to this language, a structure

$$S = \langle \bar{I}, \bar{f}, \bar{P} \rangle$$

consists of the following things:

6.1.1. A (possibly empty) set $\bar{I}$ of individuals.

6.1.2. An assignment to the function symbol f of a function $\bar{f}:\bar{I} \rightarrow \bar{I}$. (If f were an individual constant we would of course assign to f an individual $\bar{f} \in \bar{I}$).

6.1.3. An assignment to the predicate symbol P of a set $\bar{P} \subseteq \bar{I}$.

We extend our language by introducing names of the individuals in $\bar{I}$. To each individual we introduce exactly one name. We use a,b, c,... as syntactical notations for these names. $\bar{a},\bar{b},\bar{c}$, ... denote the

the individuals whose names are a,b,c,... . These names are individual constants so we also add the following axioms

$$a \in I$$

for each name a, to the axioms and rules of section 3. The notions of a term-expression, formula-expression, derivation, term, formula and theorem will, unless otherwise stated, refer to this enlarged language which we call <u>the language of S</u>. The language of section 2 will be called the original language.

We shall define a <u>partial function</u> V_S on the set of all closed term-expressions and formula-expressions. The values of V_S will be either individuals or one of the truth-values 1 (truth) or 0 (falsity). The definition of V_S is by induction on the construction of the term-expressions and the formula-expressions.

6.1.4. $V_S(a) = \bar{a}$, for each individual constant a.

6.1.5. $V_S(ft)$ is defined iff $V_S(t)$ is defined. If $V_S(ft)$ is defined, then

$$V_S(ft) = \bar{f}(V_S(t)).$$

6.1.6. $V_S(\imath x A(x))$ is defined iff $V_S(A(a))$ is defined for all $\bar{a} \in \bar{I}$ and $\{\bar{a} \mid V_S(A(a)) = 1\}$ is a singleton set. If $V_S(\imath x A(x))$ is defined, then

$$V_S(\imath x A(x)) = \bar{b} \text{ iff } \{\bar{b}\} = \{\bar{a} \mid V_S(A(a)) = 1\}.$$

6.1.7. $V_S(Pt)$ is defined iff $V_S(t)$ is defined. If $V_S(t)$ is defined, then

$$V_S(Pt) = 1 \text{ if } V_S(t) \in \bar{P}$$

and

$$V_S(Pt) = 0 \text{ if } V_S \notin \bar{P}.$$

$V_S(t = s)$ is defined iff $V_S(t)$ and $V_S(s)$ are both defined, and if this is the case, then

$$V_S(t = s) = 1 \text{ if } V_S(t) = V_S(s)$$

and

$$V_S(t = s) = 0 \text{ otherwise.}$$

6.1.8. $V_S(\bot) = 0$.

6.1.9. $V_S(A \& B)$ is defined iff $V_S(A)$ and $V_S(B)$ are both defined. If $V_S(A \& B)$ is defined, then

$$V_S(A \& B) = 1 \text{ if } V_S(A) = V_S(B) = 1$$

and

$$V_S(A \& B) = 0 \text{ otherwise.}$$

6.1.10. $V_S(A \to B)$ is defined iff

(i) $V_S(A)$ is defined

and

(ii) $V_S(A) = 1$ only if $V_S(B)$ is defined.

If $V_S(A \to B)$ is defined, then

$$V_S(A \to B) = 1 \text{ if } V_S(A) = 0 \text{ or } V_S(B) = 1$$

and

$$V_S(A \to B) = 0 \text{ otherwise.}$$

6.1.11. $V_S(\forall x A(x))$ is defined iff $V_S(A(a))$ is defined for all $\bar{a} \in \bar{I}$. If $V_S(\forall x A(x))$ is defined, then

$$V_S(\forall x A(x)) = 1 \text{ if } V_S(A(a)) = 1 \text{ for all } \bar{a} \in \bar{I},$$

and

$$V_S(\forall x A(x)) = 0 \text{ otherwise.}$$

6.1.12. A closed term-expression t is said to _have a reference in S_ if $V_S(t)$ is defined; a closed formula-expression A is said to _have a truth-value in S_ if $V_S(A)$ is defined and S is said to _be valid in S_ if $V_S(A)$ is defined and = 1.

The soundness and the completeness of the deductive system of section 3 then means the following three things (where we refer to the original language).

6.2. _THEOREM. For each closed term-expression and each closed formula-expression we have_

6.2.1. _t is a term iff t has a reference in each structure S_

6.2.2. _A is a formula iff A has a truth-value in each structure S_

6.2.3. _A is a theorem iff A is valid in each structure S._

6.2.4. _Remark._ Let us for the moment confine our attention to the description-free system. In this system each closed term-expression is a term and each closed formula-expression is a formula. Hence, V_S becomes a total function when restricted to this system. All talk of V_S:s being defined or not becomes superfluous and the definition of V_S above is a definition of a standard valuation function (despite the fact that we also include the empty domain). The propositions 6.2.1 and 6.2.2 become trivially true and the truth of 6.2.3 is also well-known. The fact that we permit the empty domain offers no special difficulty. An ordinary Henkin-style completeness proof goes through.

In what follows I shall therefore take the completeness of the description-free system for granted.

In order to prove the soundness we need the following result.

6.2.5. Substitution property. Let S be a structure and let t be a closed term-expression in the language of S. Let u(x) and A(x) be a term-expression and a formula-expression, respectively, in the language of S containing only x free.

If $V_S(t)$, $V_S(u(t))$ and $V_S(A(t))$ are all defined and if $V_S(t) = \bar{a}$, then so are $V_S(u(a))$ and $V_S(A(a))$ and

$$V_S(u(t)) = V_S(u(a))$$

and

$$V_S(A(t)) = V_S(A(t)).$$

The proof, by induction over u(x) and A(x), is straightforward and will be omitted here.

In order to exhibit more fully the meaning of the deductive rules of section 3, I shall give the proof of the soundness in detail.

6.2.6. Proof of soundness. We prove the statements

If $t \in I$ is derivable, then $V_S(t)$ is defined for all S

If $A \in F$ is derivable, then $V_S(A)$ is defined for all S

If A is derivable, then $V_S(A) = 1$ for all S

simultaneously by induction on the lengths of the derivations of $t \in I$, $A \in F$ and A.

Let S be a fixed structure and consider closed term-expressions and formula-expressions in the language of S.

Case 1. t is a name, then $V_S(t)$ is defined by 6.1.4.

Case 2.

$$\frac{t \in I}{ft \in I}$$

By the induction hypothesis $V_S(t)$ is defined and hence, so is $V_S(ft)$ by 6.1.5.

Case 3.

$$\frac{\exists_1 x A(x)}{\imath x A(x) \in I}$$

By the induction hypothesis $V_S(\exists_1 x A(x))$ is defined and = 1. This means in particular that $V_S(\neg \forall x \neg A(x))$ is defined, so $V_S(A(a))$ is defined for all $\bar{a} \in \bar{I}$. $V_S(\exists_1 x A(x)) = 1$ means also that the set $\{\bar{a} \mid V_S(A(a)) = 1\}$ contains exactly one member, so $V_S(\imath x A(x))$ is defined by 6.1.6.

Case 4. $V_S(\perp)$ is defined by 6.1.8.

Case 5.

$$\frac{t \in I}{Pt \in I}$$

By the induction hypothesis $V_S(t)$ is defined and hence so is $V_S(Pt)$ by 6.1.7.

The rule

$$\frac{t \in I \quad s \in I}{t = s \in F}$$

is treated similarly.

Case 6.

$$\frac{A \in F \quad B \in F}{A \& B \in F}$$

$V_S(A)$ and $V_S(B)$ are defined by the induction hypothesis, hence so is $V_S(A \& B)$ by 6.1.9.

Case 7.

$$\frac{A \in F \quad \begin{matrix}[A] \\ \vdots \\ B \in F\end{matrix}}{A \rightarrow B \in F}$$

By the induction hypothesis $V_S(A)$ is defined. Also, since we have a derivation

$$\begin{matrix}A \\ \vdots \\ B \in F\end{matrix}$$

it follows by the induction hypothesis that $V_S(A) = 1$ only if $V_S(B)$ is defined. Hence, $V_S(A \rightarrow B)$ is defined by 6.1.10.

Case 8.

$$\frac{\begin{matrix}[x \in I] \\ \vdots \\ A(x) \in F\end{matrix}}{\forall x\, A(x) \in F}$$

For an arbitrary name a, we have a derivation

$$a \in I$$
$$\vdots$$
$$A(a) \in F$$

By the induction hypothesis it follows that $V_S(\bar{a})$ is defined for all $\bar{a} \in \bar{I}$, hence $V_S(\forall x A(x))$ is defined by 6.1.11.

Case 9.

$$\frac{\exists_1 x A(x)}{A(\iota x A(x))}$$

Using the substitution property this case follows like case 3.

The cases in which the result follows by &-introduction, &-elimination or $\rightarrow$-elimination, the result follows immediately by the induction hypothesis.

Case 10.

$$\frac{A \in F \qquad \begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \rightarrow B}$$

By the induction hypothesis $V_S(A)$ is defined and $V_S(A) = 1$ only if $V_S(B)$ is defined and $= 1$. By 6.1.10 it follows that $V_S(A \rightarrow B)$ is defined and $= 1$.

Case 11.

$$\frac{\begin{matrix}[x \in I] \\ \vdots \\ A(x)\end{matrix}}{\forall x A(x)}$$

For each name a, we have a derivation

$$\begin{array}{c} a \in I \\ \vdots \\ A(a) \end{array}$$

so by the induction hypothesis $V_S(A(a))$ is defined and = 1 for all $\bar{a} \in \bar{I}$ which means that $V_S(\forall x A(x)) = 1$.

Case 12.

$$\frac{\forall x A(x) \qquad t \in I}{A(t)}$$

By the induction hypothesis $V_S(t)$ is defined. Suppose that $V_S(t) = \bar{a} \in \bar{I}$. Also, $V_S(\forall x A(x)) = 1$ which means that $V_S(A(b)) = 1$ for all $\bar{b} \in \bar{I}$. By the substitution property we have $V_S(A(t)) = V_S(A(a)) = 1$, and the result follows.

Case 13.

$$\frac{\neg A \in F \qquad \begin{array}{c}[\neg A] \\ \vdots \\ \bot\end{array}}{A}$$

By the induction hypothesis $V_S(\neg A)$ is defined, hence so is $V_S(A)$. The assumption that $V_S(A) = 0$ implies that $V_S(\bot) = 1$ which contradicts 6.1.8. Hence $V_S(A) = 1$.

Case 14.

$$\frac{t \in I}{t = t}$$

By the induction hypothesis $V_S(t)$ is defined and since $V_S(t) = V_S(t)$, $V_S(t = t) = 1$ follows.

Case 15.

$$\frac{t = s \qquad A(t)}{A(s)}$$

By the induction hypothesis $V_S(t = s) = 1$ and $V_S(A(t)) = 1$. The former means that $V_S(t) = V_S(s) = \bar{a} \in \bar{I}$ for some name a. Hence $V_S(A(s)) = V_S(A(a)) = V_S(A(t)) = 1$ by the substitution property.

This completes the proof.

The following lemma gives us (non-constructively) the other half of 6.2.3.

6.2.7. LEMMA. If A is a formula which is not a theorem, then there is a structure in which A is not valid.

Proof. Assume that A is a formula which is not derivable. By the eliminability theorem 5.1.4, A^o is not derivable in the description-free system. By the completeness of the description-free system there is a structure S such that

$$V_S(A^o) = 0.$$

By theorem 5.1.3 and the soundness result we have

$$V_S(A) = V_S(A^o) = 0$$

and the lemma follows.

6.3. In order to prove 6.2.1 and 6.2.2 from right to left we need the following result which is a model theoretic analogue of theorem 5.2.9.

6.3.1. LEMMA. Let S be an arbitrary structure. For each closed term-expression t and each closed formula-expression A we have

$$V_S(t) \text{ is defined iff } V_S(I(t)) = 1$$

and

$$V_S(A) \text{ is defined iff } V_S(F(A)) = 1.$$

The proof of this lemma by induction over t and A is straightforward and is left to the reader.

With the following lemma, the proof of theorem 6.2 is complete.

6.3.2. LEMMA. Let t be a closed term-expression and A a closed formula-expression. If $t \in I$ is not derivable, then there is a structure S such that $V_S(t)$ is not defined, and if $A \in F$ is not derivable, then there is a structure S such that $V_S(A)$ is not defined.

Proof. Suppose that $t \in I$ is not derivable. By theorem 5.2.9 it follows that I(t) is not derivable. Since I(t) is description-free it is not derivable in the description-free system. By the completeness of the description-free system, there is a structure S such that $V_S(I(t)) = 0$. By lemma 6.3.1 it follows that $V_S(t)$ is undefined.

The case in which $A \in F$ is not derivable is treated similarly and the lemma follows.

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