Notes

- Please read
  - Kass and Miller, “Rapid, Stable Fluid Dynamics for Computer Graphics”, SIGGRAPH’90
- Blank in last class:
  - At free surface of ocean, p=0 and u^2 negligible, so Bernoulli’s equation simplifies to ϕ_t=-gh
  - Plug in the Fourier mode of the solution to this equation, get the dispersion relation

Dispersion relation again

- Ocean wave has wave vector K
  - K gives the direction, k=|K| is the wave number
  - E.g. the wavelength is $2\pi/k$
- Then the wave speed in deep water is $c = \sqrt{\frac{g}{k}}$
- Frequency in time is $\omega = \sqrt{gk}$
  - For use in formula

$$h(x,z,t) = A(K)\cos\big(K \cdot (x,z) - \omega t\big)$$

Simulating the ocean

- Far from land, a reasonable thing to do is
  - Do Fourier decomposition of initial surface height
  - Evolve each wave according to given wave speed (dispersion relation)
    - Update phase, use FFT to evaluate
- How do we get the initial spectrum?
  - Measure it! (oceanography)

Energy spectrum

- Fourier decomposition of height field:

$$h(x,z,t) = \sum_{i,j} \hat{h}(i,j,t) e^{\sqrt{-1}(i,j)\cdot(x,z)}$$

- “Energy” in K=(i,j) is $S(K) = \left|\hat{h}(K)\right|^2$
- Oceanographic measurements have found models for expected value of S(K) (statistical description)

Phillips Spectrum

- For a “fully developed” sea
  - wind has been blowing a long time over a large area, statistical distribution of spectrum has stabilized
- The Phillips spectrum is: [Tessendorf…]

$$S(K) = A\frac{1}{k^4}\exp\left(\frac{-1}{(kL)^2} - (kl)^2\right)\left(\frac{|K \cdot W|}{|K||W|}\right)^2$$

  - A is an arbitrary amplitude
  - $L=|W|^2/g$ is largest size of waves due to wind velocity W and gravity g
  - Little l is the smallest length scale you want to model

Fourier synthesis

- From the prescribed S(K), generate actual Fourier coefficients

$$\hat{h}(K,0) = \tfrac{1}{\sqrt{2}}\left(X_1 + X_2\sqrt{-1}\right)\sqrt{S(K)}$$

  - X_i is a random number with mean 0, standard deviation 1 (Gaussian)
  - Uniform numbers from unit circles aren’t terrible either
- Want real-valued h, so must have

$$\hat{h}(K) = \hat{h}(-K)^*$$

  - Or give only half the coefficients to FFT routine and specify you want real output

Time evolution

- Dispersion relation gives us ω(K)
- At time t, want $h(x,t) = \sum_{K=(i,j)} \hat{h}(K,0)e^{\sqrt{-1}(K\cdot x - \omega t)}$
  $= \sum_{K=(i,j)} \hat{h}(K,0)e^{-\sqrt{-1}\omega t}e^{\sqrt{-1}K\cdot x}$
- So then coefficients at time t are
  - For j≥0: $\hat{h}(i,j,t) = \hat{h}(i,j,0)e^{-\sqrt{-1}\omega t}$
  - Others: figure out from conjugacy condition (or leave it up to real-valued FFT to fill them in)

Picking parameters

- Need to fix grid for Fourier synthesis (e.g. 1024x1024 height field grid)
- Grid spacing shouldn't be less than e.g. 2cm (smaller than that - surface tension, nonlinear wave terms, etc. take over)
  - Take little l (cut-off) a few times larger
- Total grid size should be greater than but still comparable to L in Phillips spectrum (depends on wind speed and gravity)
- Amplitude A shouldn't be too large
  - Assumed waves weren't very steep

Note on FFT output

- FFT takes grid of coefficients, outputs grid of heights
- It's up to you to map that grid (0…n-1, 0…n-1) to world-space coordinates
- In practice: scale by something like L/n
  - Adjust scale factor, amplitude, etc. until it looks nice
- Alternatively: look up exactly what your FFT routines computes, figure out the "true" scale factor to get world-space coordinates

Tiling issues

- Resulting grid of waves can be tiled in x and z
- Handy, except people will notice if they can see more than a couple of tiles
- Simple trick: add a second grid with a non-rational multiple of the size
  - Golden mean (1+sqrt(5))/2=1.61803… works well
  - The sum is no longer periodic, but still can be evaluated anywhere in space and time easily enough

Choppy waves

- See Tessendorf for more explanation
- Nonlinearities cause real waves to have sharper peaks and flatter troughs than linear Fourier synthesis gives
- Can manipulate height field to give this effect
  - Distort grid with (x,z) -> (x,z)+λD(x,z,t)

$$D(x,t) = \sum_K -\sqrt{-1}\frac{K}{|K|}\hat{h}(K,t)e^{\sqrt{-1}K\cdot x}$$

Choppiness problems

- The distorted grid can actually tangle up (Jacobian has negative determinant - not 1-1 anymore)
  - Can detect this, do stuff (add particles for foam, spray?)
- Can't as easily use superposition of two grids to defeat periodicity… (but with a big enough grid and camera position chosen well, not an issue)

Shallow Water

Shallow water

- Simplified linear analysis before had dispersion relation

$$c = \sqrt{\frac{g}{k}\tanh kH}$$

  - For shallow water, kH is small (that is, wave lengths are comparable to depth)
  - Approximate tanh(x)=x for small x:

$$c = \sqrt{gH}$$

- Now wave speed is independent of wave number, but **dependent** on depth
  - Waves slow down as they approach the beach

What does this mean?

- We see the effect of the bottom
  - Submerged objects (H decreased) show up as places where surface waves pile up on each other
  - Waves pile up on each other (eventually should break) at the beach
  - Waves refract to be parallel to the beach
- We can't use Fourier analysis

PDE's

- Saving grace: wave speed independent of k means we can solve as a 2D PDE
- We'll derive these "shallow water equations"
  - When we linearize, we'll get same wave speed
- Going to PDE's also let's us handle non-square domains, changing boundaries
  - The beach, puddles, …
  - Objects sticking out of the water (piers, walls, …) with the right reflections, diffraction, …
  - Dropping objects in the water

Kinematic assumptions

- We'll assume as before water surface is a height field y=h(x,z,t)
- Water bottom is y=-H(x,z,t)
- Assume water is shallow (H is smaller than wave lengths) and calm (h is much smaller than H)
  - For graphics, can be fairly forgiving about violating this…
- On top of this, assume velocity field doesn't vary much in the y direction
  - u=u(x,z,t), w=w(x,z,t)
  - Good approximation since there isn't room for velocity to vary much in y(otherwise would see disturbances in small length-scale features on surface)
- Also assume pressure gradient is essentially vertical
  - Good approximation since p=0 on surface, domain is very thin

Conservation of mass

- Integrate over a column of water with cross-section dA and height h+H
  - Total mass is ρ(h+H)dA
  - Mass flux around cross-section is ρ(h+H)(u,w)
- Write down the conservation law
- In differential form (assuming constant density):

$$\frac{\partial}{\partial t}\left(h+H\right)+\nabla\cdot\left((h+H)u\right)=0$$

  - Note: switched to 2D so u=(u,w) and ∇=(∂/∂x, ∂/∂z)

Pressure

- Look at y-component of momentum equation:

$$v_t + u \cdot \nabla v + \frac{1}{\rho}\frac{\partial p}{\partial y} = -g + \nu \nabla^2 v$$

- Assume small velocity variation - so dominant terms are pressure gradient and gravity:

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -g$$

- Boundary condition at water surface is p=0 again, so can solve for p:

$$p = \rho g (h - y)$$

Conservation of momentum

- Total momentum in a column:

$$\int_{-H}^{h} \rho \vec{u} = \rho \vec{u}(h + H)$$

- Momentum flux is due to two things:
  - Transport of material at velocity u with its own momentum: $\int_{-H}^{h} (\rho \vec{u})\vec{u}$
  - And applied force due to pressure. Integrate pressure from bottom to top:

$$\int_{-H}^{h} p = \int_{-H}^{h} \rho g (h - y) = \frac{\rho g}{2}(h + H)^2$$

Pressure on bottom

- Not quite done… If the bottom isn't flat, there's pressure exerted partly in the horizontal plane
  - Note p=0 at free surface, so no net force there
- Normal at bottom is: $n = (-H_x, -1, -H_z)$
- Integrate x and z components of pn over bottom
  - (normalization of n and cosine rule for area projection cancel each other out)

$$-\rho g (h + H) \nabla H \, dA$$

Shallow Water Equations

- Then conservation of momentum is:

$$\frac{\partial}{\partial t}(\rho \vec{u}(h + H)) + \nabla \cdot \left(\rho \vec{u}\vec{u}(h + H) + \frac{\rho g}{2}(h + H)^2 \right) - \rho g (h + H) \nabla H = 0$$

- Together with conservation of mass

$$\frac{\partial}{\partial t}(h + H) + \nabla \cdot ((h + H)u) = 0$$

we have the Shallow Water Equations

Note on conservation form

- At least if H=constant, this is a system of conservation laws
- Without viscosity, "shocks" may develop
  - Discontinuities in solution (need to go to weak integral form of equations)
  - Corresponds to breaking waves - getting steeper and steeper until heightfield assumption breaks down

Simplifying Conservation of Mass

- Expand the derivatives:

$$\frac{\partial (h + H)}{\partial t} + u \cdot \nabla (h + H) + (h + H) \nabla \cdot u = 0$$

$$\frac{D(h + H)}{Dt} = -(h + H) \nabla \cdot u$$

- Label the depth h+H with η:

$$\frac{D\eta}{Dt} = -\eta \nabla \cdot u$$

- So water depth gets advected around by velocity, but also changes to take into account divergence

Simplifying Momentum

- Expand the derivatives:

$$(\rho\eta u)_t + \nabla\cdot\left(\rho u u\eta + \frac{\rho g}{2}\eta^2\right) - \rho g\eta\nabla H = 0$$
$$\rho\eta u_t + \rho u\eta_t + \rho u\nabla\cdot(\eta u) + \rho\eta u\cdot\nabla u + \rho g\eta\nabla\eta - \rho g\eta\nabla H = 0$$

- Subtract off conservation of mass times velocity:

$$\rho\eta u_t + \rho\eta u\cdot\nabla u + \rho g\eta\nabla\eta - \rho g\eta\nabla H = 0$$

- Divide by density and depth:

$$u_t + u\cdot\nabla u + g\nabla\eta - g\nabla H = 0$$

- Note depth minus H is just h:

$$u_t + u\cdot\nabla u + g\nabla h = 0$$
$$\frac{Du}{Dt} = -g\nabla h$$

Interpreting equations

- So velocity is advected around, but also accelerated by gravity pulling down on higher water
- For both height and velocity, we have two operations:
  - Advect quantity around (just move it)
  - Change it according to some spatial derivatives
- Our numerical scheme will treat these separately: "splitting"

Linearization

- Again assume not too much velocity variation (i.e. waves move, but water basically doesn't)
  - No currents, just small waves
  - Alternatively: inertia not important compared to gravity
  - Or: numerical method treats the advection separately (see next week!)
- Then drop the nonlinear advection terms
- Also assume H doesn't vary in time

$$h_t = -(h+H)\nabla\cdot u$$
$$u_t = -g\nabla h$$

Wave equation

- Only really care about heightfield for rendering
- Differentiate height equation in time

$$h_{tt} = -h_t\nabla\cdot u - (h+H)\nabla\cdot u_t$$

- Plug in u equation

$$h_{tt} = -h_t\nabla\cdot u + g(h+H)\nabla^2 h$$

- Finally, neglect nonlinear (quadratically small) terms on right to get

$$h_{tt} = gH\nabla^2 h$$

Deja vu

- This is the linear wave equation, with wave speed $c^2=gH$
- Which is exactly what we derived from the dispersion relation before (after linearizing the equations in a different way)
- But now we have it in a PDE that we have some confidence in
  - Can handle varying H, irregular domains…
- Caveat: to handle H going to 0 or negative, we'll in fact use

$$h_{tt} = g(h+H)\nabla^2 h$$

Initial + boundary conditions

- We can specify initial h and h_t
  - Since it's a second order equation
- We can specify h at "open" boundaries
  - Water is free to flow in and out
- Specify $\partial h/\partial n=0$ at "closed" boundaries
  - Water does not pass through boundary
  - Equivalent to reflection symmetry
  - Waves reflect off these boundaries
- Note: dry beaches etc. don't have to be treated as boundaries -- instead just have h=-H initially

Example conditions

- Start with quiet water h=0, beach on one side of domain
- On far side, specify h by 1D Fourier synthesis (e.g. see last lecture)
- On lateral sides, specify $\partial h/\partial n=0$ (reflect solution)
- Keep beach side dry h=-H
- Start integrating

Space Discretization

- In space, let's use finite-differences on a regular grid
- Need to discretize $\nabla^2 h=h_{xx}+h_{zz}$
- Standard 5-point approximation good:

$$\left(\nabla^2 h\right)_{ij} \approx \frac{h_{i+1j}-2h_{ij}+h_{i-1j}}{\Delta x^2}+\frac{h_{ij+1}-2h_{ij}+h_{ij-1}}{\Delta z^2}$$

- At boundaries where h is specified, plug in those values instead of grid unknowns
- At boundaries where normal derivative is specifed, use finite difference too
  - Example $h_{i+1j}-h_{ij}=0$ which gives $h_{i+1j}=h_{ij}$

Surface tension

- Let's go back to nonlinear shallow water equations for a moment
- If we include surface tension, then there's an extra normal traction (i.e. pressure) on surface
  - Proportional to the mean curvature
  - The more curved the surface, the more it wants to get flat again
  - Actually arises out of different molecular attractions between water-water, water-air, air-air
- We can model this by changing pressure BC to $p=\sigma\kappa$ from p=0 at surface y=h

Mean curvature

- If surface is fairly flat, can approximate

$$\kappa \approx -\nabla^2 h$$

- Plugging this pressure into momentum gives

$$u_t + u\cdot\nabla u + g\nabla h - \frac{1}{\rho}\nabla\sigma\nabla^2 h = 0$$

Simplifying

- Doing same linearization as before, but now in 1D (forget z) get

$$h_{tt} = gHh_{xx} - \frac{\sigma H}{\rho}h_{xxxx}$$

- Should look familiar - it's the bending equation from long ago
- Capillary (surface tension) waves important at small length scales

Other shallow water eq's

- General idea of ignoring variation (except linear pressure) in one dimension applicable elsewhere
- Especially geophysical flows: the weather
- Need to account for the fact that Earth is rotating, not an inertial frame
  - Add Coriolis pseudo-forces
- Can have several shallow layers too