Notes

- No extra class tomorrow

PDE's

- Subject of Partial Differential Equations is vast
- We'll focus on one particularly important equation: $\nabla \bullet \nabla \phi = f$
  - Called Poisson's equation (if right hand side is zero, Laplace's equation)
- Arises almost everywhere
  - Minimization of norm of gradient (see RBF's)
  - Gravitational/electrostatic potential
  - Steady state of heat flow and other diffusion processes
  - Stochastic processes (Brownian motion)
  - Fluid dynamics

Reduce to 1D first

- Typical Boundary Value Problem (BVP):

$$\frac{\partial^2 \phi}{\partial x^2} = f \quad \text{for } x \in (0,1)$$

$$\phi(0) = a, \quad \frac{\partial \phi}{\partial x}(1) = b$$

- Boundary conditions:
  specify solution value: "Dirichlet"
  specify solution derivative: "Neumann"
- Can't directly solve as a time integration

Finite Difference Method

- Discretize unknown solution on a grid:

$$\phi(x_i) \approx \phi_i \quad (\text{where } x_i = i\Delta x\ )$$

- Use Taylor series to estimate derivatives from values on grid

$$\frac{\partial^2 \phi}{\partial x^2}(x_i) = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} + O\left(\Delta x^2\right)$$

Discretized Boundary Conditions

- Dirichlet: substitute in known values

$$\phi_0 = a$$

- Neumann: discretize boundary condition, use it to extrapolate

$$\frac{\phi_{N+1} - \phi_N}{\Delta x} \approx \frac{\partial \phi}{\partial x}(1) = b$$

$$\Rightarrow \quad \phi_{N+1} = \phi_N + \Delta x\, b$$

Solve!

- At each grid point we have a linear equation
- Combine into one large linear system (solve for all solution values simultaneously)
- Resulting matrix is symmetric (negative) definite, and **sparse**
  - In fact, in 1D, just tridiagonal…

Higher dimensions

- Lay down a regular grid as before
- Matrices get even bigger, but not quite as simple structure
- Notion of **stencil**: shorthand for matrix

Stability

- The preceding methods work, but not every stencil does
- Need a notion of stability ~ conditioning
- Example problem with central differences
  - Matrix could be singular, or worse
- Example problem with one-sided differences
  - Information propagation is wrong

Finite Difference Limitations

- Accurately treating boundary conditions that don’t line up with the grid
- Adaptivity