Notes

- Class time/place changed
- Next class (Tuesday Sept 18) is **cancelled**
  - We'll find a time to make it up

Interpolation in 1D

- Linear interpolation
- Polynomial interpolation
  - Lagrange formula
  - Dangers
- Splines and more
- Can be extended to 2d etc with tensor products

Function Space Interpretation

- Rather than think of it as a discrete problem:

  "How do I estimate function at a given point given data at nearby points"

  think in terms of function space:

  "Which function from my chosen space fits the data?"

- This perspective will come up again and again

1D Function Spaces

- Piecewise-linear interpolation
- Polynomial interpolation
- Splines
- Choice of basis

Scattered Data Interpolation

- Look at something more interesting
- Arbitrarily scattered data points in multiple dimensions
- How do we fit a smooth surface through them?

- Triangulation
- Radial Basis Functions
- Moving Least-Squares

Triangulation

- First construct a mesh through the data points
- Then interpolate on each triangle separately
- In 2D this works fine
- In 3D (tetrahedralization) it gets tricky
- Doesn't scale well past 3D

- We'll talk about mesh generation later

Radial Basis Functions (RBF)

- Assume data is in the span of a set of radial basis functions
- Solve for the right coefficients

RBF + polynomials

- Problem: RBF's can't even get constants correct
- Fix by adding a low order polynomial term
- Under-constrained system!
- What extra constraints should we add?

PDE Interpretation

- Think of the problem physically: what problem are we really solving? (another perspective which will come up again and again)
- We want the smoothest surface which goes through the given points
- Define smoothest, then solve the problem exactly (gives a differential equation…)

First Try in 1D

- Try to minimize a measure of how 'wobbly' the function is:

$$\min_{f(x_i)=f_i} \int_{-\infty}^{\infty} \tfrac{1}{2}\left(f'(x)\right)^2 dx$$

- Calculus of variations gives:

$$f''(x) = 0 \quad \text{in } (x_{i-1}, x_i)$$
$$f(x_i) = f_i$$

Exact Solution

- Direct method: differential equation has piecewise linear solution
  - But this is too hard to do directly in more dimensions…
- Method of fundamental solution:
  - First find a function that satisfies

$$\phi''(x) = \delta(x)$$

  - Simplest (symmetric) answer is:

$$\phi(x) = \tfrac{1}{2}|x|$$

Fundamental Solution…

- Then we know the exact solution is of the form

$$f(x) = A + \sum_{i=1}^{n} \lambda_i \phi(x - x_i)$$
$$= A + \sum_{i=1}^{n} \tfrac{1}{2} \lambda_i |x - x_i|$$

- The constant A doesn't affect f'
- Also want f' → 0 at infinity
- Away from all data points, f' is proportional to $\sum_{i=1}^{n} \lambda_i$

Simple 1D RBF

- Putting this together, our equations are then:

$$A+\sum_{i=1}^{n}\lambda_i\phi(x_j-x_i)=f_j \quad j=1,\ldots,n$$

$$\sum_{i=1}^{n}\lambda_i=0$$

- n+1 linear equations for n+1 unknowns

Smoother Interpolation

- Minimize curvature

$$\min_{f(x_i)=f_i}\int_{-\infty}^{\infty}\tfrac{1}{2}\left(f''(x)\right)^2 dx$$

- Calculus of variations gives us:

$$f^{(iv)}(x)=0 \quad \text{in } (x_{i-1},x_i)$$
$$f(x_i)=f_i$$

Smoother Fundamental Solution

- Our basis function is now

$$\phi(x)=\tfrac{1}{12}|x|^3$$

and we also include a linear term in polynomial (doesn't change f'')

- So our solution is of the form:

$$f(x)=A+Bx+\sum_{i=1}^{n}\lambda_i\phi(x-x_i)$$

Boundedness

- Outside of the data points, f'' is:

$$f''(x)=\pm\sum\tfrac{1}{2}\lambda_i(x-x_i)$$
$$=\pm\tfrac{1}{2}\left(\sum\lambda_i\right)x\mp\tfrac{1}{2}\left(\sum\lambda_i x_i\right)$$

- Setting this to zero gives two conditions on the coefficients…
- n+2 equations for n+2 unknowns

More Dimensions

- Same approach generalizes
- E.g. "thin-plate spline" comes out of

$$\min_{f(x_i)=f_i}\int\tfrac{1}{2}\left(\nabla\bullet\nabla f\right)^2$$

- The Laplacian $\nabla\bullet\nabla$ is a measure of mean curvature
- Calculus of variations gives the "biharmonic equation":

$$\nabla\bullet\nabla\ \nabla\bullet\nabla f=0$$

Thin-Plate Spline

- In 2D, fundamental solution is proportional to

$$\phi(x)=|x|^2\log|x|$$

and in 3D: $\phi(x)=|x|$

- Include a linear polynomial
- Boundedness conditions are:

$$\sum\lambda_i=0,\quad\sum\lambda_i x_i=0,\quad\sum\lambda_i y_i=0,\quad\sum\lambda_i z_i=0$$

Other RBF's

- Other basis functions can be used of course
- Usual alternatives:
  - Triharmonic basis function: $\phi(x) = |x|^3$
  - Multiquadric: $\phi(x) = \sqrt{|x|^2 + c^2}$
  - Gaussian: $\phi(x) = \exp\left(-\frac{x^2}{c^2}\right)$
- With or without low order polynomial

The Equations

- In all cases, we end up with a linear system to solve
- How do we solve it?
- Gaussian Elimination is the usual answer